Debt Structure under Limited Commitment

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Abstract

We provide a model of the term structure of corporate debt. The optimal debt contract balances the need to provide sufficient termination threat to make repayments incentive compatible (favoring early repayments) with the desire to avoid costly early liquidation (favoring late repayments). This simple trade-off endogenously determines (i) the number of repayment dates, (ii) their timing, and (iii) promised repayment amounts. Firms with stable risky cash flows and large outside financing needs make debt payments earlier and more often, effectively a sequence of short-term debt contracts. For firms with cash-flow growth or significant risk-free cash-flow component, on the other hand, adding risky repayment dates can decrease pledgeable income. In some cases, pledgeability is maximized with one risky bullet repayment far in the future, effectively a long-term debt contract.

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1 Introduction

A large literature has investigated why firms use debt to raise financing for investments.[1] However, much less is known about the determinants of the term structure of corporate debt: the number of repayment dates, their optimal timing, and the respective repayment amounts. This paper develops a simple model of multi-period debt financing, in which a rich term structure of optimal debt contracts emerges from a simple trade-off between providing the firm with incentives to repay and preventing costly early liquidation.

The main friction in our model is classic incomplete contracting: Because cash flow is not verifiable, the entrepreneur can abscond with the cash flow instead of repaying debt. Therefore, debt repayments have to be incentive compatible. Incentive compatibility is achieved by scheduling debt repayments before all of the firm's cash flows have realized, thereby creating a termination threat that induces the firm to repay its debt. For example, in the classic two-period models of Bolton and Scharfstein (1990, 1996) and Hart and Moore (1998), debt repayments occur at the intermediate date—leading to the general insight that debt must be short term relative to the maturity of the underlying assets. However, the two-period nature makes it hard to say anything about debt structure more generally: What if, as in practice, multiple repayment dates are possible? Will the firm continue to make one bullet repayment, or will it optimally pick a number of repayment dates. If the latter, what should be the timing and size of those repayments?

In this paper, we characterize the optimal debt repayment schedule for a firm that produces a stream of unobservable cash flows over many periods. The main trade-off that deter-

[1] Classic contributions to this literature include models of costly state verifiation (Townsend (1979); Gale and Hellwig (1985)), termination threat models of debt(Bolton and Scharfstein (1990, 1996); Hart and Moore (1994, 1998)), incentive-based theories of debt (Innes 1990), and theories based on information sensitivity (Gorton and Pennacchi (1990); Dang et al. (2012)).

mines the optimal debt repayment schedule balances default risk with the necesary incentives to ensure repayment. If the firm defaults on any of its contractually agreed debt repayments, it is terminated by its creditors. Early termination is costly, and the entrepreneur therefore would like to schedule debt payments as late as possible. However, as in all termination threat models, the entrepreneur faces limits in how late he can credibly promise to make repayments to investors: Towards the end of the project, the entrepreneur's continuation value is lower, leading to larger incentives to default and divert the cash flow. As we will show, this simple trade-off generates a rich term structure of optimal debt repayments, characterizing the optimal number of repayment dates, their timing, and optimal repayment amounts.

We first develop a baseline model, in which the firm generates a constant expected cash flow that follows a binary distribution (zero or positive) in each period. In this setting, we show that a repayment profile with constantly spaced debt payments towards the end of the project is optimal. On each repayment date, the firm pays back the entire realized period cash flow to minimize the number of risky payments. The cash flows in between payment dates accrue to the firm, thereby providing the firm with incentives to honor the contractual payments. This baseline model also generates a number of key comparative statics on how leverage, cash-flow risk, and the profitability of the firm affect the optimal debt repayment structure. For example, when per-period cash flows are riskier, it is optimal to space debt repayments further apart. A larger need for outside financing (higher leverage) is associated with a larger number of repayment dates and earlier largest average repayment times.

A key feature of our baseline model with constant expected cash flows is that pledgeability is maximized by scheduling as many repayments as possible, subject to spacing these repayments such that they satisfy incentive compatibility. One natural implementation of this contract is a sequence of short-term debt contracts, where the rollover frequency is determined by the riskiness of the firm's cash flows. Interestingly, this result no longer holds with more general cash flow distributions. For example, when there is growth in the firm's expected cash flow, pledgeability is maximized with a limited number of repayment dates—even though the firm could add more repayment dates, pledgeability is largest under a debt contract with only a few risky repayments towards the end of the project's life. In some cases, pledgeability is maximized with a single (bullet) repayment. For growth firms, the optimal debt contract therefore resembles long-term debt with a maturity close to the lifespan of the firm's assets.

When the firm generates a positive minimum cash flow in each period, the debt contract that maximizes pledgeability depends on the riskiness of the firm. When the safe minimum cash flow is large relative to total cash flow, pledgeability is maximized by offering a safe repayment in every period—essentially safe short-term debt. If, on the other hand, the risky part of the cash flow makes up a significant fraction of the firm's overall cash flow, pledgeability is maximized by alternating between safe and risky repayments. While safe repayments occur throughout the lifetime of the firm's assets, risky repayments are scheduled towards the end of the project and need to be appropriately spaced to preserve incentive compatibility. Moreover, adding more of these risky payments only increases pledgeability up to a point: Pledgeability is generally maximized with a fixed number of risky repayments, independent of the number of periods. In this case, the optimal debt contract therefore resembles a combination of safe short-term debt and a number of risky long-term bonds or loans.

Our paper contributes to the literature on optimal debt contracts. We directly build on the literature on debt as a termination threat (in particular, Bolton and Scharfstein (1990, 1996); Hart and Moore (1995, 1998); Berglöf and von Thadden (1994)). While these papers highlight the importance of short-term debt (relative to asset maturity), the two-period nature of these models does not lend itself to study the optimal repayment structure when multiple repayment dates are possible. The papers that have extended termination-threat models to more periods, generally do not focus on the optimal term structure of debt repayments. For example, Hart and Moore (1994) characterize the fastest and slowest way to repay in a deterministic multi-period setting, but because of the absence of default risk, the key trade-off that determines the optimal repayment structure in our paper does not arise. Gromb (1994) shows that, in a multi-period setting, the ability to repeatedly renegotiate the debt contract can severely constrain pledgeability. Our approach also differs from the literature on optimal financial contracting in dynamic settings (e.g., DeMarzo and Fishman (2007); DeMarzo and Sannikov (2006)). Whereas these papers are interested in the optimal financing contract, our objective is to determine the optimal structure of multi-period debt contracts. Restricting our focus on debt contracts allows us to derive rich predictions on optimal multi-period debt structure but comes at the cost of restricting the contracting space. Finally, our approach also differs from Rampini and Viswanathan (2010), who develop a multi-period model of financing subject to enforcement contraints, in which the optimal contract can be implemented by one-period state-contingent debt contracts.

2 Model Setup

Consider an entrepreneurial firm that can undertake a project that requires an initial investment outlay of I. If funded, the project lasts for T discrete periods. At each $t \in \mathcal{T}$ the project generates a cash flow X_t, where $\mathcal{T} = \{0, 1, 2, \ldots, T\}$ denotes the set of potential cash flow dates. The cash flow distribution at each date t is binary. With probability $\frac{1}{K}$, the project generates positive cash flow of $K\Delta$, where $\Delta > 0$ and $K \in \mathbb{Z}_+$. With probability $1-\frac{1}{K}$, the cash flow at date t is zero. Therefore, at each date t, the project yields an expected cash flow of Δ, while the parameter K captures the riskiness of the project's cash flows (in the sense of a mean-preserving spread). We assume that the cash flow realizations X_t are serially uncorrelated. The assumption of binary cash flows makes the analysis tractable and highlights the key tradeoffs. We extend our analysis to more general cash flow distributions in Section 6.2.

The entrepreneur has cash c at hand and must therefore finance the remainder $D \equiv I - c$ by raising outside financing. We restrict our attention to debt contracts as the only means of outside financing available to the firm. A debt contract is a sequence of promised repayments $\mathcal{R} = \{R_t\}$, $t \in \mathcal{T}$. If at any date t the entrepreneur has promised a positive repayment $R_t > 0$ but does not pay, the project is terminated. Effectively, this means that the investor can commit to terminating the project in the case of default by the firm. When the project is terminated, neither the investor nor the entrepreneur receive any future cash flows. This implies that the project's liquidation value is zero and the entrepreneur cannot undertake another investment after being terminated, either because he is shut out of lending markets or because he has lost access to his project. For simplicity, we assume that the entrepreneur does not save. Rather, at each date t, the entrepreneur consumes the cash flow X_t net of any debt payments made at that date (i.e., consumption is given by $X_t - R_t$). As a result, the firm can only use the contemporaneous cash flow X_t to make the debt payment R_t. This no-savings assumption will greatly simplify the analysis. However, we will relax this assumption in Section 6.1, where we show that adding savings to the model does not affect the main economic insights from our model.

The main contracting friction in our model is standard. We assume that cash flows are not verifiable. Therefore, at any date t the entrepreneur can abscond with the cash flow that realized in that period. The entrepreneur may therefore default even when the realized cash flow is sufficient to make the promised repayment R_t. This implies that any credible sequence of debt repayments $\mathcal{R}$ must be incentive compatible. Formally, denote by V_t the entrepreneur's expected payoff at the beginning of period t. Then, the debt contract $\mathcal{R}$ is incentive compatible if and only if for every t, $R_t \leq V_{t+1}$.

We assume both the entrepreneur and the debt investors are risk neutral. In addition, investors are perfectly competitive, such that any incentive compatible contract that provides an total expected repayment of D will be acceptable to investors. The entrepreneur then chooses such a repayment schedule $\mathcal{R}$ to maximize her expected payoff at day 0. Mathematically,

$$\begin{aligned} &\max\nolimits_{\mathcal{R}} V_0 \\ \text{s.t.} \quad & R_t \leq V_{t+1}, \\ & \mathcal{D}(\mathcal{R}) = D, \end{aligned} \tag{1}$$

where the entrepreneur's payoff V_t satisfies the following recursive and explicit formulations:

$$V_t = \Delta + \Pr(X_t \geq R_t)\,(-R_t + V_{t+1}) \tag{2}$$

$$= \sum_{i=t}^{T} \prod_{s=t}^{i-1} \Pr\left(X_s \geq R_s\right) \Delta - \sum_{i=t}^{T} \prod_{s=t}^{i} \Pr\left(X_s \geq R_s\right) R_i. \tag{3}$$

The value of a debt contract $\mathcal{R}$ is given by

$$\mathcal{D}(\mathcal{R}) = \sum_{t=0}^{T} \prod_{s \leq t} \Pr(X_s \geq R_s) R_t. \tag{4}$$

Using investors' IR condition (1), the expression (2), and (4), we can then simplify the entrepreneur's value function V_0 to

$$V_0 = \sum_{i=0}^{T} \prod_{s=0}^{i-1} \Pr\left(X_s \geq R_s\right) \Delta - D. \tag{5}$$

In addition to incentive compatibility, the optimal debt contract must satisfy the following feasibility condition, which states that any promised payment R_t must be weakly smaller than the cash flow in that period:

$$R_t \leq K\Delta. \tag{6}$$

Suppose, in contrast, that the optimal contract contains a promised repayment $R_t > K\Delta$ for some t. Then the entrepreneur will default with certainty at date t, even if the positive cash flow realizes. Then, the investor's IR constraint (1) holds only if the expected total repayments before period t are D. However, if this is the case, then the entrepreneur would adjust R_t to 0. Such an adjustment will not change the investors IR constraint (1), but will provide the entrepreneur at least the same total expected cash flows (strictly more if $t \leq T - 1$).

The main choice varible in our model is whether the firm promises a positive repayment at a particular date t. To see this, denote the set of (positive) repayment dates to be $\mathcal{Q} \equiv \{t \in \mathcal{T} | R_t > 0\}$. Then, as the following lemma shows, it is the timing of repayments that matters for the firm, while the size of each repayment can usually not be uniquely determined.

Lemma 1 *For any two incentive compatible repayment schedules, $\mathcal{R}$ and $\mathcal{R}'$, if $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}')$ and $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, then the entrepreneur is indifferent between $\mathcal{R}$ and $\mathcal{R}'$.*

Intuitively, Lemma 1 states that if two debt contracts $\mathcal{R}$ and $\mathcal{R}'$ have identical repayment dates and same expected values, then they yield the same expected payoffs to the entrepreneur. This result follows directly from the binary cash flow assumption. The probability of making a repayment $R_t \in (0, K\Delta]$ is $\Pr(X = K\Delta) = \frac{1}{K}$, the probability of positive cash flow realization, regardless of the size of the repayment. From equation (5), we see that any positive R_t enters the entrepreneur's payoff only through this probability. Therefore, only the timing of payments $\mathcal{Q}$ is important, but not the size of each individual repayment.

3 Optimal Debt Structure

The main trade-off in determining the optimal debt structure is as follows. On the one hand, the entrepreneur likes to make debt payments as late as possible. By doing so, the project is less likely to be terminated early on, providing the entrepreneur higher expected cash flows. On the other hand, the entrepreneur faces limits in how late he can credibly promise to make repayments to investors: Towards the end of the project, the entrepreneur's continuation value is lower, such that he has larger incentives to default and divert the cash flow.

To see what determines the optimal repayment structure, it is instructive to start with low outside financing needs D. The nature of the optimal debt structure then emerges as we increase D. We will build up these results using several case, before moving to a general proposition that fully characterizes the optimal debt repayment structure. The numbering of the cases will become clear as we move from case to case below.

Case 1-1: $D \in \left(0, \frac{\Delta}{K}\right]$. We first assume that the amount of outside financing needed is less than $\frac{\Delta}{K}$. Of course, because $V_{T+1} = 0$, the entrepreneur cannot credibly promise to make a payment to investors at date T. Therefore, incentive compatibility requires that $R_T = 0$. At date $T-1$, however, the entrepreneur can credibly promise to repay $R_{T-1} = KD$. This is incentive compatible, because

$$V_T = \Delta \geq KD = R_{T-1}. \tag{7}$$

Given the assumption that $D \in \left(0, \frac{\Delta}{K}\right]$, this single repayment also satisfies the investor's IR constraint (1). From equation (2), the entrepreneur's continuation value at the beginning of date $T-1$ is given by

$$V_{T-1} = \Delta + \frac{1}{K}(\Delta - KD),$$

and the overall payoff to the entrepreneur by

$$V_0 = \Delta + V_1 = ... = (T-1)\Delta + V_{T-1} = T\Delta + \frac{1}{K}(\Delta - KD). \tag{8}$$

Note that even though the entrepreneur can potentially choose to make multiple repayments or a single one before period $T-1$, this is never optimal. Intuitively, promising multiple repayments inefficiently increases default risk. Promising repayment earlier than date $T-1$ risks that the project is terminated prematurely, and unnecessarily risks loss of cash flows. Therefore, any alternative schedule with a single repayment of KD at $t' < T-1$, which yields a payoff to the entrepreneur of

$$(t'+1)\Delta + \frac{1}{K}[(T-t')\Delta - KD],$$

is dominated by (8).

Case 1-2: $D \in \left(\frac{\Delta}{K}, \frac{2\Delta}{K}\right]$. When the required amount of financing D exceeds $\frac{\Delta}{K}$, a single repayment of KD at day $T-1$ is no longer incentive compatible because it would violate the IC constraint (7). So to support a higher repayment, the entrepreneur moves the single repayment date to $T-2$. In this case, because the final two periods' cash flows are left to the entrepreneur, $V_{T-1} = 2\Delta$, which provides the upper bound for the incentive compatible repayment at $T-2$:

$$R_{T-2} \leq V_{T-1} = 2\Delta.$$

Therefore, when the required amount of financing D lies in the interval $\left(\frac{\Delta}{K}, \frac{2\Delta}{K}\right]$, the entrepreneur can raise the required financing with a single repayment at date $T-2$. Because any additional repayment date would add additional default risk, the single payment at $T-2$ is the optimal way to finance the project.

