A Theory of Liquidity Spillover Between Bond and CDS Markets

I build a dynamic search model of bond and CDS markets and show that allowing short positions through CDS contracts increases liquidity of the underlying bond market. Existing theories on derivatives, in contrast, show that derivatives fragment traders across the derivative and underlying markets and thereby decrease liquidity in the underlying cash market. They conclude this assuming that the total number of investors stays constant when derivatives are introduced. I show the opposite is true if the number of investors is instead endogenous. My results help explain how sovereign bond markets reacted to naked CDS bans.

During the recent European debt crisis, the controversy surrounding credit default swaps (CDS) culminated in bans on “naked” purchases of CDS. A credit default swap is a financial derivative instrument that resembles an insurance protection against a firm or a government default. It allows investors to trade the credit risk of the bond issuer without trading the bonds themselves. A naked CDS purchase refers to a CDS purchase where the protection buyer does not own the underlying bonds. As they are a short and, possibly, a speculative position against the bond issuer, regulators blamed them for exacerbating the European debt crisis. Ultimately, in May 2010, Germany banned naked CDS purchases written on Eurozone government bonds and gave investors a date when the ban would be lifted. The ban was thus to be temporary. Later in October 2011, the European Union voted to permanently ban naked CDS referencing EU sovereigns.

What is interesting about these bans is how the underlying bond market reacted. After the temporary ban, liquidity in the underlying bond market temporarily increased (as measured by bond bid-ask spreads). The reaction after the permanent ban, however, reversed: Bond market liquidity decreased. These patterns are documented in Sambalaibat (2014).

Such bond market reactions—in particular, the reaction after the permanent ban—challenge existing theories. Models on the interaction between
derivative and the underlying markets show that introducing derivatives fragments traders across the derivative and the underlying markets and thereby attracts liquidity away from the underlying market.\textsuperscript{1} The implication is banning derivatives—by reversing the fragmentation derivatives cause—improves liquidity in the underlying. Existing theories thus predict the effect after the temporary ban: an increase in liquidity. They do not, however, reconcile with the observed decrease in liquidity after the permanent ban. We thus need to understand: How does CDS trading affect the underlying bond market? Why would a permanent CDS ban decrease bond market liquidity? And can we explain the opposite bond market reactions jointly?

In this paper, I build a dynamic search model of bond and CDS markets and show that relaxing a key assumption present in a variety of models reconciles these facts. The assumption is fixed aggregate number of traders. Existing theories keep the aggregate number of investors fixed, and, as a result, introducing derivatives, by construction, fragments traders across multiple markets and attracts liquidity away from the underlying market.\textsuperscript{2} I show that when the aggregate number of investors is instead endogenous, the effect is the opposite: Introducing CDS increases liquidity in the bond market.

I refer to this effect as a liquidity spillover effect, and it works as follows. When investors can trade CDS with one another, it increases trading opportunities for long investors—investors who want to be exposed to credit risk. Before they could just buy bonds, but now they can also sell CDS. The increased trading opportunity attracts additional long investors into the economy. These investors, however, because of search frictions in the CDS market, wait to get matched in the CDS market. And here is the key ingredient: While they wait, they simultaneously trade as a bond buyer and search for a bond seller in the bond market. If they get matched first in the bond market, they just buy bonds. Bond sellers, in turn, benefit from the additional buyers: They sell their bonds faster. The result is an increase in bond market liquidity. This is the spillover effect and is the main insight of the paper.


\textsuperscript{2}The fragmentation arises not only in models of an underlying and derivative markets but also in any model with multiple markets and a fixed total number of investors. For example, results in information-based frameworks of Admati and Pfleiderer (1988), Pagano (1989), and Chowdhry and Nanda (1991) and search-theoretic frameworks of Vayanos and Wang (2007), Vayanos and Weill (2008), and Weill (2008) imply that traders endogenously concentrate in one market and trading in the other market either deteriorates or disappears.
Garleanu, and Pedersen (2005, 2007) and Vayanos and Weill (2008). A fraction of bond owners, upon a liquidity shock, try to sell their bonds. Finding a buyer, however, involves search. When a seller finds a buyer, the difficulty of finding another buyer forces the seller to accept a discounted price. Search frictions thus create an illiquidity discount in the bond price. The illiquidity discount, the length of the search process, and the volume of trade all depend on the relative number of sellers and buyers. The number of buyers and sellers are, in turn, endogenous.