Case 1-K: $D \in \left(\frac{(K-1)\Delta}{K}, \Delta\right]$. It is easy to see that as D continues to increase, the single payment date continues moving forward in time. This is possible, as long as the required single repayment satisfies the feasibility constraint (6). This leads to the final case, in which financing with one repayment date is possible, Case 1-K. In this case, a maximum repayment of KD is made at date $T-K$. As before, the entrepreneur's continuation value $V_{T-K+1} = K\Delta$ gives the maximum repayment $R_{T-K} = K\Delta$, but note that in this case the feasibility condition (6) also binds. Therefore, $D = \Delta$ is the maximum amount that can be funded with a single repayment date. Cases 1-1 to 1-K are illustrated in Figure 1.

Case 2-1: $D \in \left(\Delta, \Delta + \frac{\Delta}{K^2}\right]$. As we just saw, when D exceeds Δ, a single repayment of KD at any time is no longer a feasible way to finance the project because it would violate the feasibility condition (6). Therefore, the entrepreneur must now promise repayments at two dates. The optimal way to do this is to move forward the payment from date $T-K$ to $T-K-1$ and add a second payment at date $T-1$, resulting in optimal repayment dates $\mathcal{Q} = \{T-K-1, T-1\}$. Note that once there are two repayment dates, the optimal debt contract is not unique, except for when D is at the upper boundary of the interval (i.e., $D = \Delta + \frac{\Delta}{K^2}$). One such a contract is given by $R_{T-K-1} = K\Delta$ and $R_{T-1} = K^2(D-\Delta)$. One

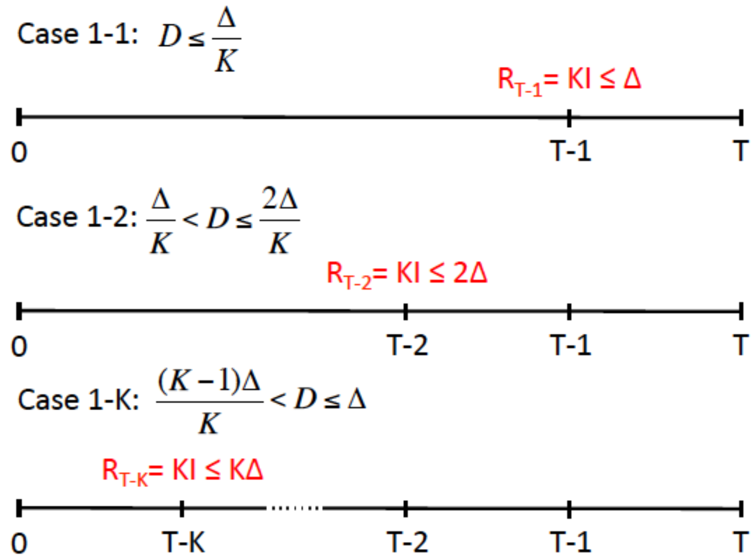

Figure 1: This figure illustrates the range of outside financing needs for which financing is possible with one repayment date, Case 1-1 to Case 1-K.

can easily verify that this contract satisfies investor's IR condition (1):

$$\mathcal{D}(\mathcal{R}) = \frac{1}{K} R_{T-K-1} + \frac{1}{K^2} R_{T-1} = D.$$

Moreover, the contract is incentive compatible: For any $D \in \left(\Delta, \Delta + \frac{\Delta}{K^2}\right]$ the IC constraint at date $T-1$ is satisfied,

$$R_{T-1} = K^2(D - \Delta) \le \Delta = V_T.$$

In addition, we can show that also the IC constraint at date $T-K-1$ holds. Using (2), we can write the continuation value after date $T-K-1$ as

$$V_{T-K} = \Delta + V_{T-K+1} = ... = (K-1)\Delta + V_{T-1} = K\Delta + \frac{1}{K}(-R_{T-1} + \Delta).$$

By incentive compatibility of R_{T-1}, we have $V_{T-K} \geq K\Delta = R_{T-K-1}$. Intuitively, leaving K periods' worth of cash flow between two repayment dates to the entrepreneur makes sure that the repayment of $K\Delta$ at date $T-K-1$ is incentive compatible. The second repayment at date $T-1$ is bounded by Δ, by exactly the same intuition in Case 1-1. It is also easy to verify that the schedule $\mathcal{R}$ with $R_{T-K-1} = K\Delta$ and $R_{T-1} = \Delta$ attains the upper bound of $D = \Delta + \frac{\Delta}{K^2}$ in this case.

Case 2-2: $D \in \left(\Delta + \frac{\Delta}{K^2}, \Delta + \frac{2\Delta}{K^2}\right]$. As we increase D further, the optimal repayment dates shift forward to $\mathcal{Q} = \{T-K-2, T-2\}$. Specifically, compared with Case 2-1, both repayment dates are moved forward by one period. This increases pledgeability at the second (and last) repayment date, while maintaining incentives to repay at the first repayment date. Similar to Case 1-2, the maximum incentive compatible repayment at $T-2$ is $V_{T-1} = 2\Delta$. By keeping K periods between repayments maintains the incentive compatibility of the first repayment, which is at most $K\Delta$. The debt contract with $R_{T-K-2} = K\Delta$ and $R_{T-2} = 2\Delta$ then attains the upper bound of this case, $D = \Delta + \frac{2\Delta}{K^2}$.

Case 2-K: $D \in \left(\Delta + \frac{(K-1)\Delta}{K^2}, \Delta + \frac{\Delta}{K}\right]$. As we continue to increase D, at some point we arrive at Case 2-K. This is the last case, in which the required amount of financing can be raised with two repayment dates, which occur at $\mathcal{Q} = \{T-2K, T-K\}$. The maximum incentive compatible debt contract has promised repayments of $R_{T-2K} = R_{T-K} = K\Delta$, which attains the maximum debt value of $\Delta + \frac{\Delta}{K}$ in this case. At this point, the feasibility condition for both repayments binds, such that if the entrepreneur needs to borrow more, she must increase the number of repayment dates. Cases 2-1 to 2-K are illustrated in Figure 2.

Given the pattern that emerges above, we can now characterize the **general Case N-j**,

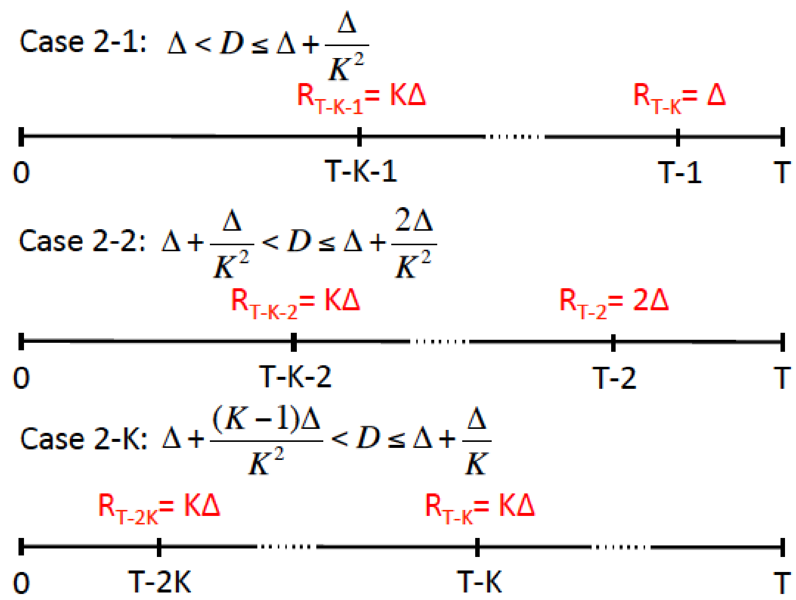

Figure 2: This figure illustrates the range of outside financing needs for which financing is possible with two repayment dates, Case 2-1 to Case 2-K.

which has N repayments with the final repayment at date $T-j$:

$$D \in \left(\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}, \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N} \right].$$

In this general case, the optimal repayment dates are $\mathcal{Q} = \{T-(N-1)K-j, T-(N-2)K-j, ..., T-j\}$. The maximum feasible and incentive compatible repayments are $R_{T-nK-j} = K\Delta$ for all $n = 1, 2, ..., N-1$, and $R_{T-j} = j\Delta$ for the final repayment at date $T-j$. The expected value of this debt contract is

$$\mathcal{D} = \frac{1}{K}K\Delta + \frac{1}{K^2}K\Delta + ... + \frac{1}{K^{N-1}}K\Delta + \frac{1}{K^N}j\Delta,$$

which is equal to the maximum amount of financing that can be raised in Case N-j. The

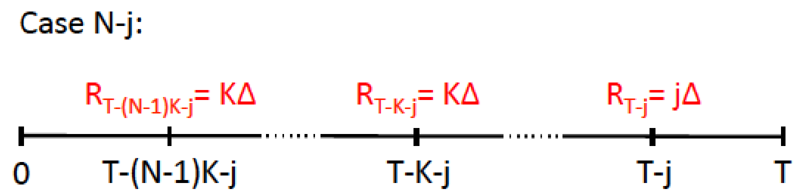

Figure 3: This figure illustrages the general case N-j.

general case N-j is illustrated in Figure 3.

With this general case in hand, we are now in a position to give a full characterization of the optimal repayment schedule $\mathcal{Q}$, which is formally summarized in Proposition 1.

Proposition 1 *In an optimal debt contract, the set of repayment dates is*

$$\mathcal{Q}_{N,j} \equiv \{T-j, T-K-j, T-2K-j, ..., T-(N-1)K-j\}$$

if and only if the required investment

$$D \in \left(\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}, \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N} \right], \tag{9}$$

which is a partition of all feasible investment amounts when (N, j) is any pair of positive integers such that

$$T-(N-1)K-j \geq 0.$$

In addition, when D equals any one of the cutoff values $\sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$, the unique optimal *debt repayment schedule is given by*

$$R_t = \begin{cases} K\Delta, & \textit{if } t \in \mathcal{Q} \backslash \{T-j\} \\ j\Delta, & \textit{if } t = T-j \\ 0, & \textit{otherwise.} \end{cases}$$

Proposition 1 uniquely pins down the optimal repayment dates (debt structure) of the optimal debt contract. It also uniquely pins down the optimal payment amounts at each repayment date at the boundary of each of the intervals in (9). At the boundaries, the incentive compatibility constraints bind at each of the repayment dates, such that it is impossible to shift repayments between repayment dates in $\mathcal{Q}$. In between the boundaries of the intervals in (9), the repayment dates are still uniquely determined, but the repayment amounts are not uniquely determined. As shown in Lemma 1, the entrepreneur is indifferent between all incentive compatible (and feasible) repayment patterns on the repayment dates $\mathcal{Q}$.

One key feature of the optimal repayment schedule is payments are constantly spaced, separated by K periods. Moreover, denote by $PI(N)$ the maximum pledgeable income of a debt contract with exactly N risky repayment dates. The maximum pledgeable income with $N < \lfloor \frac{T}{K} \rfloor$ repayment dates takes the form of a simple geometric sum:

$$PI(N) = \Delta \sum_{n=0}^{N-1} \frac{1}{K^n}, \tag{10}$$

where $\lfloor x \rfloor$ denotes the floor function (i.e, the largest integer weakly smaller than x). Intuitively, the maximum the firm can pledge with N repayment dates is N repayments of

$R_{T-nK} = K\Delta$, each weighted by the probability of making the n^{th} repayment $\frac{1}{K^n}$, where $n = 1, 2, ..., N$. Given that, in order to minimize default risk, the firm prefers to limit its debt repayment schedule to a minimum number of risky repayments. $PI(N)$ also pins down the number of payment dates of the optimal debt contract:

Corollary 1 *The optimal debt contract has exactly N payments, if*

$$D \in \left(\Delta \sum_{i=0}^{n-2} \frac{1}{K^i}, \Delta \sum_{i=0}^{n-1} \frac{1}{K^i} \right]. \tag{11}$$

4 Empirical Implications

In this Section, we perform a number of comparative statics to illustrate some of the key empirical implications of our model. Specifically, we investigate how leverage (D), the riskiness of cash flows (K), and profitability (Δ) affect the number of repayments (N), the spacing between repayments ($t_{i+1} - t_i$, where $t_i, t_{i+1} \in \mathcal{Q}$ are two consecutive repayment dates), and the duration of the optimal debt contract.

For our analysis of debt duration, we focus on the (expected) average repayment time (ART) of the debt contract. Formally, for a debt contract with repayment dates $\mathcal{Q} = \{t_i : i = 1, \ldots, N\}$, we define the average repayment time (ART) as

$$ART \equiv \frac{\sum_{i=1}^{N} t_i \frac{R_{t_i}}{K^i}}{D}. \tag{12}$$

One slight complication when analyzing duration is that, as shown in Proposition 1, only the equilibrium repayment dates $\mathcal{Q}$ are uniquely determined, whereas the repayment amounts at each repayment date, and therefore the average repayment time ART, are not uniquely pinned down. However, for a given set of parameters, we can characterize the longest and shortest ART as follows.

Lemma 2 *Given any D that satisfies* (9)*, the longest average repayment time*

$$\sup ART = [T - j - (N-1)K] + \frac{1}{D}\left[\sum_{i=1}^{N-2} \frac{\Delta}{K^{i-1}} i + \frac{j\Delta}{K^{N-1}}(N-1)\right] \tag{13}$$

is attained by back loading repayments: $R_{t_i} = K\Delta$ for all $2 \leq i \leq N-1$, $R_{t_N} = j\Delta$, and $R_{t_1} = K(D - \sum_{i=1}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N})$.

The shortest average repayment time

$$\inf ART = (T - j) - \frac{1}{D}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) \tag{14}$$

is attained by front loading repayments: $R_{t_i} = K\Delta$ for all $i \leq N-1$ and $R_{t_N} = K^N \left(D - \sum_{i=0}^{N-2} \frac{\Delta}{K^i}\right)$.

4.1 Leverage

We first analyze the effect of leverage. The easiest way to analyze higher leverage in our model is through a reduction of the firm's cash resources c. This translates into a higher amount $D = I - c$ that needs to be raised through debt, while leaving the NPV (assuming no default) of the investment project constant. From Proposition 1, we know that the entrepreneur can raise more outside debt financing in three ways: (i) increasing payments within a given case N-j, (ii) adding more repayment dates, and (iii) moving existing repayment dates forward. In the latter two scenarios, the optimal debt contract becomes more short-term, in the sense that the average repayment time of the optimal debt contract decreases. However, in the first scenario, increasing payments have opposite effects on $\inf ART$ and $\sup ART$, because the weights $\frac{R_{t_i}}{K^i}$ in (12) are affected differently. Taken together, we therefore have the following proposition.

Proposition 2 *Holding all other parameters constant, an increase in debt financing D (i.e., higher leverage) weakly increases the number of repayments $\#\mathcal{Q}$ and strictly decreases the longest average repayment time* $\sup ART$*. In addition, the shortest average repayment time* $\inf ART$ *is strictly increasing in D within any case N-j as specified by* (9)*, but is strictly decreasing across cases.*

Empirically, our model therefore predicts that highly levered firms exhibit debt structures with more repayment dates. Moreover, when leverage increases sufficiently to increase the required number of repayment dates, debt duration decreases.

4.2 Riskiness of Cash Flow

We now turn to the riskiness of cash flows, which in our model is captured by the parameter K. The effect of cash-flow risk on the optimal debt contract is an immediate corollary of Proposition 1: When cash flow becomes riskier, the repayment profile becomes lumpier, with higher individual promised payments ($K\Delta$) and longer time intervals (K) between two payments. In addition, maximum pledgeable income with N repayments ($PI(N)$) is decreasing in cash-flow risk. As a result, to support the same amount of outside financing, a firm with riskier cash flows needs to spread its debt repayments across more repayment dates. Combined with the longer intervals between repayments, this implies that the entire repayment profile of the optimal debt contract extends forward. Consequently, debt duration decreases when risk increases. The following proposition summarizes these findings.

Proposition 3 *As cash-flow risk K increases, holding all other parameters including D constant, the number of repayments $\#\mathcal{Q}$ weakly increases; the time between two repayments* *increases; and debt duration (in terms of both* $\inf ART$ *and* $\sup ART$*) decreases.*

The prediction that cash flow risk is associated with earlier debt repayment has broad support in the empirical literature on debt maturity. Stohs and Mauer (1996) find that riskier firms (as measured by lower EBITDA volatility) have shorter maturity debt. Barclay and Smith (1995) document that higher volatility of asset returns (implied from equity returns) correlates negatively with the fraction of debt that matures in more than three years. Guedes and Opler (1996) document that higher industry volatility of ROA growth is negatively correlated with maturity. Ortiz-Molina and Penas (2008) find that riskier small business owners with more personal delinquency records receive shorter credit line. In addition, the prediction that cash flow risk is associated with more repayment dates is consistent with the empirical findings documented by Choi et al. (2016). However, our model highlights an alternative channel. While in Choi et al. (2016) firms choose a larger number of repayment dates to smooth out rollover risk, in our framework a larger number of repayment dates is necessary because in the presence of increased cash flow risk each individual repayment is less likely to be made.

4.3 Profitability

Finally, we examine the effect of the profitability of the investment project on debt structure. The expected per-period cash flow Δ is a natural measure for profitability. However, given the binary cash-flow structure, changing Δ also affects the variance of the period cash flow, which is given by $\Delta^2 (K-1)$. To analyze the marginal effect of profitability, we therefore vary Δ while holding cash flow variance constant. This exercise is equivalent to reducing K, and therefore the results in Proposition 3 are flipped. In particular, the debt profile of a more profitable company features fewer payments and shorter intervals between payments.

Profitable firms therefore structure their debt into fewer repayments that are concentrated towards the end of the project's life. Consequently, debt duration is longer.

Proposition 4 *As the expected period cash flow Δ increases, holding cash flow variance ($\Delta^2 (K-1)$) and all other parameters constant, the number of repayments $\#\mathcal{Q}$ weakly decreases, the time between risky repayments decreases, and* $\sup ART$ *increases. Finally, when* $K \geq 4$*, also* $\inf ART$ *increases.*

The main prediction of Proposition 4 is that higher profitability is associated with more backloaded repayments (i.e., fewer repayment dates concentrated towards the end of the project's lifetime). This prediction is consistent with the empirical literature, which finds that profitbility is generally associated with longer debt maturity. For example, Qian and Strahan (2007) find that more profitable firms (as measured by net income divided by assets) borrow longer-term. Similarly, Guedes and Opler (1996) show that less profitable firms (as measured by larger operating loss carryforwards) tend to have debt of shorter maturity.

5 Cash Flow Growth and Positive Low Cash Flow

In this Section, we add two features to our baseline model. First, in Section 5.1, we allow for growth in the project's cash flow over time. Second, in Section 5.2 we allow for a positive low cash flow, rather than a zero cash flow in the low state. In both of these cases, we will see that pledgeability is generally no longer maximized with the largest possible number of risky repayment dates. Rather, pledgeability may be maximized with a contract that resembles long-term debt, or a combination of safe short-term debts and long-term debts.

5.1 Growth in Cash Flow

Suppose that positive cash flow realizations of the project grow at the rate $\mu > 1$. Specifically, at date $t \in \mathcal{T}$, the cash flow is given by $X_t \in \{K\mu^t\Delta, 0\}$. As in the baseline model, the probability of having a positive cash flow $K\mu^t\Delta$ at date t is still $\frac{1}{K}$, such that the expected date-t cash flow is $\mu^t\Delta$. This cash flow distribution differs from the baseline model mainly in that the maximum feasible repayment now depends on the time when the particular repayment is made. (In contrast, in the baseline model, the maximum feasible repayment is $K\Delta$, which is time-invariant.) This change of assumption may be particularly relevant for young growth firms, whose capability to produce cash flow increases over time.

Similar to the baseline model, where we simply assumed that K is an integer, we now make the following (essentially equivalent) assumption on the pair of (K, μ) only in this section for tractability.

Assumption 1 *There exists $m \in \mathbb{Z}_+$ such that*

$$K = \sum_{s=1}^{m} \mu^s. \tag{15}$$

Under this assumption, it is incentive compatible for the firm to repay the maximum feasible amount $K\mu^t\Delta$ at t, if the next m periods' cash flows are left to the entrepreneur:

$$K\mu^t\Delta = \mu^{t+1}\Delta + \mu^{t+2}\Delta + ... + \mu^{t+m}\Delta.$$

As a result, it is incentive compatible for the entrepreneur to repay $K\mu^t\Delta$ every m periods.

Some of the main intuitions of the baseline model remain valid with growth in cash flow. The entrepreneur would like to make as few repayments as possible and schedule them as late

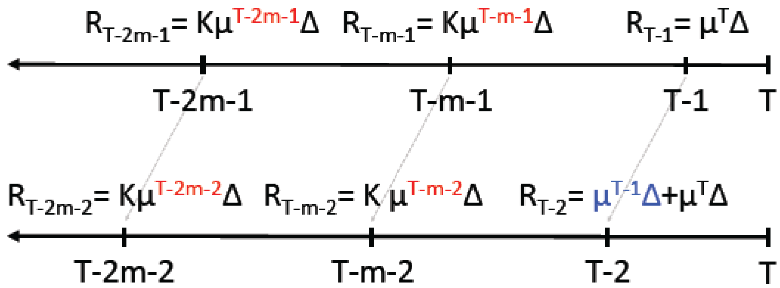

Figure 4: Under cash-flow growth, shifting forward the existing N repayments reduces the total expected repayments on these repayment dates.

as possible, subject to maintaining incentive compatibility. As in the baseline model, once the firm starts making risky repayments, these are constantly spaced. The slight difference is that growth in cash flow allows risky repayments to be scheduled closer to each other, given that $m < K$.

However, one key implication that changes with growth in cash flow is that increasing the number of repayments may no longer increase pledgeability. To see this, note that when the number of repayments increases from N to $N+1$, the first N repayments have to be shifted forward, as illustrated in Figure 4. When there is growth in cash flow, positive cash-flow realizations are smaller in earlier periods, such that the maximum total expected repayments on the first N repayment dates is now strictly smaller than under the original debt contract with N repayments. On the other hand, the new contract has an extra repayment $N+1$. Therefore, whether the new contract with $N+1$ repayments can generate a higher total expected repayment depends on which effect dominates, the decrease in the first N repayments or the added $N+1$ repayment.

Intuitively, the reduction in expected repayments from the first N repayment dates dom-

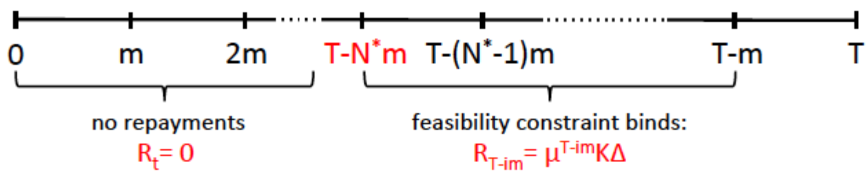

Figure 5: When there is growth in cash flow, pledgeability is generally maximized with a fixed number of N^* repayments towards the end of the project.

inates when N is large: Shifting forward the first N repayment dates approximately reduces the value of the first N repayments by the growth factor μ. On the other hand, the effect of the added repayment date $N+1$ is weighted by the survival probability $\frac{1}{K^N}$, it becomes small when N is sufficiently large. Therefore, when N is large, an increase in the number of repayments decreases the total expected repayment. Pledgeability is then maximized with N^* repayments, where $N^* > 0$ is given by the smallest integer such that

$$\left[(\mu^{-1}-1)K\sum_{j=1}^{N^*+1}\left(K\mu^{-m}\right)^j\right]+1<0. \tag{16}$$

Because $\mu^{-1}-1<0$ and $K\mu^{-m}>1$, N^* is well defined and unique. Importantly, N^* is independent of T. Therefore, even when the number of possible repayment dates T grows large, pledgeability continues to be maximized with a fixed number of N^* repayments, as illustrated in Figure 5.

Proposition 5 *In the model with growing cash flows ($\mu > 1$),*

1. *the maximum pledgeable income $PI(N)$ is maximized at N^* for any T sufficiently large;*

2. *for any $N \leq N^*$, the maximum pledgeable income with N repayments $PI(N)$ is*

$$PI(N) = \sum_{i=0}^{N-1} \frac{\mu^{T-(N-i)m}}{K^i} \Delta;$$

3. *for any $N \leq N^*$, if $D \in (PI(N-1), PI(N)]$, the optimal debt contract has N repayment dates and has the first repayment date $t_1 \geq T - Nm$;*

4. *for any $N \leq N^*$, if $D = PI(N)$, there is a unique optimal debt contract characterized by*

$$R_t = \begin{cases} \mu^t K \Delta, & \text{if } t \in \{T-m, T-2m, ..., T-Nm\} \\ 0, & \text{otherwise.} \end{cases}$$

One implication of Proposition 5 is that for some firms it may never be optimal to have more than one repayment, regardless of the financing need D:

Corollary 2 *When $1 > \frac{1}{K} + \mu^{-m}$, the maximum number of repayment N^* in Proposition 5 is* 1.

From Proposition 5 and Corollary (2), we see that in the presence of cash-flow growth, the optimal debt contract resembles long-term debt. Independent of the project's horizon T, all repayments occur in the final N^*m periods. Therefore, as T becomes very large, the earliest possible repayment time $T - N^*m$ in the optimal debt contract approaches T, in the sense that

$$\lim_{T \to \infty} \frac{T - N^*m}{T} = 1.$$

This finding complements our understanding of short-term debt as a disciplinary device to improve pledgeability. In two period models, for the threat of termination to be credible,

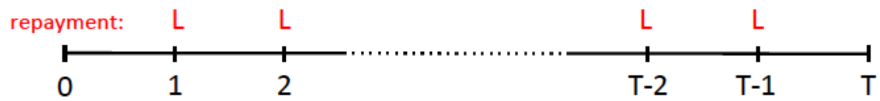

Figure 6: When $L \geq \frac{\Delta}{K-1}$, pledgeable income is maximized by making a risk-free repayment of L every period.

debt essentially has to be short-term (i.e., one-period debt). When many repayment dates are possible, on the other hand, it is possible that the financing horizon (debt duration) roughly matches the project's horizon, especially when T is large.

5.2 Positive Low Cash Flow

In this section, we analyze the model in which the project can generate a risk-free cash flow $L > 0$ in every period. Assume the cash flow distribution X_t is binary with a high cash flow of $L + K\Delta$ with probability $\frac{1}{K}$ and L with complementary probability. The average per-period cash flow is $\Delta + L$. Again, assume that $\Delta > 0$, and $K > 1$ is an integer.

Obviously, if $I \leq TL$, the optimal contract is to repay by risk-free cash flows up to L at every $t \in \{0, 1, 2, ..., T-1\}$. One can also easily verify that any risk-free repayment profile is indeed incentive compatible. More generally, Proposition 6 shows that when the risk-free cash-flow component is sufficiently large, it is never optimal to use risky debt, and pledgeability is maximized by making a risk-free repayment at every date, as illustrated in Figure 6.

Proposition 6 *If*

$$L \geq \frac{\Delta}{K-1}, \tag{17}$$

then the risk-free schedule $R_t = L$ for all t maximizes pledgeable income.

The intuition of Proposition 6 is as follows. The benefit of increasing the repayment beyond the risk-free level is that the entrepreneur pays back more when the high cash flow realizes, which improves pledgeability. However, this risky repayment also generates default risk, which hurts the expected value of the current as well as the subsequent repayments. Therefore, risky repayment is never optimal when the risk-free cash flow L is large compared to the expected benefit of risky repayment Δ, as implied by condition (17). In addition, condition (17) is more likely to hold when cash flow risk is large (high K). In this case, default risk is more costly in expectation, such that risk-free debt is more likely to be optimal.

From here to the end of the section, we assume (17) does not hold in order to focus on the case where introducing risky repayments can enhance pledgeability. Similar to the case with cash flow growth in Section 5.1, Proposition 7 shows that some of the baseline results still hold. Specifically, all risky repayments are scheduled towards the end of the project. In addition, in order to minimize the number of risky repayments, every risky repayment is the entire high cash flow of $K\Delta + L$. However, similar to the case with cash-flow growth described Proposition 5, with a risk-free cash-flow component, we also find that pledgability is usually maximzied by limiting the number of risky repayments to $N^{**} > 0$, where N^{**} is the smallest integer such that

$$\frac{\Delta + L}{K^{N^{**}+1}} < L.$$

Proposition 7 *In the model with riskfree cash flows,*

1. *the maximum pledgeable income $PI(N)$ is maximized at N^{**} for any T sufficiently large;*

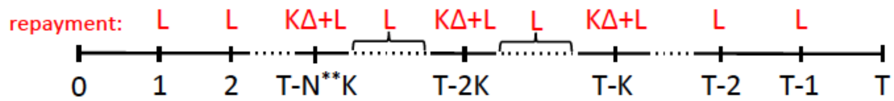

Figure 7: When $L < \frac{\Delta}{K-1}$, pledgeability is maximized by limiting the number of risky repayment dates to N^{**}.

2. *for any $N \leq N^{**}$, the maximum pledgeable income is*

$$PI(N) = (T - NK)L + \sum_{j=1}^{N} \frac{\Delta + L}{K^{j-1}};$$

3. *for any $N \leq N^{**}$, if $D \in (PI(N-1), PI(N)]$, the optimal debt contract has N repayment dates, and the first risky repayment will be made at $t_1 \geq T - NK$;*

4. *for any $N \leq N^{**}$, if $D = PI(N)$, there exists a unique optimal contract that is characterzied by*

$$R_t = \begin{cases} K\Delta + L, & \text{if } t \in \{T-K, T-2K, ..., T-NK\} \\ L & \text{otherwise.} \end{cases}$$

One interesting implication of Proposition 7 is that the entrepreneur always pays out the entire risk-free cash-flow component L, even after she begins to make risky repayments. This result arises from the trade-off between an increase in a (relatively) safe repayment and a decrease in a (relative) risky repayment of moving the risky repayment one day later. Let's consider the one risky repayment case. Before such a risky repayment, all repayments are safe, so we can call them safe repayments; after the first risky repayment, all repayments are risky, so we can call them risky repayments. When the entrepreneur delays a risky repayment one date, she can make an extra safe repayment L, but the maximum risky repayment she can make will decrease by $\frac{\Delta+L}{K}$, due to the incentive compatibility constraint. Therefore, the contract that leads to the maximum pledgeable income either have the risky repayment date as early as possible (until the incentive compatibility constraint binding) or has no risky repayments at all. In the latter case, it is obvious that the entrepreneur needs to pay out all safe cash flows to maximize the pledgeable income. In the former case, to make the risky repayment date as early as possible (and the incentive compatibility constraint binding), the entrepreneur will pay off all safe cash flows even after then risky repayment date.