I add to this environment first, CDS contracts. CDSs pay when the underlying bond defaults. A CDS buyer—who stands to benefit if the bond defaults—has a short exposure to the underlying credit risk. The CDS seller has the opposite long exposure. CDS are in zero net supply; bonds are in fixed supply. Trading CDS contracts, as with trading bonds, involves search and bilateral bargaining. Second, I endogenize entry and, consequently, the aggregate number of investors. The total mass of long investors, in particular, has a natural interpretation as funding liquidity because long investors, in practice, take the capital intensive side of bond and CDS trades and, as a result, supply liquidity into credit markets. The model thereby features distinct notions of funding liquidity and market liquidity, and both are endogenous.

How does the model explain the bond market reactions? A permanent naked CDS ban reverses the liquidity spillover effect. Investors can no longer sell CDS because their counterparties, the naked CDS buyers, are banned from buying CDS. Long investors exit the CDS market, but by exiting the CDS market they pull out from the bond market also. The result is a decrease in bond market liquidity, consistent with the bond market reaction after the permanent EU ban. If we interpret the number of long investors as resources allocated to credit markets, the intuition for this result is: Financial institutions respond to a permanent CDS ban by scaling back their credit market operations. They fire their CDS and bond traders, shut down their credit trading desks, and switch into equity or currency markets.

I explain the reaction after the temporary ban as follows. The key ingredient for the liquidity spillover effect is endogenous entry. If entry and, consequently, the number of long investors is fixed, the introduction of CDS decreases bond market liquidity. This opposite implication, in fact, rationalizes the reaction after the temporary ban. The idea is if investors expect the ban to be temporary, it is unlikely that they downsize their resources just to scale it all back up after the ban is lifted. More likely, they keep the capital allocated to credit markets fixed and just reshuffle resources locally within the credit market, specifically, from the CDS into the bond market.

Afonso (2011) and Lagos and Rocheteau (2009) also endogenize entry and the aggregate number of investors in a search framework but do so in a single market setting.
The result is a temporary increase in bond market liquidity, consistent with the reaction after the temporary German ban.

The paper offers two contributions in addition to showing the spillover effect. Existing microstructure models of derivatives focus on informational frictions as the source of illiquidity. We hence lack models of derivatives where illiquidity arises from a key friction in trading assets over-the-counter: search costs. Search models are the current workhorse environment of endogenous liquidity frictions and asset prices of OTC traded assets. But, so far, they feature either a single asset or multiple assets with identical cash flows. Thus, my first contribution is to provide, to my knowledge, the first theoretical framework of OTC trading in both the underlying and derivative markets. I model the endogenous interaction between multiple OTC traded assets where one asset is a derivative of the other. I thereby shed light on agents’ incentive to search and trade the underlying versus the derivative asset, the number of long and short interest in each market, market liquidity of each asset, and the prices negotiated between counterparties.

Second, I shed light on naked CDS purchases and thereby fill a gap in the CDS literature that focuses on covered CDS purchases (where investors buy CDS protection on bonds they own). Allowing investors to trade the issuer’s credit risk without trading or owning the bonds is what defines CDS, why they proliferated, and why they were controversial.

The paper is organized as follows. Section 1 presents the model envi-

---

4 They include Subrahmanyam (1991), Gorton and Pennacchi (1993), and John, Koticha, Subrahmanyam, and Narayanan (2003) that I mentioned previously. Additionally, Back (1993) develops a framework based on Kyle (1985) to study the effect of options on price volatility. Biais and Hillion (1994) provide another information-based model of options and study their effect on price informativeness of the underlying asset. For information-based frameworks dealing with themes of complementarity versus substitutability, see Goldstein, Li, and Yang (2013) in the context of multiple markets and Goldstein and Yang (2015) in the context of multiple dimensions of information (My results can be interpreted as: Bonds and CDS are complements when investors adjust their participation rate but are substitutes when they do not).