Part 3 of Proposition 7 confirms the intuition of Part 3 of Proposition 5 that having as many risky repayments as possible does not generally maximize pledgeability. However, the intuition here is different. As the entrepreneur moves the repayment schedule forward by one period to allow for higher (or more) risky repayments, she sacrifices one period with a risk-free repayment of L. The contribution to the debt value from the final risky repayment is weighted by the probability of making this repayment $\frac{1}{K^N}$ (if there are N risky repayments). Therefore as N becomes very large, the benefit from the last risky repayment diminishes exponentially, and the cost of sacrificing a risk-free repayment of L dominates. It turns out that this result is very robust. For example, we show in Section 6.2, that for general cash flow distributions, pledgeability is maximized by limiting the number of risky repayments. In particular, the entrepreneur generally does not "smooth out" risky repayments over the entire lifespan of the project.

6 Extensions

6.1 Allowing for Savings

Up to now, we have assumed that the entrepreneur can only use contemporaneous cash flow to make repayment. In this section, we assume that the entrepreneur can save, and so he can use previously realized cash flows to make current repayment. The main difference from the baseline model is that the probability that the entrepreneur can make a repayment depends on the number of the period before this repayment.

We show that although saving can enlarge the feasibility of any single repayment (that is, R_t could be strictly greater than $K\Delta$), in the optimal repayment schedule, any single repayment $R_t \leq K\Delta$.

Proposition 8 *In the optimal debt contract, any individual repayment R_t is weakly smaller than $K\Delta$.*

With the possibility of savings, one may think that, unlike the evenly space repayments of $K\Delta$ in the baseline model, it could be optimal to allow the entrepreneur to save for some time and demand a single large repayment to consolidate the default risks. This intuition is incorrect. To see flaw in this argument, let's consider how to improve upon a single repayment of $R_t = 2K\Delta$. For simplicity, suppose this is the last repayment. For this repayment to be incentive compatible, it can only occur weakly before $T - 2K$, otherwise the expected cash flow left to the entrepreneur is smaller than this repayment and she will default. We can then split up the single repayment into two repayments of $K\Delta$ at $T - 2K \geq t$ and $T - K$. First it is easy to see the new schedule is still incentive compatible. Next, we argue this adjustment also strictly improves the value of debt.

For any realization of historical cash flows, if the entrepreneur can repay $2K\Delta$ at date t, she can also make the two repayments of $K\Delta$ in the new schedule. On the other hand, the new schedule allows the entrepreneur strictly more time to pay back the second $K\Delta$. Therefore, there exists some cash flow paths on which the entrepreneur defaults on the original schedule, but honors the adjusted schedule. Hence, any debt contract with any individual repayment greater than $K\Delta$ is strictly dominated.

6.2 General Cash Flow Distribution

In this section, we abandon the binary cash flow assumption and consider more general cash flow distributions. We show that in general the optimal debt contract has at most a fixed number of risky repayments that is independent of T. As a result, when T is large, it is never optimal for the entrepreneur to fill the entire lifespan of the project with risky repayments.

Proposition 9 *Suppose the cash flow distribution $F(X)$ has the following properties:*

There exists some positive cash flow level $L > 0$ such that

1. *$X \geq L$ holds with probability 1;*
2. *$F(L) = \epsilon > 0$;*
3. *X has finite expectation.*

Then for any T sufficiently large, any repayment profile that maximizes pledgeable income has strictly less than $2N^*$ risky repayments, where $N^* > 0$ is the smallest integer that satisfies

$$\frac{L}{E(X)} > \max\{(1-\epsilon)^{N^*-2}, 2(1-\epsilon)^{2N^*-1}\}. \tag{18}$$

Assumption 1 says the cash flow has a minimum level of L. Assumption 2 states that the distribution has a point mass on L. These assumptions are trivially satisfied for any positive discrete cash flow distributions.

The intuition for Proposition 9 is essentially the same as Part 3 in Proposition 7. The benefit of having one more risky repayment is weighted by the survival probability, which decreases exponentially with the number of risky repayments. The cost is a constant sacrifice of a risk-free repayment L. Therefore, when we have sufficient number of risky repayments, adding more does improve the value of debt.

7 Conclusion

This paper provides a model of the optimal term structure of corporate debt. Building on the insights of the literature on debt as a termination threat, which as mostly worked in two-date settings, our multi-period model generates rich implications on the optimal number and timing of payments to creditors. Optimal repayment structure is determined by a simple trade-off between providing the firm with incentives to repay and preventing costly early liquidation.

The model generates a rich set of empirical predictions: Depending on the required amount of outside financing and the cash-flow charasteristics of the firm, the resulting debt structures can resemble a sequence of risky short-term debt contracts (firms with stable expected cash flow and large outside financing needs), long-term debt (growth firms) and safe short-term debt (firms with a significant safe cash-flow component), or a combination of safe short-term debt and risky bonds or loans (firms with a moderate safe cash-flow component).

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A Omitted Proofs

Proof of Lemma 1

Because $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}')$, and X_t is either $K\Delta$ or 0, equation (6) implies that $\Pr(X_t \geq R_t) = \Pr(X_t \geq R'_t)$ for any t. Therefore,

$$\sum_{t=0}^{T} \prod_{s=0}^{t-1} \Pr\left(X_s \geq R_s\right) \Delta = \sum_{t=0}^{T} \prod_{s=0}^{t-1} \Pr\left(X_s \geq R'_s\right) \Delta.$$

In addition, since $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, it follows from equation (5) that $\mathcal{R}$ and $\mathcal{R}'$ will lead to the same V_0. Therefore, the entrepreneur is indifferent between $\mathcal{R}$ and $\mathcal{R}'$.

Q.E.D.

Proof of Proposition 1

We prove this proposition by a series of claims.

Claim 1: For any two incentive compatible debt contracts, $\mathcal{R}$ and $\mathcal{R}'$, if $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$ and $\mathcal{Q} \subset \mathcal{Q}'$, then the entrepreneur strictly prefers $\mathcal{R}$. Put differently, other things equal, the entrepreneur wants to reduce the number of repayments.

To see this, first note that since $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, it follows from Lemma 1 that $V_0(\mathcal{R}) > V_0(\mathcal{R}')$ if and only if

$$\sum_{t=0}^{T} \prod_{s=0}^{t-1} \Pr\left(X_s \geq R_s\right) \Delta - \sum_{t=0}^{T} \prod_{s=0}^{t-1} \Pr\left(X_s \geq R'_s\right) \Delta > 0.$$

Because $\mathcal{Q} \subset \mathcal{Q}'$, $\Pr\left(X_s \geq R_s\right) \geq \Pr\left(X_s \geq R'_s\right)$ for all $s \in \mathcal{T}$. However, since at least one element in $\mathcal{Q}'$ does not belong to $\mathcal{Q}$, there is at least one $s' \in \mathcal{T}$ such that $R_{s'} = 0$ and

$R'_{s'} \in (0, K\Delta]$. Hence, at s', $\Pr(X_{s'} \geq R_{s'}) = 1 > 1/K = \Pr(X_{s'} \geq R'_{s'})$. Therefore, the entrepreneur strictly prefers $\mathcal{R}$.

Claim 2: Denote by $\#\mathcal{Q}(\mathcal{R})$ the number of repayments of the debt contract $\mathcal{R}$ and by ϱ the vector of the repayment dates. If $\mathcal{D}(\mathcal{R}) = \mathcal{D}(\mathcal{R}')$, $\#\mathcal{Q}(\mathcal{R}) = \#\mathcal{Q}(\mathcal{R}')$, and $\varrho > \varrho'$ (that is, any elment of ϱ is greater than or equal to ϱ', and at least one element of ϱ is strictly greater than the corresponding element of ϱ'), then the entrepreneur strictly prefers $\mathcal{R}$. Put differently, if two incentive compatible debt contracts have the same value and the same number of repayments, the entrepreneur prefers the one with late repayments.

Because $\varrho > \varrho'$, for any t, $\prod_{s=0}^{t-1} \Pr(X_s \geq R_s) \geq \prod_{s=0}^{t-1} \Pr(X_s \geq R'_s)$, and there exists a repayment date $t_j \in \mathcal{Q}(\mathcal{R})$ that comes strictly later than the corresponding repayment date $t'_j \in \mathcal{Q}(\mathcal{R}')$. Then, at t_j, $\prod_{s=0}^{t_j-1} \Pr(X_s \geq R_s) > \prod_{s=0}^{t_j-1} \Pr(X_s \geq R'_s)$. Therefore, the entrepreneur strictly prefers $\mathcal{R}$.

We next prove Corollary 1 as a lemma for Proposition 1, even though for exhibitional purposes, the result is stated in the paper as a corollary.

Before proving the corollary, we state a repeatedly used adjustment procedure to the debt contract as a lemma.

Lemma 3 *Suppose $t_i, t_j \in \mathcal{Q}(\mathcal{R})$ are two repayment dates, with $R_{t_j} < K\Delta$. Define "(t_i, t_j, ϵ) adjustment" to be the following procedure to construct a new contract $\mathcal{R}'$: $R'_{t_i} = R_{t_i} - \epsilon$ and $R'_{t_j} = R_{t_j} + \frac{\epsilon}{K^{i-j}}$, leaving all other repayments unchanged. Then, the value of debt is unchanged, $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. In addition, if $t_i > t_j$, then $\mathcal{R}'$ is also incentive compatible.*

Proof of Lemma 3: First, it is straight forward from (4) that $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. Next, if $t_i > t_j$, let V' be the entrepreneur's continuation value under contract $\mathcal{R}'$. It is clear from (3) that $V'_t \geq V_t$ holds for all t and strictly for $t_j < t \leq t_i$. In particular, $V'_{t_j+1} = V_{t_j+1} + \frac{\epsilon}{K^{i-j}}$, so condition $R'_{t_j} \leq V'_{t_j+1}$ still holds. IC conditions for other repayments are trivially satisfied.

Proof of Corollary 1: Consider any debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) \leq n-1$ first. Let the i^{th} repayment date be $t_i \in \mathcal{Q}$. Note that the maximum amount of any single repayment is $K\Delta$; and R_{t_i} is actually paid if and only if $X_{t_\tau} = K\Delta$ for all $\tau \leq i$, which happens with probability $1/K^i$. Therefore, the maximum total expected repayments of $\mathcal{R}$ with at most $n-1$ repayments is

$$\sum_{i=1}^{n-1}\left[\frac{1}{K^i}(K\Delta)\right] = \Delta\sum_{j=0}^{n-2}\frac{1}{K^i} < D$$

by equation (11). Hence, investor's IR constraint (1) implies that there must be at least n repayments: $\#\mathcal{Q}(\mathcal{R}) \geq n$.

Next, we show that any contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = n+k$, where $k \geq 1$, can be strictly improved. Suppose there is a positive integer $j < n+k$ such that $R_{t_j} < K\Delta$. Then, we can apply $(n+k, j, \epsilon)$ adjustment until either all initial $n+k-1$ repayments equal $K\Delta$ or the last one repayment $R'_{t_{n+k}} = 0$. In the first case,

$$\sum_{j=1}^{n+k}\left[\frac{1}{K^j}R_{t_j}\right] > \sum_{j=1}^{n+k-1}\left[\frac{1}{K^j}(K\Delta)\right] \geq D,$$

so the total expected value of repayments exceeds D, contradicting the IR constraint (1). In the second case, the entrepreneur can eliminate the last repayment without affecting the value of debt. By Claim 1, the adjusted repayment schedule is strictly preferred. Hence, we establish Corollary 1.

Now, we prove Proposition 1. If D satisfies (9), then it automatically satisfies (11). Corollary 1 implies that the optimal debt contract will include exactly N repayments. Denote the optimal debt contract by $\mathcal{R}^*$.

Next, we inductively prove that the ith repayment occurs at $t_i^* = T - j - (N-i)K$. We first establish the statement for $i = N$, namely $t_N^* = T - j$. Consider any debt contract $\mathcal{R}$

whose last repayment date is later than $T-j$. The incentive compatibility constraint implies that $R_{t_N} \le (j-1)\Delta$. However, since the expected value of the first $N-1$ repayments is at most $\sum_{i=0}^{N-2} \frac{\Delta}{K^i}$ (attained when every repayment is $K\Delta$), $D \le \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j-1)\Delta}{K^N}$, violating equation (9).

Consider any debt contract $\mathcal{R}$ with the last repayment date $t_N < T-j$. We can apply (t_N, t_i, ϵ) adjustment until $R_{t_i} = K\Delta$ for all $i < N$. After such an adjustment, condition (9) implies $R_{t_N} \le j\Delta$. So, the entrepreneur can delay R_{t_N} to $T-j$ without affecting incentive compatibility and the value of debt, which by Claim 2, makes entrepreneur strictly better off. Hence, in $\mathcal{R}^*$, $t_N^* = T-j$.

Suppose $t_s^* = T-j-(N-s)K$ for all $s \ge i+1$. We now prove the statement for t_i^*. First, starting from $\mathcal{R}^*$, we can apply (t_i^*, t_l^*, ϵ) adjustment for all $l < i$, until $R_{t_l} = K\Delta$. Next, apply (t_i^*, t_l^*, ϵ) adjustment for all $l > i$, until $R_{t_l^*} = K\Delta$ (if $l < N$) or $j\Delta$ (if $l = N$). By Lemma (3), both adjustments do not affect the value of debt, and the first one is incentive compatible. It is easy to see that the second adjustment is also incentive compatible because the induction assumption implies that $V_{t_l^*} = K\Delta$ (or $j\Delta$) for all $i < l < N$ (or $l = N$).

After the adjustment,

$$R_{t_i^*} = \left(D - \sum_{l=0, l\neq i-1}^{N-2} \frac{\Delta}{K^l} - \frac{j\Delta}{K^N} \right) K^i.$$

From (9), $R_{t_i^*} \in \left(K\Delta - \frac{\Delta}{K^{N-i}}, K\Delta\right] \subset ((K-1)\Delta, K\Delta]$. Therefore, IC condition for $R_{t_i^*}$ implies that $t_i^* \le t_{i+1}^* - K$. If $t_i^* < t_{i+1}^* - K$, we can simply move $R_{t_i^*}$ to a later date: $t_{i+1}^* - K$. It is easy to see that such an adjustment does not affect the value of debt and is still incentive compatible. By Claim 2, the new contract dominates $\mathcal{R}^*$, contracting with the optimality of $\mathcal{R}^*$. Therefore, the induction conclusion holds for i and $t_n^* = T-j-(N-n)K$

for any $n \leq N$ in $\mathcal{R}^*$.

Finally, when $D = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$, we know from the previous proof that $\mathcal{Q}(\mathcal{R}^*) = \{T-j, T-j-K, \ldots, T-j-(N-1)K\}$. IC conditions imply that $R^*_{t_i} \leq K\Delta$ for $i < N$ and $R^*_{t_N} \leq j\Delta$. As a result, $\mathcal{D}(\mathcal{R}^*) \leq D$, with equality holding if and only if $R^*_{t_i} = K\Delta$ for $i < N$ and $R^*_{t_N} = j\Delta$. This establishes the uniqueness and completes the proof.

Q.E.D.