6 Oehmke and Zawadowski (2013) explore how CDS trading (both covered or naked) affects bond prices in Amihud and Mendelson (1986) type framework with exogenous trading frictions. In contrast, my model features endogenous trading costs and, thereby, an endogenous interaction and spillover between the underlying and derivative markets. For models on issues surrounding covered CDS purchases specifically, see, for example, Thompson (2007), Arping (2014), Bolton and Oehmke (2011), Sambalaibat (2012), and Parlour and Winton (2013).
vironment, and Section 2 characterizes the equilibrium. Section 3 derives the main theoretical result. In Section 4, I model bond shorting and contrast its effect to that of CDS. I also endogenize investors’ search intensities. Proofs of all the propositions are in the Appendix.

1 Model Environment

Time is continuous and goes from zero to infinity. Agents are risk-averse, live infinitely, have idiosyncratic stochastic endowments, and can invest in a risk-free asset with return $r$. They hold and trade bilaterally a risky bond and a derivative “CDS” contract with a cash flow based on the risky bond. Finding a trading counterparty involves search. Agents enter this credit market economy if it makes them better off than their outside option.

Assets

The bond is in supply $S$, trades at price $p_b$, and has a cumulative cash flow process $D_b^t$ satisfying:

$$dD_b^t = \delta dt - JdN_t,$$  

(1)

In (1), $\delta > 0$ is a constant and represents the contractual coupon flow of the bond. I model default as a random deviation from (specifically, a decrease in) the contractual cash flow. In particular, the process $\{N_t, t \geq 0\}$ is a Poisson counting process with intensity parameter $\eta > 0$, and $J > 0$ is a constant and is the default size. The process $N_t$ counts the number of defaults in $[0, t]$, and its increment, $dN_t$, is 0 or 1. Thus, (1) says, in a small interval $[t, t + dt]$, with probability $\eta dt$, the bond defaults, and the rate of the coupon flow decreases by $J$; otherwise, it pays the coupon. I restrict agents’ bond position to $\theta_b \in \{0, 1\}$ units and assume that agents cannot short bonds. In Section 4.1, however, I model bond shorting and compare its effects with that of CDS.

In a CDS contract, the buyer of the contract pays a premium flow $p_c$ to the seller of the contract; the seller, in turn, pays the buyer $\delta$ if the bond defaults. The CDS buyer’s cumulative cash flow $D_c^t$, as a result, follows:

$$dD_c^t = JdN_t.$$  

(2)

Comparing (2) with (1), $D_b^t$ and $D_c^t$ are perfectly negatively correlated. The CDS buyer, as a result, has a short exposure to the underlying credit risk. The CDS seller, on the other hand, has a cash flow that is positively correlated

---

7To simplify notation, I denote the continuous time dependence $y(t)$ as $y_t$.

8As in standard search models, default risk is exogenous; the focus is instead on changes in asset prices through changes in asset liquidity. See He and Milbradt (2014) for a model of endogenous feedback loop between credit risk and liquidity.
with the bond \((-JdN_t)\) and is thus long credit risk. From hereon, when I refer to a long or a short position, I will mean with respect to the underlying credit risk.\(^9\) I denote agents’ CDS position with \(\theta_c \in \{-1, 0, 1\}\), where each denotes a short, a neutral, and a long position, respectively. I rule out simultaneous long positions in both assets.

An investor terminates a CDS contract by paying the other party a fee. The fees are endogenous and are such that the nonterminating party is indifferent between (a) continuing the contract and (b) accepting the fee, searching for a new counterparty, and, upon a match, entering a new position. I denote with \(T_s\) and \(T_b\) the fees the seller and the buyer pay, respectively.

### Agents

Agents have time preference rate \(\beta\) and CARA utility preferences with risk aversion parameter \(\alpha\): \(u(C) = -\exp(-\alpha C)\). Agent \(i\)’s cumulative endowment process \(e_{i,t}\) follows:

\[
d e_{i,t} = \mu_e e_{i,t} dt + \rho_{i,t} \sigma_e (-dN_t) + \sqrt{1 - \rho_{i,t}^2} \sigma_z dZ_t^e, \tag{3}
\]

where \(\mu_e > 0\), \(\sigma_e > 0\), and \(\sigma_z > 0\) are constants, \(Z_t\) is a standard Brownian motion, and \(\rho_{i,t}\) is the instantaneous correlation process between the bond cash flow and the agent’s endowment process. The processes \(\{Z_t, \rho_{i,t}, N_t\}\) are pairwise independent.

The correlation process \(\rho_{i,t}\) is a three-state Markov chain with states \(\rho_{i,t} \in \{-\rho, 0, \rho\}\), where \(\rho > 0\). From hereon, I refer to an agent with \(\rho_{i,t} = -\rho\) as a high-valuation, with \(\rho_{i,t} = 0\) as an average-valuation, and with \(\rho_{i,t} = \rho\) as a low-valuation type. The labels capture agents’ equilibrium valuation for the bond: The investor whose endowment is negatively correlated with the bond is the most willing to buy the bond, while those with a positively correlated endowment are the least willing to buy the bond. High- and low-valuation agents switch to an average-valuation with Poisson intensities \(d\) and \(u\), respectively. The intensity of switching from an average-valuation type to either a high or low-type is zero (that is, the average type is an absorbing state). In Section 2, as I describe how different agents trade in equilibrium, I explain why I need three types and why the average type is an absorbing state.

### Agents’ Decisions

Agents first decide whether to enter the economy. At any point, fixed flows of agents \(F_h\) and \(F_l\) are born as high- and low-valuation agents, respectively.

\(^9\)Thus, a long position through the CDS market, for example, does not mean an investor has bought CDS but means she has sold CDS and, hence, is (long-) exposed to the underlying default risk.
An agent $i \in \{h, l\}$ enters the credit market economy if the expected utility of doing so, $V_{i[0,0]}$, is at least greater than her outside option, $O_i$. A fraction $\nu_i$ of $i$-valuation investors thus enter according to:

$$
\nu_i = \begin{cases} 
1 & V_{i[0,0]} > O_i \\
[0,1] & V_{i[0,0]} = O_i \\
0 & V_{i[0,0]} < O_i 
\end{cases}
$$

(4)

for $i \in \{h, l\}$. As a result, the flows of high and low-valuation entrants are $\nu_h F_h$ and $\nu_l F_l$, and their steady state measures are $\frac{\nu_h F_h}{\gamma_d}$ and $\frac{\nu_l F_l}{\gamma_u}$, respectively.\(^{10}\)

Once in the economy, agents choose their consumption, $C$, and asset portfolios $\{\theta_b, \theta_c\}$.\(^{11}\) Categorizing agents into types $\tau \in \mathcal{T}$ by their valuation (high, average, or low) and asset position, transitions across types occur according to a $K(\tau_i)$-dimensional counting process $\hat{N}_t(\tau)$, where the intensity associated with dimension $k$ is $\gamma(k, \tau_i)$ for agent type $\tau_i$.\(^{12}\) At the arrival times associated with dimension $k$, the agent chooses between types $\tau_i' \in \mathcal{T}(\tau_i, k) \subset \mathcal{T}$. Let us denote with $U(W_t, \tau_t)$ the indirect utility of type $\tau_t$ agent with wealth $W_t$ at time $t$. The agent’s optimization problem is:

$$
U(W_0, \tau_0) = \max_{\{C_t \in \mathbb{R}, \tau_t' \in \mathcal{T}(\tau_t, k)\}} \mathbb{E} \left[ \int_0^\infty e^{-\beta t} u(C_t) dt \right]
$$

(5)

subject to the agent’s wealth process, $W_t$:

$$
dW_t = (r W_t - C_t) dt + d\theta_t + dD^b_t \theta_{b,t} - p_b d\theta_{b,t} + (p_c dt - dD^c_t) \theta_{c,t},
$$

(6)

and the transversality condition $\lim_{T \to \infty} E[e^{-\beta T} e^{-\alpha W_T}] = 0$.