Proof of Lemma 2

Suppose D satisfies (9). Proposition (1) uniquely determines the set of repayment dates. Suppose $\sup ART$ is attained by some schedule $\mathcal{R}$ other than the one specified in the lemma. Then, there must exist an $i \in [2, N-1]$ such that $R_{t_i} < K\Delta$, or $i = N$ and $R_{t_i} < j\Delta$. Given such an i, consider an alternative schedule $\mathcal{R}'$ given by a (t_1, t_i, ϵ) adjustment. It is clear that when $\epsilon < \frac{K\Delta - R_{t_i}}{K^N}$ (or $\frac{j\Delta - R_{t_N}}{K^N}$ if $i = N$), schedule $\mathcal{R}'$ is still incentive compatible. By Lemma 3, $\mathcal{D}(\mathcal{R}') = \mathcal{D}(\mathcal{R})$. The adjusted schedule $\mathcal{R}'$ increases ART:

$$ART(\mathcal{R}') - ART(\mathcal{R}) = \frac{\epsilon(t_i - t_1)}{KD} > 0.$$

Contradiction! So $\sup ART$ is uniquely attained by $R_{t_N} = j\Delta$, $R_{t_i} = K\Delta$ for all $i \in [2, N-1]$, and $R_{t_1} = K(D - \sum_{i=1}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N})$. Substituting these into equation (12) implies

$$\begin{aligned}
&\sup ART \\
=&\frac{1}{D}\left\{(D - \sum_{i=1}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N})(T - j - (N-1)K)\right. \\
&\qquad \left. + \sum_{i=1}^{N-2} \frac{\Delta}{K^i}\left(T - j - (N-1-i)K\right) + \frac{j\Delta}{K^N}(T-j)\right\} \\
=&(T-j) - \frac{1}{D}\left[\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}\left(N-i-1\right) - \left(N-1\right)\left(K\Delta - R_{t_1}\right)\right] \\
=&\,(T-j) - \frac{1}{D}\left[\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}\left(N-i-1\right) + K\left(N-1\right)\left(D - \sum_{i=0}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N}\right)\right],
\end{aligned}$$

which is exactly equation (13).

Similarly, in order to attain inf ART, the entrepreneur wants to front-load repayments, so that the weights on earlier repayment dates are as large as possible. This is done by setting $R_{t_i} = K\Delta$ for all $i \leq N-1$ and $R_{t_N} = K^N\left(D - \sum_{i=0}^{N-2} \frac{\Delta}{K^i}\right)$. Substituting these into equation (12) implies

$$\begin{aligned}
&\inf ART(D) \\
=&\frac{\sum_{i=0}^{N-2} \frac{\Delta}{K^i}\left(T - j - (N-1-i)K\right) + (D - \sum_{i=0}^{N-2} \frac{\Delta}{K^i})(T-j)}{D} \\
=&(T-j) - \frac{1}{D}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i),
\end{aligned}$$

which is exactly equation (14).

Q.E.D.

Proof of Proposition 2

Corollary 2 directly implies that $\#\mathcal{Q}$ is weakly increasing in D. In the rest of the proof, we study $\inf ART(D)$ and $\sup ART(D)$ as D increases.

Let's begin with $\inf ART$. There are two cases depending on whether D is at the boundary of (9).

Case 1: Both D and $D+\epsilon$ satisfy (9) for some common N and j; that is, the increase in D does not change $\mathcal{Q}$. From equation (14), we have

$$\inf ART(D+\epsilon) = (T-j) - \frac{1}{D+\epsilon}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) > \inf ART(D).$$

Hence, when D increases and the set of repayment dates does not change, $\inf ART(D)$ is strictly increasing in D.

Case 2: Suppose $D = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$ for some N and $j = 0, 1, ..., K-1$. In this case, when D increases to $D+\epsilon$, the last repayment moves one period forward from $T-j$ to $T-(j+1)$. Note that here we slightly abuse notation by equivalenting Case (N-1)-K in (9) and "Case N-0". We then have

$$\lim_{\epsilon \to 0} \inf ART(D+\epsilon) = (T-(j+1)) - \frac{1}{D}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) = \inf ART(D) - 1.$$

Therefore, when the increase in D leads to an earlier last repayment date (but keeps the number of repayments), $\inf ART$ discretely drops by 1.

In sum, $\inf ART$ is not continuous in D. In particular, when D is in the interior of a case, $\inf ART(D)$ is continously increasing in D. At any boundary of (9), $\inf ART(D)$ is left-continuous, but drops discretely by one when D increases marginally.

Let's now turn to $\sup ART$. Similarly, there are again two cases.

Case 1: Both D and $D+\epsilon$ satisfy (9) for some common N and j. It then follows from equation (13) that

$$\sup ART(D+\epsilon) - \sup ART(D) < 0,$$

because

$$\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}} (N-i-1) + K(N-1) \left(-\sum_{i=0}^{N-2} \frac{\Delta}{K^i} - \frac{j\Delta}{K^N} \right) < 0.$$

Therefore, if the increase in D does not change $\mathcal{Q}$, $\sup ART(D)$ is strictly decreasing in D.

Case 2: Suppose $D = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$ for some N and $j = 0, 1, ..., K-1$. When D marginally increases to $D+\epsilon$, the equilibrium debt contract then features $\#\mathcal{Q} = N$ and $t_N = T-(j+1)$. Therefore,

$$\begin{aligned}
&\lim_{\epsilon \to 0} \sup ART(D+\epsilon) \\
=&\lim_{\epsilon \to 0} (T-(j+1)) - \frac{1}{D+\epsilon} \left[\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}} (N-i-1) + K(N-1) \left((D+\epsilon) - \sum_{i=0}^{N-2} \frac{\Delta}{K^i} - \frac{(j+1)\Delta}{K^N} \right) \right] \\
=&\sup ART(D) - 1 + \frac{\frac{\Delta}{K^N}(N-1)K}{D}.
\end{aligned}$$

Therefore,

$$\lim_{\epsilon \to 0} \sup ART(D+\epsilon) \in (\inf ART(D) - 1, \inf ART(D))$$

because $\sup ART(D) = \inf ART(D)$ and $\frac{\frac{\Delta}{K^N}(N-1)K}{D} < 1$.

Therefore, we conclude that $\sup ART(D)$ is strictly decreasing in D if the marginal change of D does not change $\mathcal{Q}$; however, when the marginal increase in D leads to a different set of repayment dates, $\sup ART$ has a discrete drop. The magnitude of the drop, however, is smaller than that of $\inf ART$.

We finally study ART at different boundary levels of (9). From Proposition 1, the optimal debt schedule is unique at these boundaries, so ART is also unique. We prove that a higher boundary level of D is associated with lower ART. Suppose that $D = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{j\Delta}{K^N}$ and $D' = \sum_{i=0}^{N-2} \frac{\Delta}{K^i} + \frac{(j+1)\Delta}{K^N}$. We have

$$
\begin{aligned}
& ART(D') - ART(D) \\
=& \left[(T-(j+1)) - \frac{1}{D'}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i)\right] - \left[(T-j) - \frac{1}{D}\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i)\right] \\
=& -1 + \left(\frac{1}{D} - \frac{1}{D'}\right)\sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) \\
<& -1 + \frac{K(N-1)\frac{\Delta}{K^N}}{D} \frac{\sum_{i=0}^{N-2} \frac{\Delta}{K^i}}{D'} \\
<&0.
\end{aligned}
$$

Here, the last inequality is due to the fact that $K^N D > (N-1)K\Delta$. Therefore, ART decreases in D when restricted to boundary levels of (9). This completes the proof.

Q.E.D.

Proof of Proposition 3

Corollary 2 together with the fact that $\Delta \sum_{i=0}^{N-1} \frac{1}{(K+1)^i}$ is decreasing in K directly imply that $\#\mathcal{Q}$ is weakly increasing in K. It follows from Proposition 1 that if $t_i, t_{i+1} \in \mathcal{Q}$, then $t_{i+1} - t_i = K$. So, it is obvious that the time interval between two consecutive repayments is strictly increasing in K.

We now study $\inf ART$ and $\sup ART$ as K increases to $K+1$. Let's first consider $\inf ART$. Similarly to Proposition 2, there are two cases.

Case 1: Suppose $\mathcal{Q}$ does not change as K increases to $K+1$. When $N=1$, $\inf ART$ does not change, because j does not change. For $N \geq 2$, we have (noting that $K \geq 2$ by assumption)

$$
\begin{aligned}
& D\left[\inf ART(K) - \inf ART(K+1)\right] \\
=& \sum_{i=0}^{N-2} \frac{\Delta}{(K+1)^{i-1}}(N-1-i) - \sum_{i=0}^{N-2} \frac{\Delta}{K^{i-1}}(N-1-i) \qquad (19) \\
=& (N-1)\Delta + \sum_{i=2}^{N-2} \left[\frac{\Delta}{(K+1)^{i-1}} - \frac{\Delta}{K^{i-1}}\right](N-1-i) \\
\geq& \Delta(N-1)\left[1 + \sum_{i=2}^{N-2}\left[\frac{1}{(K+1)^{i-1}} - \frac{1}{K^{i-1}}\right]\right] \\
>& \Delta(N-1)\left[1 - \sum_{i=1}^{\infty} \frac{1}{K^i}\right] \geq 0. \qquad (20)
\end{aligned}
$$

Therefore, when K increases to $K+1$, if $\mathcal{Q}$ does not change, $\inf ART$ decreases.

Case 2: Suppose $\mathcal{Q}$ changes as K increases to $K+1$. It directly follows equation (14) that $\inf ART$ is a decreasing with respect to N and j respectively. Denote by N_K and j_K the equilibrium outcome in Proposition 1 given K. If $N_{K+1} \geq N_K$ and $j_{K+1} \geq j_K$, then $\inf ART(K+1) < \inf ART(K)$. Because we have established $N_{K+1} \geq N_K$, so we only need to show that if $N_{K+1} > N_K$ and $j_{K+1} < j_K$, $\inf ART(K+1) < \inf ART(K)$. This is proved below. From equation (14), we have

$$
\begin{aligned}
&\inf ART(K+1) - \inf ART(K) \\
=&(j_K - j_{K+1}) - \frac{1}{D}\left[\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-1-i) - \sum_{i=0}^{N_K-2} \frac{\Delta}{K^{i-1}}(N_K-1-i)\right] \\
=&(j_K - j_{K+1}) - \frac{1}{D}\left[\sum_{i=N_K-1}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-1-i) + \sum_{i=0}^{N_K-2} \frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-N_K)\right. \\
&\left.\sum_{i=0}^{N_K-2} \frac{\Delta}{(K+1)^{i-1}}(N_K-1-i) - \sum_{i=0}^{N_K-2} \frac{\Delta}{K^{i-1}}(N_K-1-i)\right] \\
<&K - \frac{1}{D}\left[\sum_{i=N_K-1}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-1-i) + \sum_{i=0}^{N_K-2} \frac{\Delta}{(K+1)^{i-1}}(N_{K+1}-N_K)\right] \\
\leq&K - \frac{K}{D}\left[\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i}} + \frac{1}{K}\sum_{i=0}^{N_{K+1}-2} \frac{\Delta}{(K+1)^{i}}\right] \\
\leq&K - \frac{K}{D}\left[\sum_{i=0}^{N_{K+1}-1} \frac{\Delta}{(K+1)^{i}}\right] \leq 0
\end{aligned}
$$

Here, the first inequality is true because of equation (20), and the last inequality is due to the fact that $D \leq \sum_{i=0}^{N_{K+1}-1} \frac{\Delta}{(K+1)^i}$. Therefore, $\inf ART$ decreases in K. This concludes the proof of Case 2 and the analysis of $\inf ART$.

We now turn to $\sup ART$. It follows from equation (13) that $\sup ART$ can be rewritten as

$$
\sup ART = [T - j - (N-1)K] + \frac{1}{D}\left[\sum_{i=1}^{N-2} \frac{\Delta}{K^{i-1}} i + \frac{j\Delta}{K^{N-1}}(N-1)\right].
$$

Obviously, if the increase in K does not change $\mathcal{Q}$, $\sup ART$ will decrease. Otherwise, suppose the increase in K leads to a change of $\mathcal{Q}$. Similar to the proof for $\inf ART$, one can verify $\sup ART$ is decreasing in N and j respectively. As a result, if $N_{K+1} \geq N_K$

and $j_{K+1} \geq j_K$, then $\sup ART(K+1) < \sup ART(K)$. Finally, we prove the result for $N_{K+1} = N_K + 1$ and $j_{K+1} < j_K$:

$$
\begin{aligned}
&\sup ART(K+1) - \sup ART(K) \\
<&(j_K - j_{K+1}) - K + \frac{1}{D}\left[\sum_{i=1}^{N_K-1} \frac{\Delta}{(K+1)^{i-1}} i + \frac{j_{K+1}\Delta}{(K+1)^{N_K}} N_K - \sum_{i=1}^{N_K-2} \frac{\Delta}{K^{i-1}} i - \frac{j_K \Delta}{K^{N_K-1}}(N_K - 1)\right] \\
\leq& -1 + \frac{1}{D}\left[\frac{\Delta(N_K-1)}{(K+1)^{N_K-2}} + \frac{j_K\Delta}{(K+1)^{N_K}}\right] \\
=& -1 + \frac{1}{D}\left[\frac{\Delta(N_{K+1}-2)}{(K+1)^{N_{K+1}-3}} + \frac{j_K\Delta}{(K+1)^{N_{K+1}-1}}\right] \\
<& -1 + \frac{1}{D}\left[\frac{\Delta(N_{K+1}-2)}{(K+1)^{N_{K+1}-3}} + \frac{\Delta}{(K+1)^{N_{K+1}-2}}\right] \leq 0
\end{aligned}
$$

Simple mathematical induction on N_{K+1} establishes the same conclusion for any $N_{K+1} > N_K$ and $j_{K+1} < j_K$. In all, $\sup ART$ decreases in K.

Q.E.D.

Proof of Proposition 4

Assume the variance of per-period cash flow is a constant: $\Delta^2 (K-1) = \alpha^2$ for some constant $\alpha > 0$. Denote the solution by $\Delta_K = \frac{\alpha}{\sqrt{K-1}}$, which is decreasing in K. It then follows from Proposition 3 that $\#\mathcal{Q}$ and $t_{i+1} - t_i$ are both decreasing in Δ. In the remainder of the proof, we show both $\inf ART$ and $\sup ART$ increase as Δ increases. We only consider the increase in Δ that decreases K to a smaller integer. Without loss of generality, we focus on the comparison between $ART(\Delta_{K+1})$ and $ART(\Delta_K)$.