The Bond and the CDS Market

Establishing new asset positions or rebalancing existing ones involves search. A market prescribes which asset an investor can trade given a match with a counterparty. In market $m \in \{b, c\}$, where $b$ stands for the bond market and

\(^{10}\)We can ignore the entry decision of average type agents because, in equilibrium, the expected utility of participating in the economy as an average type agent is zero: $V_{a[0,0]} = 0$. Thus, for any positive outside option ($O_a > 0$ even if it is small), their entry rate is zero. Moreover, the results depend not on absolute levels of $O_h$ and $O_l$ but on their magnitudes relative to $O_a$ (i.e. the model can be recast in terms of $O_h - O_a$ and $O_l - O_a$). Thus, without loss of generality, I set $O_a = 0$. As for the outside options of high and low types, my main results hold for any $O_h$ and $O_l$ including for the special case where $O_h = O_l$. I denote them separately to be general and to later show where the effects come from.

\(^{11}\)Implicit in agents’ choice of an asset position is a choice of which markets to search in: bond, CDS, or both.

\(^{12}\)The appendix explains the counting process in more detail.
c for the CDS market, buyers and sellers meet at a total rate

\[ M_m \equiv \lambda_m \mu_{m,b} \mu_{m,s}, \tag{7} \]

where \( \lambda_m \) is the exogenous matching efficiency of market \( m \in \{b, c\} \), and \( \mu_{m,b} \) and \( \mu_{m,s} \) are the masses of B–yers and S–ellers in market \( m \in \{b, c\} \), respectively. Given the total meeting rate, buyers find a seller with intensity \( \lambda_m \mu_{m,s} \), and sellers find a buyer with intensity \( \lambda_m \mu_{m,b} \). For now, the search intensities \( \lambda_b \) and \( \lambda_c \) are exogenous, but, in Section 4.3, I endogenize them.\(^\text{13}\)

2 Equilibrium, Prices, and Liquidity

I focus my analysis on the steady state equilibrium defined in this section. We first need value functions, \( V_\tau \), from Proposition 1.

**Proposition 1.** Solutions for \( U(W, \tau) \) are of the form:

\[ U(W, \tau) = -e^{-\rho_\tau(W+V_\tau+\bar{\alpha})} \tag{8} \]

where \( \bar{\alpha} \) is given by (B.58). The term \( V_\tau \) is given by

\[ rV_\tau = ((\delta - \eta J) - x_\tau) \theta_b - y|\theta_b| + (p_c - (\eta J + x_\tau)) \theta_c - y|\theta_c| \tag{9} \]

where \( x_\tau = -x \) for high, \( x_\tau = 0 \) for average, and \( x_\tau = x \) for low-valuation investors,

\[ x \equiv r\alpha \rho_e \eta J, \tag{10} \]

\[ y \equiv \frac{r\alpha}{2} \eta J^2, \tag{11} \]

and \( P(\tau, \tau') \), given in (A.27), is the instantaneous payoff of switching from \( \tau \) to \( \tau' \).

The term \( x_\tau = r\alpha \rho_e \sigma_e \eta J \) in (9) captures hedging benefits. It increases in agents’ risk aversion, the correlation between the agents’ endowment and the bond, the volatility of agents’ endowment, \( \sigma_e \), and the bond default risk (both the default intensity, \( \eta \), and the size of the default, \( J \)).

\(^\text{13}\)I separate the search efficiencies in the bond and the CDS market, \( \lambda_b \) and \( \lambda_c \), to be general. The distinction allows for, in a reduced form, potential institutional and regulatory differences between trading bonds vs. CDS contracts (e.g. unlike bonds, CDS trading involves setting up ISDA agreements between counterparties). The main results do not depend on their difference but hold for any \( \lambda_b \) and \( \lambda_c \) including for the special case where \( \lambda_c = \lambda_b \). Later, the distinction helps me to shut down frictions in the bond and the CDS market one by one and show where the effects come from.
The term $y = \frac{\eta r}{2} J^2$ affects both long and short exposures and in the same direction; thus, it captures a holding cost. It increases in the risk aversion parameter and default risk (again, both the default intensity and the size of default).

Since the indirect utility $U(W, \tau)$ is a monotone transformation of $V_\tau$, from hereon, I will work directly with $V_\tau$'s and refer to $V_\tau$'s as value functions. Also, to simplify derivations and to work with linearized versions of (9), I assume the risk aversion parameter $\alpha$ is small (see Duffie, Garleanu, Pedersen (2007) and Vayanos and Weill (2008) for similar approximations).