We first show that when $K \geq 4$, for any fixed N and j, when Δ increases from Δ_{K+1} to

Δ_K, $\inf ART$ increases. From (14),

$$
\begin{aligned}
&\inf ART(\Delta_{K+1}) - \inf ART(\Delta_K)\\
=& -\frac{1}{D}\left[\sum_{i=0}^{N-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N-1-i) - \sum_{i=0}^{N-2}\frac{\Delta_K}{K^{i-1}}(N-1-i)\right]\\
=& -\frac{1}{D}\left[(N-1)\left(\frac{K+1}{\sqrt{K}} - \frac{K}{\sqrt{K-1}}\right) + (N-2)\left(\frac{1}{\sqrt{K}} - \frac{1}{\sqrt{K-1}}\right)\right.\\
&\qquad \left. + \left(\sum_{i=2}^{N-2}\frac{1}{\sqrt{K}(K+1)^i}(N-1-i) - \sum_{i=2}^{N-2}\frac{1}{\sqrt{K-1}K^i}(N-1-i)\right)\right]\\
<& -\frac{1}{D}\left[(N-1)\left(\frac{K+1}{\sqrt{K}} - \frac{K}{\sqrt{K-1}}\right) + (N-2)\left(\frac{1}{\sqrt{K}} - \frac{1}{\sqrt{K-1}}\right)\right.\\
&\qquad \left. +(N-3)\sum_{i=2}^{N-2}\left(\frac{1}{\sqrt{K}(K+1)^i} - \frac{1}{\sqrt{K-1}K^i}\right)\right]\\
<& -\frac{1}{D}(N-1)\left[\left(\frac{K+1}{\sqrt{K}} - \frac{K}{\sqrt{K-1}}\right) + \left(\frac{1}{\sqrt{K}} - \frac{1}{\sqrt{K-1}}\right) + \sum_{i=2}^{\infty}\left(\frac{1}{\sqrt{K}(K+1)^i} - \frac{1}{\sqrt{K-1}K^i}\right)\right]\\
=& -\frac{1}{D}(N-1)\left[\frac{1}{\sqrt{K}}\frac{K+1}{1-\frac{1}{K+1}} - \frac{1}{\sqrt{K-1}}\frac{K}{1-\frac{1}{K}}\right]\\
=& -\frac{1}{D}(N-1)\left[\frac{1}{\sqrt{K}}\frac{(K+1)^2}{K} - \frac{1}{\sqrt{K-1}}\frac{K^2}{K-1}\right]
\end{aligned}
$$

When $K \geq 4$, $\inf ART(\Delta_{K+1}) - \inf ART(\Delta_K) < 0$. That is, for fixed N and j, when Δ increases from Δ_{K+1} to Δ_K, $\inf ART$ increases. This directly implies that if the increase in Δ does not change $\mathcal{Q}$, $\inf ART$ will increase. Then, by the same arguments in Proposition 3, we only need to show that when $j_K - j_{K+1} \leq K-1$ and $N_{K+1} = N+1$ ($N_K = N$), $\inf ART$

will increase as Δ increases from Δ_{K+1} to Δ_K. This can be seen frome below.

$$\begin{aligned}
&\inf ART(K+1) - \inf ART(K)\\
=&(j_K - j_{K+1}) - \frac{1}{D}\left[\sum_{i=0}^{N-1}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N-i) - \sum_{i=0}^{N-2}\frac{\Delta_K}{K^{i-1}}(N-1-i)\right]\\
\leq&(K-1) - \frac{1}{D}\left[\sum_{i=0}^{N-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N-i) - \sum_{i=0}^{N-2}\frac{\Delta_K}{K^{i-1}}(N-1-i) + \frac{\Delta_{K+1}}{(K+1)^{N-2}}\right]\\
=&(K-1) - \frac{1}{D}\left[\sum_{i=0}^{N-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}}(N-1-i) - \sum_{i=0}^{N-2}\frac{\Delta_K}{K^{i-1}}(N-1-i)\right.\\
&\qquad\qquad\left. + \sum_{i=0}^{N-2}\frac{\Delta_{K+1}}{(K+1)^{i-1}} + \frac{\Delta_{K+1}}{(K+1)^{N-2}}\right]\\
<&(K-1) - \frac{K+1}{D}\left[\sum_{i=0}^{N-2}\frac{\Delta_{K+1}}{(K+1)^{i}} + \frac{\Delta_{K+1}}{(K+1)^{N-1}}\right]\\
<&0.
\end{aligned}$$

Therefore, when Δ increases from Δ_{K+1} to Δ_K, $\inf ART$ increases.

We now show that the same property holds for $\sup ART$. Indeed, we do not need the assumption that $K \geq 4$ now. Let's first consider the case that $\mathcal{Q}$ does not change. Then, from (13),

$$\sup ART = [T - j - (N-1)K] + \frac{1}{D}\left[\sum_{i=1}^{N-2}\frac{1}{K^{i-1}\sqrt{K-1}}i + \frac{j}{K^{N-1}\sqrt{K-1}}(N-1)\right].$$

When Δ increases from Δ_{K+1} to Δ_K, for any fixed j and N, $\sup ART$ will increase, because $\sup ART$ is strictly decreasing in K. Similarly, in the case where $N_{K+1} \geq N_K$ and/or $j_{K+1} \geq j_K$, $\sup ART$ also increases in Δ. Hence, we only need to show that the same property holds in the case that $j_K - j_{K+1} = K-1$ and $N_{K+1} = N_K + 1 = N+1$. Note that

$$\begin{aligned}
&\sup ART(K+1) - \sup ART(K)\\
=&(K-1) + \frac{1}{D}\left[\sum_{i=1}^{N-1}\frac{1}{(K+1)^{i-1}\sqrt{K}}i + \frac{j_{K+1}}{(K+1)^N\sqrt{K}}N\right.\\
&\qquad\qquad\left. - \sum_{i=1}^{N-2}\frac{1}{K^{i-1}\sqrt{K-1}}i - \frac{j_K}{K^{N-1}\sqrt{K-1}}(N-1)\right]\\
<&-1-N+\frac{N-1}{D}\left[\frac{1}{(K+1)^{N-2}\sqrt{K}} + \frac{j_{K+1}}{(K+1)^N\sqrt{K}}\right]\\
<&0.
\end{aligned}$$

This completes the proof.

Q.E.D.

Proof of Proposition 5

Part 1: Consider a debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = 1$ first. We claim that the repayment date $t = T-m$ maixmizes $PI(1)$. To see this, we note that R_t has two upper bounds. First, $R_t \leq K\mu^t\Delta$, due to the feasibility constraint; and second, $R_t \leq V_{t+1}$ for the contract to be incentive compatible. Note that

$$V_{t+1} = \sum_{s=t+1}^{T}\mu^s\Delta = \mu^t\sum_{s=1}^{T-t}\mu^s\Delta.$$

Hence, $\forall t \in [T-m, T)$,

$$V_{t+1} \leq \left(\sum_{s=1}^{m}\mu^s\right)\mu^t\Delta = K\mu^t\Delta,$$

and so the incentive compatibility constraint must be binding to attain the maximum pledgeable income; that is, $R_t = V_{t+1}$. Since V_{t+1} is strictly decreasing in t, when $t \in [T-m, T)$, in order to achieve the maximum pledgeability, the repayment date must be $T-m$, and the maximum pledgeable income is $K\mu^{T-m}\Delta$.

Now, consider $t \leq T-m$. Then,

$$V_{t+1} \geq \left(\sum_{s=1}^{m} \mu^s\right) \mu^t \Delta \geq K\mu^t \Delta.$$

So the feasibility constraint must be binding to maximize $PI(1)$. Because $K\mu^t\Delta$ is strictly increasing in t, in order to achieve the maximum pledgeability, the repayment date must be $T-m$, and the maximum pledgeable income is $K\mu^{T-m}\Delta$. Combining both cases, if $\#\mathcal{Q}(\mathcal{R}) = 1$, $PI(1) = K\mu^{T-m}\Delta$, which is attained by making the only repayment at date $T-m$. By the same arguments, we show that for a debt contract $\mathcal{R}$, if $t_j \in \mathcal{Q}$ and $R_{t_j} = V_{t_j+1}$ (i.e., $V_{t_j} = \mu^{t_j}\Delta$), then the maximum repayment at the $(j-1)^{\text{th}}$ repayment date occurs at $t_j - m$.

Now, let's consider $PI(2)$. For any debt contract $\mathcal{R}$ with $\#\mathcal{Q}(\mathcal{R}) = 2$, suppose $t_2 = T-q$. It follows from the proof of the one repayment contract that $q \leq m$. Then, in order to attain the maximum pledgeable income, the entrepreneur can first set

$$R_{t_2} = V_{t_2+1} = \sum_{s=T-q+1}^{T} \mu^s \Delta.$$

As a consequence, $t_1 = t_2 - m$. Denote by $PI^q(N)$ the maximum pledgeable income of a contract with N repayments and the last repayment occurs at date $T-q$. Then,

$$PI^q(2) = \frac{K\mu^{T-q-m}\Delta}{K} + \frac{1}{K^2}\sum_{s=T-q+1}^{T}\mu^s\Delta.$$

So,

$$PI^{q+1}(2) - PI^q(2) > 0 \Leftrightarrow \mu^{-1} + \frac{\mu^m}{K^2} > 1. \tag{21}$$

Suppose equation (21) holds, then $PI^q(2)$ is strictly increasing in q. Hence,

$$PI(2) = PI^m(2) = \mu^{T-2m}\Delta + \frac{\mu^{T-m}\Delta}{K}.$$

Let's now compare $PI(2)$ and $PI(1)$ under equation (21).

$$I(2) > I(1) \Leftrightarrow \mu^{-m} + \frac{1}{K} > 1. \tag{22}$$

Note that, by equation (15), we have

$$K - \frac{K}{\mu} = \sum_{s=1}^{m}\mu^s - \sum_{s=0}^{m-1}\mu^s = \mu^m - 1. \tag{23}$$

Then, equation (22) is equivalent to $1-\mu^m+\frac{\mu^m}{K} > 0$, which holds if and only if $\frac{K}{\mu}-K+\frac{\mu^m}{K} > 0$. The last inequality is equivalent to equation (21). So, when $n=2$, $PI(2) = PI^m(2)$ if and only if $PI(2) > PI(1)$.

We now use induction. Assume that $PI(n) = PI^m(n)$ if and only if $PI(n) > PI(n-1)$, where $n \geq 2$. Let's consider $n+1$. Fix any $t_{n+1} = T-q$. When the contract can attain the largest pledgeable income, $R_{t_{n+1}} = V_{T-q+1}$. Then, by the assumption that $PI(n) = PI^m(n)$, we have

$$PI^q(n+1) = \left[\sum_{j=1}^{n} \frac{\mu^{T-q-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}} \sum_{j=T-q+1}^{T} \mu^j \Delta\right].$$

Then,

$$\begin{aligned}
& PI^{q+1}(n+1) - PI^q(n+1) \\
= & \left[\sum_{j=1}^{n} \frac{\mu^{T-(q+1)-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}} \sum_{j=T-(q+1)+1}^{T} \mu^j \Delta\right] \\
& \qquad - \left[\sum_{j=1}^{n} \frac{\mu^{T-q-(n-j+1)m}\Delta}{K^{j-1}} + \frac{1}{K^{n+1}} \sum_{j=T-q+1}^{T} \mu^j \Delta\right] \\
= & \mu^{T-q}\left[(\mu^{-1}-1)\sum_{j=1}^{n} \frac{\mu^{-(n-j+1)m}\Delta}{K^{j-1}}\right] + \frac{\mu^{T-q}\Delta}{K^{n+1}}.
\end{aligned}$$

So, $PI^{q+1}(n+1) > PI^q(n+1)$ if and only if

$$\left[(\mu^{-1}-1)\sum_{j=1}^{n} \frac{\mu^{-(n-j+1)m}}{K^{j-1}}\right] + \frac{1}{K^{n+1}} > 0. \tag{24}$$

Now, suppose equation (24) holds, then $PI(n+1) = PI^m(n+1)$. We then have

$$PI(n+1) - PI(n) = \sum_{j=1}^{n+1} \frac{\mu^{T-(n-j+2)m}}{K^{j-1}} - \sum_{j=1}^{n} \frac{\mu^{T-(n-j+1)m}}{K^{j-1}} > 0$$

if and only if

$$(1-\mu^m)\sum_{j=1}^{m} \frac{\mu^{-(n-j+1)m}}{K^{j-1}} + \frac{1}{K^n} > 0. \tag{25}$$

It then follows from equation (23) that

$$1 - \mu^m = \frac{K}{\mu} - K.$$

So, equation (25) and equation (24) are equivalent. Therefore, if $PI(n+1) = PI^m(n+1)$, $PI(n+1) > PI(n)$.

Note that equation (24) is equivalent to

$$\left[(\mu^{-1}-1)K\sum_{j=1}^{n}\left(K\mu^{-m}\right)^{n-j+1}\right]+1>0.$$

So, it follows from equation (16) that if and only if $N \leq N^*$, $PI(N) \geq PI(N-1)$. Therefore, $PI(N)$ is maximized at $N = N^*$.

Part 2: Because for any $N \leq N^*$, $PI(N) = PI^m(N)$. Therefore,

$$PI(N) = \sum_{i=0}^{N-1}\frac{\mu^{T-(N-i)m}}{K^i}\Delta.$$

Part 3: Now, suppose $D \in (PI(N-1), PI(N)]$. By the definition of $PI(N)$, since $D > PI(N-1)$, it is impossible to design a contract with at most $N-1$ repayment dates such that the investor's IR constraint holds. But $D \leq PI(N)$, so there exists a contract with N repayment dates such that the investor's IR constraint holds.

Now, consider any contract with $N+p$ repayment days ($p \in \mathbb{Z}_+$). Without loss of generality, we only consider contracts with $R_t = K\mu^t\Delta$, $\forall t \in \mathcal{Q}$. (Otherwise, the entrepreneur can just move the last repayment to previous repayment dates, when $R_t < K\mu^t\Delta$. Such an adjustment will not affect the continuation value V_{t+1} if R_{n+p} is still positive. Then, if the original contract is incentive compatible, the adjusted contract is also incentive compatible.) Such a contract must have the j^{th} repayment date earlier than $t_{N+p} - (N+p-j)m$. This is true because when $R_t = K\mu^t\Delta$, $\forall t \in \mathcal{Q}$, any two consecutive repayments must be separated by at least m dates. Since $t_{N+p} < T$ and $p \geq 1$, $t_{N+p} - (N+p-j)m < T - (N-j+1)m$.

Note, if the entrepreneur uses a contract with N repayment dates, the j^{th} repayment day could be $T-(N-j+1)m$. Therefore, the contract with $N+p$ repayment dates is worse than a contract with N repayment days and the j^{th} repayment date being $T-(N-j+1)m$. Hence, when $D \in (PI(N-1), PI(N)]$, the optimal contract has exactly N repayment dates. In addition, since $t_{j+1} - t_j = m$ and $T - t_N \leq m$, the first repayment date $t_1 \geq T - Nm$.

Part 4: When $D = PI(N)$, the optimal debt contract must have $t_N = T - m$, because $PI(N) = PI^m(N)$. Then, $\#\mathcal{Q} = \{T-m, T-2m, ..., T-Nm\}$. In addition, to attain the maximum pledgeable income, the entrepreneur must make the largest repayment at each repayment date, so $R_t = K\mu^t\Delta$, implying that the schedule proposed is the unique one to attain $D = PI(N)$, which can be attained by the repayment schedule

$$R_t = \begin{cases} \mu^t K\Delta, & \text{if } t \in \{T-m, T-2m, ..., T-Nm\} \\ 0, & \text{otherwise.} \end{cases}$$

Q.E.D.

Proof of Corollary 2

Consider the left-hand side of equation (16) when $N = 1$, we have

$$\left[(\mu^{-1}-1)K\sum_{j=1}^{N}\left(K\mu^{-m}\right)^{N-j+1}\right] + 1 = (\mu^{-1}-1)K(K\mu^{-m}). \tag{26}$$

Note that $(\mu^{-1}-1)K = 1 - \mu^m$, equation (26) becomes $K\mu^{-m} - K + 1$, which is negative because $1 > \frac{1}{K} + \mu^{-m}$. So when $1 > \frac{1}{K} + \mu^{-m}$, $N^* = 1$.

Q.E.D.