2.1 Conjectured Agent Types and Transitions

To characterize the equilibrium, I adopt the standard approach in the search literature: I conjecture the equilibrium agent types and their optimal trading strategies.\footnote{Then, the proof of the equilibrium existence shows that the conjectured strategies are indeed optimal.} See Figure 1 to follow along the next discussion.

For high-valuation investors, the optimal exposure is long credit risk. Since an investor goes long by either buying a bond or selling CDS, at any point, a high-valuation investor is one of three asset positions: (1) owns a bond: $\{\theta_b, \theta_c\} = \{1, 0\}$, (2) sold CDS: $\{\theta_b, \theta_c\} = \{0, 1\}$, or (3) without a position: $\{\theta_b, \theta_c\} = \{0, 0\}$. High-valuation investors with the first two positions are inactive: They have reached their terminal optimal position. The investor without a position, on the other hand, seeks a long exposure and does so by searching for a counterparty simultaneously in both the bond and the CDS market.

Low-valuation investors short credit risk. Since they achieve this by buying CDS, they are the naked CDS buyers in the model. At any point, they are one of two asset positions: (1) bought CDS: $\{\theta_b, \theta_c\} = \{0, -1\}$ or (2) looking to buy CDS: $\{\theta_b, \theta_c\} = \{0, 0\}$.

The third valuation type, the average type, allows me to model exit. To model investors’ entry, I also have to model their exit, and the third type provides an exit point for high and low types when they get a valuation shock. Two conditions have to be met for exit to be optimal. First, since investors cannot exit with an existing position, the terminal optimal position for an average type has to be no position: $\{\theta_b, \theta_c\} = \{0, 0\}$. Assumption 1 ensures this in equilibrium (The Appendix gives a detailed intuition for the Assumption). Second, to rule out any incentive to wait to switch to a different valuation type, the valuation type has to be an absorbing state, hence this assumption earlier. See Vayanos and Wang (2007), Weill (2008), Rocheteau and Weill (2011), and Afonso (2011) for similar setups.\footnote{In the appendix, I explain further why I need three types.}

Assumption 1. $2y > x - (r + \gamma_d) O_h > - (x - 2y) (r + \gamma_u) O_l > 0$. 

$14Then, the proof of the equilibrium existence shows that the conjectured strategies are indeed optimal.
Put together, new high and low-valuation entrants ensure the existence of agents seeking long and short exposures. The random changes in valuations (from high and low to an average type) ensure the existence of agents looking to unwind their exposures. Specifically, the only average types are bond owners ($\{\theta_b, \theta_l\} = \{1, 0\}$) because investors with CDS positions unwind their positions immediately upon a valuation shock. The equilibrium agent types, as a result, are $\mathcal{T} = \{h[0,0], h[1,0], h[0,1], a[1,0], l[0,0], l[0,-1]\}$. The mass of bond buyers and CDS sellers each equals the mass of $h[0,0]$ agents: $\mu_{b,n} = \mu_{h[0,0]}$ and $\mu_{c,s} = \mu_{h[0,0]}$, where $\mu_\tau$ denotes the mass of type $\tau$ agents. The mass of bond sellers equals the mass of $a[1,0]$ agents: $\mu_{b,n} = \mu_{a[1,0]}$, while the mass of CDS buyers equals the mass of $l[0,0]$ types: $\mu_{c,n} = \mu_{h[0,0]}$.

2.2 Equilibrium Definition and Existence

**Definition 1.** A steady state equilibrium is value functions $\{V_\tau\}_{\tau \in \mathcal{T}}$, population measures $\{\mu_\tau\}_{\tau \in \mathcal{T}}$, prices $\{p_b, p_c\}$, termination fees $\{T_b, T_c\}$, and entry decisions $\{\nu_h, \nu_l\}$ such that (i) agents’ value functions $\{V_\tau\}_{\tau \in \mathcal{T}}$ solve their optimization problem (5), (ii) entry decisions $\{\nu_h, \nu_l\}$ solve (4), (iii) the flow of agents switching into type $\tau \in \mathcal{T}$ equals the flow of agents switching out of $\tau$, (iv) market clearing conditions (13) and (14) hold, (v) bond and CDS prices $\{p_b, p_c\}$ solve (15) and (16), and (vi) termination fees $\{T_b, T_c\}$ solve (A.34) and (A.35).

I characterized conditions (i) and (ii). I characterize next the remaining conditions. For the third condition, consider, for example, $h[0,0]$ agents. The mass of $h[0,0]$ agents, $\mu_{h[0,0]}$, evolves as:

$$\frac{\partial \mu_{h[0,0]}}{\partial t} = \nu_h F_h + \gamma_a \mu_{h[0,1]} \quad \text{inflow} - \left( \gamma_d \mu_{h[0,0]} + \left( \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]} \right) \right) \mu_{h[0,0]} \quad \text{outflow}.$$

(12)

In (12), the flow of agents turning into $h[0,0]$-type are (1) new high-valuation entrants, $\nu_h F_h$, and (2) long investors who had previously sold CDS, but are searching again because their counterparty terminated the contract, $\gamma_a \mu_{h[0,1]}$. The agents switching out of type $h[0,0]$ are those (1) hit by a valuation shock, $\gamma_d \mu_{h[0,0]}$, and (2) matched with either a bond seller or a CDS buyer. The steady state mass is characterized by $\frac{\partial \mu_{h[0,0]}}{\partial t} = 0$; that is, the inflow equals the outflow, and $\mu_{h[0,0]}$ is constant over time. The inflow-outflow equations for the other agent types are analogous and are in Appendix A.

For the bond market to clear, the total mass of bond owners has to equal to the bond supply:

$$\mu_{h[1,0]} + \mu_{a[1,0]} = S.$$  

(13)

For CDS market clearing, the number of CDSs sold has to equal the number
of CDS purchased:

\[ \mu_{h[0,1]} = \mu_{l[0,-1]} \].

(14)

Bond and CDS prices arise from Nash-bargaining between buyers and sellers. The marginal benefit of buying a bond (i.e. the buyer’s reservation value) is the difference in the expected utility of owning vs. not owning the bond: \( V_{h[1,0]} - V_{h[0,0]} \). The buyer’s gains from trade is thus \( V_{h[1,0]} - V_{h[0,0]} - p_b \).

Similarly, the seller’s reservation value is \( V_{a[1,0]} \), and her gains from trade is \( p_b - V_{a[1,0]} \). A buyer and a seller bargain over the price so that each gets half of the total surplus: \( V_{h[1,0]} - V_{h[0,0]} - V_{a[1,0]} \). The bond price, as a result, is the average between the buyer and seller’s reservation values:

\[ p_b = \frac{1}{2} V_{a[1,0]} + \frac{1}{2} (V_{h[1,0]} - V_{h[0,0]}). \]

(15)

I characterize the CDS premium, \( p_c \), analogously. A CDS buyer’s surplus is \( V_{l[0,-1]} - V_{l[0,0]} \), while the seller’s is \( V_{h[0,1]} - V_{h[0,0]} \). Thus, the CDS price is implicitly defined by

\[ V_{h[0,1]} - V_{h[0,0]} = \frac{1}{2} (V_{l[0,-1]} - V_{l[0,0]} + V_{h[0,1]} - V_{h[0,0]}). \]

(16)

**Proposition 2.** Under conditions (B.78) and (B.81), a steady state equilibrium exists where investors hold and trade CDS.

The proof is in Appendix B. The equilibria differ by their entry rate of low-valuation investors, \( \nu_l \): two corner solutions \( \nu_l = 0, \nu_l = 1 \) and a unique interior solution \( \nu_l \in (0, 1) \).\(^{16}\) The rest of the endogenous variables, including the entry rate of high-valuation investors, is unique. Thus, even if CDS trading is feasible, in the case of a corner solution \( \nu_l = 0 \), investors may not trade CDS in equilibrium. Since the paper is about the effect of CDS, I contrast the equilibria with CDS (i.e. \( \nu_l > 0 \)) to an equilibrium in which I shut down the CDS market.\(^{17}\)

The marginal effect of CDS I highlight below is the same for both levels of the entry rate (interior \( \nu_l \in (0, 1) \) and corner \( \nu_l = 1 \)), hence the equilibrium multiplicity due to the different entry rates does not matter.

### 2.3 Characterization of Liquidity and Prices

I measure bond market liquidity, first, with the trading volume defined in (7) and, second, with the illiquidity discount in the bond price. The latter arises from search frictions and is the difference between the bond price with bond market search frictions (Proposition 3) versus without (Proposition 4).