Proof of Proposition 6

Suppose on the contrary that a risky contract $\mathcal{R}$ maximizes the value of debt. Denote by $t \leq T-1$ the last risky repayment date. Construct a new contract $\mathcal{R}'$ such that $R'_s = R_s$ for all $s < t$ or $s > t+1$, $R'_t = L$, and $R'_{t+1} = R_{t+1} + \max(0, R_t - \Delta - L) < R_t$.

The new contract $\mathcal{R}'$ is incentive compatible because for any s,

$$V'_s \geq (\Delta + L) + \frac{1}{K}(V'_{s+1} - R_s) \geq \Delta + L,$$

so all risk-free repayments $R'_s \leq L$ are automatically incentive compatible. In addition, R'_{t+1} is incentive compatible because

$$V'_{t+2} = V_{t+2} = V_{t+1} + R_{t+1} - (\Delta + L) \geq R_t + R_{t+1} - (\Delta + L) \geq R'_{t+1}.$$

Next we show $\mathcal{D}(\mathcal{R}') > \mathcal{D}(\mathcal{R})$. This is because when (17) holds,

$$\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \geq L + \frac{R_{t+2} + \max(0, R_t - \Delta - L)}{K} - \frac{R_{t+1} + R_{t+2}}{K} \geq L - \frac{\Delta + L}{K} > 0.$$

Contradiction with the maximality of $\mathcal{R}$. Therefore, risk-free schedule $R_s = L$ maximizes pledgeability.

Q.E.D.

Proof of Proposition 7

Part 1: Let's first consider a contract $\mathcal{R}$ with $\mathcal{Q} = N$, where $N > N^{**}$. Then, the last repayment date is $t_N < T$. We can construct a new contract $\mathcal{R}'$: $R'_s = R_s$ for all $s < t_1$, $R'_{t_1} = L$, $R'_{s+1} = R_s$ for all $s \in [t_1, t_N]$, and $R'_{t_N+1} = R_{t_N+1} + \max\{0, R_{t_N} - \Delta - L\}$. As shown in the proof of Proposition 6, R'_{t_N} is incentive compatible. In addition, by the definition of R'_{t_N+1}, we have

$$
\begin{aligned}
&V'_{t_N+1} - V_{t_N} \\
=&\left[(\Delta + L) + \frac{V_{t_N+2} - R'_{t_N+1}}{K}\right] - \left[(\Delta + L) + \frac{V_{t_N+1} - R_{t_N}}{K}\right] \\
=&\frac{1}{K}\left[-R'_{t_N+1} - ((\Delta + L) - R_{t_N+1} - R_{t_N})\right] = 0,
\end{aligned}
$$

which implies $\mathcal{R}'$ is incentive compatible at any $s \leq t_N$. Hence, we just need to show that $\mathcal{D}(\mathcal{R}') > \mathcal{D}(\mathcal{R})$. Note that

$$
\begin{aligned}
&\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \\
\geq& R'_{t_1} + \frac{R'_{t_N+1}}{K^N} - \frac{R_{t_N} + R_{t_N+1}}{K^N} \\
\geq& L + \frac{\max\{0, R_{t_N} - \Delta - L\} - R_{t_N}}{K^N} \\
\geq& L - \frac{\Delta + L}{K^N} > 0
\end{aligned}
$$

Here, the last inequality is due to the definition of N^{**}.

In contrast, consider a contract with $\#\mathcal{Q} = N < N^{**}$. Without loss of generality, we assume that $R_s = L$ for all $s < t_1$. The entrepreneur can construct the following new contract $\mathcal{R}'$: $R'_s = R_s$ for all $s < t_1 - K$, $R'_s = R_{s+K}$ for all $s \in [t_1 - K, T - 1 - K]$, $R'_{T-K} = K\Delta + L$, and $R'_s = L$ for all $s > T - K$. Since there are K periods after date $T - K$, R'_{T-K} is incentive compatible. In addition, because $R'_{T-K} = K\Delta + L$ and $R'_s = L$ for all $s > T - K$,

$V'_{T-K} = \Delta + L$; hence, at any repayment date $s < T - K$, $\mathcal{R}'$ is incentive compatible because $\mathcal{R}$ is incentive compatible.

Note that $N < N^{**}$, and so $L < (\Delta + L)/K^{N+1}$. Therefore, we have

$$\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) = \frac{K(\Delta + L)}{K^{N+1}} - KL > 0,$$

Hence, the pledgeable income after adding the extra risky repayment increases. Therefore, $PI(N)$ is maximized at N^{**}.

Part 2: Fix an $N \leq N^{**}$ and consider any incentive compatible contract $\mathcal{R}$ with $\#\mathcal{Q} = N$. When $N = 1$, to maximize the value of the contract, we must have that the incentive compatibility binding at the unique risky repayment date t_1, and that at any date $s < t_1$, $R_s = L$. So, $t_1 \geq T - K$; otherwise, the incentive compatibility is not binding ($\sum_{s=t_1}^{T-1} R_s < (K\Delta + L) + (T - 1 - t_1)L = K\Delta + (T - t_1)L$, but $V_{t_1+1} = (T - t_1)(\Delta + L)$). Now, for any $t_1 > T - K$, we can construct a new contract $\mathcal{R}'$: $R_s = L$ for all $s \neq T - K$ and $R'_{T-K} = K\Delta + L$ (of course, $R'_T = 0$). This is incentive compatible, because $V'_{T-K+1} = K(\Delta + L) - (K - 1)L = K\Delta + L = R'_{T-K}$. Now,

$$\begin{aligned}
&\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \\
=&\frac{K(\Delta + L)}{K} - \left[(t_1 - (T - K))L + \frac{(T - t_1)(\Delta + L)}{K}\right] \\
=&(K - (T - t_1))\left(\frac{\Delta + L}{K} - L\right).
\end{aligned}$$

Hence, $\mathcal{D}(\mathcal{R}') - \mathcal{D}(\mathcal{R}) \geq 0$, because $t_1 \geq T - K$ and $N \leq N^{**}$. Note that the inequality is strict if $t_1 > T - K$; hence, $PI(1)$ is uniquely attained by $R_{t_1} = K\Delta + L$ at date $t_1 = T - K$, and $R_s = L$ at date $s \neq t_1$. Then, $PI(1) = (T - K)L + (\Delta + L)$.

Now, let's consider any $N \leq N^{**}$ and consider a contract $\mathcal{R}$ with $\#\mathcal{Q} = N$. Fixing all repayments at dates $s \geq t_2$, we have $V_{t_2+1} \geq R_{t_2}$ by the incentive compatibility of $\mathcal{R}$. Then, as the same argument of the case $N = 1$, the value of the contract is maximized when $t_1 = t_2 - K$, $R_{t_1} = K\Delta + L$, $R_s = L$ for all $s < t_2$ but $s \neq t_1$. We can now increase the value of the contract by increasing R_{t_2} to V_{t_2+1}. Similarly, fixing all repayments at dates $s \geq t_n$ for all $n \leq N$, the value of the contract is maximized by setting $t_j = t_j - K$ for all $j < n$, $R_{t_j} = K\Delta + L$ at date t_j for all $j < n$, and $R_s = L$ at dates $s < t_n$ but $s \neq t_j$ for any $t_j < t_n$. Then, finally, because $R_T = V_{T+1} = 0$, the contract's value is maximized by setting $t_N = T - K$.

Therefore, in order to maximize the pledgeable income, the entrepreneur needs to make risky repayment at dates $t = T - iK$, where $i = 1, 2, \dots, N$. In addition, at each risky repayment date, the risky repayment should be $K\Delta + L$, such that both the feasibility constraint and the incentive compatibility constraint binding; at other dates, the entrepreneur needs to repay L. Therefore, the maximum pledgeable income of a debt contract with N repayment dates is

$$PI(N) = (T - NK)L + \sum_{j=1}^{N} \frac{\Delta + L}{K^{j-1}}.$$

Part 3: For any $N \leq N^{**}$, if $D \in (PI(N-1), PI(N)]$, by the definition of $PI(N)$, the entrepreneur cannot use a contract with at most $N - 1$ repayment dates to attain D, but she can use a contract with N repayment dates to attain D.

Now, we show that a contract with $N + p$ repayments, where $p \geq 1$, can be strictly improved. The first step is to show that at all dates when the entrepreneur does not make risky repayment, the entrepreneur repays L. Suppose there is one date t such that $R_t < L$. If t is earlier than the first risky repayment t_1, the enterpreneur apply the $(t, t_{1,}\epsilon)$ adjustment by setting $R'_t = R_t + \epsilon$ and $R'_{t_1} = R_{t_1} - K\epsilon$. Then, the investor's participation constraint does not change, and the enterpreneur is at least as good as before (strictly better if $R'_{t_1} \leq L$ after the adjustment). If $t \in (t_j, t_{j+1})$, then the entrepreneur can make the (t_j, t, ϵ) adjustment by setting $R'_{t_j} = R_{t_j} - \epsilon$ and $R'_t = R_t + \epsilon$. Such an adjustment will not change the incentive compatibility and the investor's participation constraint either; again, the entrepreneur is at least not worse off. Hence, when consider the optimal debt contract, the entepreneur will repay L when she does not make risky repayments. Then, the rest of the proof is the same as that of Corollary 1 by iterated application of the (t_{N+p}, t_i, ϵ) adjustments.

As shown above, in the optimal debt contract, $t_{j+1} - t_j \leq K$ and $t_N \geq T - K$; otherwise, if some $t_j < t_{j+1} - K$ (we can denote by $t_{N+1} = T$), it is strictly better for the entrepreneur to set $R'_{t_j} = L$ and $R'_{t_j+1} = R_{t_j}$. Such a new contract is still incentive compatible, because the value between $t_j + 1$ and t_j will be greater than or equal to $K(\Delta + L)$. Therefore, $t_1 \geq T - NK$.

Part 4: For any $N \leq N^{**}$, if $D = PI(N)$, the optimal debt contract will have exactly N risky repayment dates. Then, by Part 2, to attain D by a contract with N risky repayment dates, the entrepreneur has to repay $K\Delta + L$ at dates $t_i = T - iK$ (for $i = 1, 2, \ldots, N$) and repay L at all other dates. Therefore, there is a unique optimal contract that attains D, which is

$$R_t = \begin{cases} K\Delta + L, & \text{if } t \in \{T - K, T - 2K, ..., T - NK\} \\ L & \text{otherwise.} \end{cases}$$

Q.E.D.

Proof of Proposition 8

Consider a repayment schedule with $R_t > K\Delta$ for some $t \in \mathcal{Q}$. Let R_{t_j} be the last repayment with $R_{t_j} > K\Delta$. There are two cases. First, the repayment schedule has exactly

j repayments, and so R_{t_j} is also the last repayment. Then,

$$T - t_j = \left\lceil \frac{R_{t_j}}{\Delta} \right\rceil > K;$$

otherwise, $R_{t_j} > V_{t_j+1}$, violating the incentive compatibility constraint. Then, fix all previous $j-1$ repayments (the time and the amount), the entrepreneur may consider the following adjustment: $R'_{t_j} = K\Delta$ and $R'_{t_j+K} = R_{t_j} - K\Delta$. Such a new repayment schedule is still incentive compatible; otherwise, if $R'_{t_j+K} > V_{t_j+K+1} = (T - t_j - K)\Delta$, $R_{t_j} > (T - t_j)\Delta = V_{t_j+1}$, violating the assumption that the original repayment schedule is incentive compatible.

Because the first $j-1$ repayments do not change, if the entrepreneur can repay R_{t_j} at date t_j, he is able to make the repayments R'_{t_j} and R'_{t_j+K} (because of saving). Indeed, there is a positive probability that the entrepreneur cannot repay R_{t_j} but can make the repayments R'_{t_j} and R'_{t_j+K}, because the project may generate positive cash flows between $t_j + 1$ and $t_j + K$. In addition, because of saving, the investor's participation constraint is also satisfied. Therefore, the entrepreneur can even reduce R'_{t_j+K} to a certain R''_{t_j+K} and still keep the investor's participation constraint satisfied.

Now, let's compare V_{t_j} and V'_{t_j}. Denote by S_t the total funds the entrepreneur can use to make repayment at date t. We can calculate

$$V_{t_j} = \Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R_{t_j} + (T - t_j)\Delta\right)$$

and

$$
\begin{aligned}
&V'_{t_j}\\
=&\Delta + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\Big\{-R'_{t_j} + K\Delta\\
&\qquad\qquad + \Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S'_{t_j} \geq R'_{t_j}\right)\left(-R''_{t_j+K} + (T - t_j - K)\Delta\right)\Big\}\\
=&\Delta + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\left(-R'_{t_j} + K\Delta\right)\\
&\qquad + \Pr\left(S'_{t_j} \geq R'_{t_j}\right)\Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S'_{t_j} \geq R'_{t_j}\right)\left(-R''_{t_j+K} + (T - t_j - K)\Delta\right)
\end{aligned}
$$

As we argued above, the same first $j-1$ repayments imply that $S_{t_j} = S'_{t_j}$, and then because of $R_{t_j} = R'_{t_j} + R'_{t_j+K} > R'_{t_j}$, if $\Pr\left(S_{t_j} \geq R'_{t_j}\right) > 0$,

$$
\Pr\left(S_{t_j} \geq R'_{t_j}\right) \geq \Pr\left(S_{t_j} \geq R_{t_j}\right).
$$

In addition, if $S_{t_j} \geq R_{t_j}$, then $S_{t_j} - R'_{t_j} = R'_{t_j+K} > R''_{t_j+K}$, and so

$$
\Pr\left(S_{t_j} \geq R'_{t_j}\right)\Pr\left(S'_{t_j+K} \geq R''_{t_j+K} | S_{t_j} \geq R'_{t_j}\right) > \Pr\left(S_{t_j} \geq R_{t_j}\right).
$$

These imply that

$$
\begin{aligned}
&V'_{t_j}\\
>&\Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R'_{t_j} + K\Delta\right) + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R''_{t_j+K} + (T - t_j - K)\Delta\right)\\
>&\Delta + \Pr\left(S_{t_j} \geq R_{t_j}\right)\left(-R'_{t_j} + K\Delta - R'_{t_j+K} + (T - t_j - K)\Delta\right)\\
=&V_{t_j}
\end{aligned}
$$

Hence, the adjustment makes the entrepreneur strictly better off.

In the second case, the repayment schedule has more than j repayments. Then, by assumption, all repayments after date t_j are at most $K\Delta$. The entrepreneur can then make all the repayments after date t_j as late as possible, until that delaying one of these repayments one date will violate the incentive compatibility. Then, the entrepreneur can iteratedly apply the (t_{i-1}, t_i, ϵ) adjustment for all $i \geq j$ beginning from $i = N$ (the last repayment), such that the incentive compatibility constraint binding at each repayment date after date t_j. The entrepreneur will not be worse off by this adjustment, for the same reason as in the first case. Suppose now that R_{t_j} is still strictly greater than $K\Delta$. Then, $V_{t_{j+1}} = (t_{j+1} - t_j)\,\Delta$, and $t_{j+1} - t_j = \left\lceil \frac{R_{t_j}}{\Delta} \right\rceil > K$. Hence, the same argument in the first case will prove that such a contract is not optimal for the entrepreneur.

Q.E.D.

Proof of Proposition 9

First, N^* is well defined because the LHS of (18) is a constant and the RHS decreases to 0, as $N^* \to \infty$.

The proof is by contradiction. Let $\{R_i | i = 0, 1, 2, ..., T\}$ be any repayment profile that attains the maximum pledgeable income, with risky payments (strictly greater than L) at dates $\{t_1, t_2, ..., t_N\}$. Suppose on the contrary that $N \geq 2N^*$. Before proving the proposition, we first introduce a key procedure that we use repeatedly.

For any fixed K, define risk-free modification with respect to day K to be the following procedure that constructs a new repayment profile $\tilde{\mathcal{R}}_i(K)$: For all $t > K$ or $t < t_1$, $\tilde{R}_t \equiv R_t$; $\tilde{R}_{t_1} \equiv L$; and for all $t_1 < t \leq K$, $\tilde{R}_t \equiv R_{t-1}$. Essentially, this modification removes repayment

R_K; shifts all repayments between t_1 and K one period backward; and inserts a risk-free repayment of L at date t_1. The repayment profile changes from

$$\mathcal{R} = \{R_0, R_1, ..., R_{t_1-1}, R_{t_1}, R_{t_1+1}, ..., R_{t_2}, ..., R_{K-1}, R_K, R_{K+1}, ..., R_T\}$$

to

$$\tilde{\mathcal{R}}(K) = \{R_0, R_1, ..., R_{t_1-1}, L, R_{t_1}, ..., R_{t_2}, ..., R_{K-2}, R_{K-1}, R_{K+1}, ..., R_T\}.$$

The expected payoff to the borrower at the beginning of date t is

$$V_t = \sum_{i=t}^{T} \left[E(X_i) \prod_{s=t}^{i-1} Prob(X_s \geq R_s) - R_i \prod_{s=t}^{i} Prob(X_s \geq R_s) \right],$$

and the expected payoff to the lender is still defined by equation (4).

Let $\tilde{R}(t_n)$ be the risk-free modification w.r.t. t_n with $n > N^*$, i.e. there are at least N^* prior risky payments. By the definition of such a modification and equation (4), the expected value of the modified repayment profile is

$$\begin{aligned}
\mathcal{D}(\tilde{R}(t_n)) &= \textstyle\sum_{t=0}^{t_1-1} R_t + L \\
&\quad + \textstyle\sum_{t=t_1}^{t_n-1} R_t \prod_{i=0}^{t} Prob(X_i \geq R_i) \\
&\quad + \textstyle\sum_{t=t_n+1}^{T} R_t \prod_{i=0, i\neq t_n}^{t} Prob(X_i \geq R_i).
\end{aligned}$$

Compare it with $\mathcal{D}(\mathcal{R})$ in (4), the difference is

$$\begin{aligned}
\mathcal{D}(\tilde{\mathcal{R}}(t_n)) - \mathcal{D}(\mathcal{R}) &= L - R_{t_n} \textstyle\prod_{i=0}^{t} Prob(X_i \geq R_i) \\
&\quad + \textstyle\sum_{t=t_n+1}^{T} R_t \left[\prod_{i=0, i\neq t_n}^{t} Prob(X_i \geq R_i) - \prod_{i=0}^{t} Prob(X_i \geq R_i) \right] \\
&\geq L - R_{t_n} \textstyle\prod_{i=0}^{t} Prob(X_i \geq R_i).
\end{aligned}$$

First note that $R_{t_n} Prob(X_{t_n} \geq R_{t_n})$ is weakly dominated by $E(X_{t_n})$. In addition, there are at least N^* risky repayment before t_n and the survival probability with each risky payment $Prob(X_t > R_t) \leq Prob(X_t > L) = 1 - \epsilon$. As a result, the difference $\mathcal{D}(\tilde{\mathcal{R}}(t_n)) - \mathcal{D}(\mathcal{R})$ is at least

$$\mathcal{D}(\tilde{\mathcal{R}}(t_n)) - \mathcal{D}(\mathcal{R}) \geq L - (1-\epsilon)^{N^*} E(X_{t_n}) > 0 \tag{27}$$

by the definition of N^*. Therefore, $\tilde{\mathcal{R}}(t_n)$ is a repayment profile that is strictly more valuable to the investor. However, it may not be incentive compatible.

Now we prove the original proposition. There are two cases depending whether or not there exists a risky payment $R_t \geq E(X_t)$ for some $t \in \{t_{N^*+1}, t_{N^*+2}, ..., t_N\}$.

Case 1: Suppose there exists an $t_n \in \{t_{N^*+1}, t_{N^*+2}, ..., t_N\}$ such that $R_{t_n} \geq E(X_{t_n})$.

In this case, we show that $\tilde{\mathcal{R}}(t_n)$ is actually incentive compatible, namely $\tilde{R}_t \leq \tilde{V}_{t+1}$ for all t, where $\tilde{V}$ is the payoff to the borrower with the modified schedule. Combined with condition (27), the modified schedule $\tilde{\mathcal{R}}(t_n)$ gives the desired contradiction to the optimality of $\mathcal{R}$.

First, the IC conditions $\tilde{R}_t \leq \tilde{V}_{t+1}$ are not affected when $t > t_n$ because the payments $\tilde{R}_t = R_t$ and consequently $\tilde{V}_t = V_t$. Next, we show $\tilde{V}_{t+1} \geq V_t \geq E(X_t)$ for all $t_1 < t \leq t_n$ by induction method. Note that for any t, $V_t \geq E(X_t)$, which is a direct result from IC ($R_t \leq V_{t+1}$) and the recursive formulation of V_t:

$$V_t = E(X_t) + Prob(X_t \geq R_t)(-R_t + V_{t+1}) \geq E(X_t).$$

Combined with the presumption that we are in case 1, we have:

$$
\begin{aligned}
V_{t_n} &= E(X_{t_n}) + Prob(X_{t_n} \geq R_{t_n})(-R_{t_n} + V_{t_n+1}) \\
&\leq E(X_{t_n})(1 - Prob(X_{t_n} \geq R_{t_n})) + Prob(X_{t_n} \geq R_{t_n})V_{t_n+1} \\
&\leq V_{t_n+1} = \tilde{V}_{t_n+1}.
\end{aligned}
$$

This establishes the initial step of the induction. Now suppose $V_s \leq \tilde{V}_{s+1}$ holds for some $t_1 + 1 < s \leq t_n$, and we want to show $V_{s-1} \leq \tilde{V}_s$. From the induction assumption and the fact that $\tilde{R}_s = R_{s-1}$ in this region, we have

$$
\begin{aligned}
V_{s-1} &= E(X_{s-1}) + Prob(X_{s-1} \geq R_{s-1})(-R_{s-1} + V_s) \\
&\leq E(X_s) + Prob(X_s \geq \tilde{R}_s)(-\tilde{R}_s + \hat{V}_{s+1}) \\
&= \tilde{V}_s.
\end{aligned}
$$

This completes the induction proof. IC for all $t_1 < t \leq t_n$ follows immediately:

$$
\tilde{R}_t = R_{t-1} \leq V_t \leq \tilde{V}_{t+1}.
$$

Finally, by definition of t_1, all repayments before t_1 are risk free. Mathematically, for all $t \leq t_1$, $\tilde{R}_t \leq L \leq E(X_t)$. It is easy to show by induction that $\tilde{V}_t \geq E(X_t)$ also holds for all $t \leq t_1$. Therefore, IC also holds when $t \leq t_1$. We have verified the IC condition for the modified schedule $\tilde{\mathcal{R}}(t_n)$ for all $t = 1, 2, ..., T$, and thereby completing the proof of case 1.

Case 2: Suppose all repayments at t_{N^*+1} are strictly smaller than $E(X_t)$, i.e. $R_t < E(X_t)$ for all $t \geq t_{N^*+1}$.

Let $\hat{\mathcal{R}}$ be the risk-free modification to R_{T-1}, except for $\hat{R}_{t_{N^*+1}+1}$, which is alternatively defined as

$$\hat{R}_{t_{N^*+1}+1} = R_{t_{N^*+1}} - (1-\epsilon)^{N^*-2} E(X_{T-1}). \tag{28}$$

This is well defined because $R_{t_{N^*+1}} \geq L$ is a risky payment, and condition (18) guarantees that $\hat{R}_{t_{N^*+1}+1} > 0$. Denote by $\hat{V}_t$ the corresponding expected payoff to the borrower.

First, we show that the modified schedule $\hat{\mathcal{R}}$ is incentive compatible in three exhaustive cases, $t > t_{N^*+1}$, $t_{N^*+1} \geq t > t_1$, and $t \leq t_1$. Similar to Case 1, it is easy to show by induction that $\hat{V}_t \geq E(X_t)$ for all $t > t_{N^*+1}$. Because we are in Case 2, this result also means that IC holds strictly for all $t > t_{N^*+1}$.

Next we establish IC for $t_1 < t \leq t_{N^*+1}$. First, we show $V_{t_{N^*+1}} \leq \hat{V}_{t_{N^*+1}+1}$. When $t_1 \leq t \leq T-2$ and $t \neq t_{N^*+1}$, recall $\hat{R}_{t+1} = R_t$, so

$$\begin{aligned} \hat{V}_{t+1} - V_t &= E(X_{t+1}) + Prob(X_{t+1} \geq \hat{R}_{t+1})(-\hat{R}_{t+1} + \hat{V}_{t+2}) \\ &\quad - [E(X_t) + Prob(X_t \geq R_t)(-R_t + V_{t+1})] \\ &= Prob(X_t \geq R_t)(\hat{V}_{t+2} - V_{t+1}). \end{aligned} \tag{29}$$

By iteration, we have

$$\hat{V}_{t_{N^*+1}+2} - V_{t_{N^*+1}+1} = \prod_{s=t_{N^*+1}+1}^{T-2} Prob(X_s \geq R_s)(\hat{V}_T - V_{T-1}).$$

Because $\hat{V}_T = E(X_T)$ and $V_{T-1} \leq E(X_{T-1}) + E(X_T)$, together with the fact that there are at least $N - N^* - 2 \geq N^* - 2$ risky repayments, so the above difference is bounded below by

$$\hat{V}_{t_{N^*+1}+2} - V_{t_{N^*+1}+1} \geq -(1-\epsilon)^{N^*-2} E(X_{T-1}).$$

Now consider $\hat{V}_{t_{N^*+1}+1} - V_{t_{N^*+1}}$. From the above lower bound and definition (28), we have

$$\begin{aligned}
\hat{V}_{t_{N^*+1}+1} - V_{t_{N^*+1}} &= E(X_{t_{N^*+1}+1}) + Prob(X_{t_{N^*+1}+1} \geq \hat{R}_{t_{N^*+1}+1})(-\hat{R}_{t_{N^*+1}+1} + \hat{V}_{t_{N^*+1}+2}) \\
&\quad - \left[E(X_{t_{N^*+1}}) + Prob(X_{t_{N^*+1}} \geq R_{t_{N^*+1}})(-R_{t_{N^*+1}} + V_{t_{N^*+1}+1})\right] \\
&\geq Prob(X_{t_{N^*+1}} \geq R_{t_{N^*+1}})(-\hat{R}_{t_{N^*+1}+1} + R_{t_{N^*+1}} + \hat{V}_{t+2} - V_{t+1}) \\
&\geq Prob(X_{t_{N^*+1}} \geq R_{t_{N^*+1}})[(1-\epsilon)^{N^*-2}E(X_{T-1}) - (1-\epsilon)^{N^*-2}E(X_{T-1})] \\
&= 0
\end{aligned}$$

Therefore, we have shown $\hat{V}_{t_{N^*+1}+1} \geq V_{t_{N^*+1}}$. Combined with the iterative formula (29), we have $\hat{V}_{t+1} \geq V_t$ hold for all $t \geq t_1 + 1$, which in turn implies the desired IC condition:

$$\hat{V}_{t+1} \geq V_t \geq R_{t-1} \geq \hat{R}_t.$$

Finally, similar to case 1, when $t \leq t_1$, all repayments $\hat{R}_t$ are risk free which are in turn dominated by $E(X_t)$. It is also easy to inductively prove that $\hat{V}_t \geq E(X_t)$. This completes the verification of incentive compatibility of $\hat{\mathcal{R}}$.

Next, we prove that the modified schedule has an expected value $\mathcal{D}(\hat{\mathcal{R}})$ that strictly dominates $\mathcal{D}(\mathcal{R})$, thereby contradicting with the optimality of $\mathcal{R}$ and completing the proof. Write out $\mathcal{D}(\hat{\mathcal{R}})$ explicitly:

$$\begin{aligned}
\mathcal{D}(\hat{\mathcal{R}}) &= \textstyle\sum_{t=0}^{t_1-1} R_t + L + \sum_{t=t_1}^{t_{N^*+1}-1} R_t \prod_{i=1}^{t} Prob(X_i \geq R_i) \\
&\quad + \hat{R}_{t_{N^*+1}+1} Prob(X_{t_{N^*+1}+1} \geq \hat{R}_{t_{N^*+1}+1}) \textstyle\prod_{i=1}^{t_{N^*+1}-1} Prob(X_i \geq R_i) \\
&\quad + \textstyle\sum_{t=t_{N^*+1}+1}^{T-2} R_t Prob(X_{t_{N^*+1}+1} \geq \hat{R}_{t_{N^*+1}+1}) \prod_{i=1, i\neq t_{N^*+1}}^{t} Prob(X_i \geq R_i),
\end{aligned}$$

where $\hat{R}_{t_{N^*+1}+1}$ is related to $R_{t_{N^*+1}}$ by (28). Because $\hat{R}_{t_{N^*+1}+1} < R_{t_{N^*+1}}$, so

$$Prob(X_{t_{N^*+1}+1} \geq \hat{R}_{t_{N^*+1}+1}) \geq Prob(X_{t_{N^*+1}} \geq R_{t_{N^*+1}}).$$

Thus $\mathcal{D}(\hat{\mathcal{R}})$ is bounded below by

$$\begin{aligned} \mathcal{D}(\hat{\mathcal{R}}) \quad \geq \quad & \textstyle\sum_{t=1}^{t_1-1} R_t + L + \sum_{t=t_1}^{t_{N^*+1}-1} R_t \prod_{i=0}^{t} Prob(X_i \geq R_i) \\ & +\hat{R}_{t_{N^*+1}+1} \textstyle\prod_{i=0}^{t_{N^*+1}} Prob(X_i \geq R_i) \\ & + \textstyle\sum_{t=t_{N^*+1}+1}^{T-2} R_t \prod_{i=0}^{t} Prob(X_i \geq R_i), \end{aligned} \tag{30}$$

From (4), (30), and (28), we have:

$$\begin{aligned} \mathcal{D}(\hat{\mathcal{R}}) - \mathcal{D}(\mathcal{R}) \quad \geq \quad & L - R_{T-1} \textstyle\prod_{i=0}^{T-1} Prob(X_i \geq R_i) \\ & +(\hat{R}_{t_{N^*+1}+1} - R_{t_{N^*+1}}) \textstyle\prod_{i=0}^{t_{N^*+1}} Prob(X_i \geq R_i) \\ = \quad & L - R_{T-1} \textstyle\prod_{i=0}^{T-1} Prob(X_i \geq R_i) \\ & -(1-\epsilon)^{N^*-2} E(X_{T-1}) \textstyle\prod_{i=0}^{t_{N^*+1}} Prob(X_i \geq R_i). \end{aligned}$$

Because $R_{T-1} < V_T = E(X_T)$, and there are at least $N \geq 2N^*$ and $N^* + 1$ risky repayment in $[1, T-1]$ and $[1, t_{N^*+1}]$ respectively, the above lower bounded is in turn greater than

$$\mathcal{D}(\hat{\mathcal{R}}) - \mathcal{D}(\mathcal{R}) \geq L - E(X_T)(1-\epsilon)^{2N^*} - E(X_{T-1})(1-\epsilon)^{2N^*-1} > 0,$$

where the last inequality is from the definition of N^* in (18). This completes the proof of case 2 and the proposition.

Q.E.D.