\(^{16}\) Conditions (B.78) and (B.81) ensure the existence of an interior solution for \( \nu_l \) and \( \nu_h \), respectively.

\(^{17}\) This comparison is equivalent to comparing the equilibrium with \( \nu_l > 0 \) to an equilibrium with \( \nu_l = 0 \).
Proposition 3. Absent bond market search frictions ($\lambda_b \to \infty$), the bond price is

$$p_b = \frac{(\delta - \eta J) + x - y}{r} - \frac{(r + \gamma_d) O_h}{r}.$$  \hspace{1cm} (17)

Eq. (17) shows that the bond is priced by high-valuation agents. In the absence of search frictions, a bond owner—upon a valuation shock—sells instantly to another high-valuation trader. Only high-valuation agents, as a result, own the bond. Since the bond price is the average between the marginal valuations of different bond owners, and high-valuation investors are the only bond owners, the bond price depends on their valuation only.

In particular, $\delta - \eta J$ is the expected cash flow of the bond, and the entire term $(\delta - \eta J + x - y)$ is long investors’ utility valuation of the cash flow. The term $(r + \gamma_d) O_h$ captures the value of the outside option a long investor foregoes to trade in the bond market. A lower bond price compensates long investors for their foregone outside opportunity. The better the outside option of a long investor, the more discounted the bond price.

Proposition 4. With bond market search frictions ($\lambda_b < \infty$), the bond price is given by

$$p_b = \frac{(\delta - \eta J) + x - y}{r} - \frac{(r + \gamma_d) O_h}{r} - \frac{(r + 2\gamma_d)}{r} \frac{1}{2} \omega_b,$$  \hspace{1cm} (18)

where $\omega_b$ is the total gains from a bond transaction:

$$\omega_b \equiv V_{b[1,0]} - V_{b[0,0]} - V_{a[1,0]}.$$

Comparing Propositions 3 and 4, search costs depress the bond price below the frictionless price in (17). Upon a sudden decrease in her valuation, a bond owner tries to sell her bond but, because of search frictions, is unsuccessful for some time. When she does find a buyer, she sells the bond at a discounted price accounting for the difficulty of locating another buyer. Similarly, a potential buyer anticipating the difficulty of reversing positions negotiates a discounted price. Thus, search costs create an illiquidity discount in the bond price: the difference between the price with frictions (18) versus without (17).

Definition 2. The illiquidity discount, $d_b$, in the bond price is:

$$d_b \equiv \frac{(r + 2\gamma_d)}{r} \frac{1}{2} \omega_b.$$  \hspace{1cm} (19)

Proposition 5. The CDS premium is given by:

$$p_c = (\eta J - x + y) + (r + \gamma_d) O_h + (r + 2\gamma_d) \frac{1}{2} \omega_c,$$  \hspace{1cm} (20)
where \( \omega_c \) is the total gains from a CDS transaction:

\[
\omega_c \equiv (V_{h[0,1]} - V_{h[0,0]}) + (V_{l[0,-1]} - V_{l[0,0]}).
\]

CDS contracts are priced by high-valuation investors, who are the marginal sellers of CDS. The cost of making transfers to the CDS buyer in case of default is the lowest for long investors (utility-wise). They are, as a result, more willing to sell CDS than both average and low-valuation investors. In particular, the CDS premium increases in default risk—both the default intensity, \( \eta \), and the size of the default, \( J \)—and, hence, the expected transfer. The entire term, \( (\eta J - x + y) \), then is a long investor’s utility valuation of the expected transfer.\(^{18}\) The second term shows that long investors’ option allows them to charge a higher CDS premium. The third term reflects CDS market illiquidity: It increases the CDS premium.

3 The Main Result

Before I show the main result, consider, first, how the existence of short investors affects the market choice of long investors. Lemma 1 shows that—given a choice of searching in just the bond market, just the CDS market, or in both markets simultaneously—long investors optimally choose to search in both markets simultaneously.\(^{19}\)

**Lemma 1.** Long investors optimally choose to search simultaneously in both the bond and the CDS market.

The reason is twofold. First, searching in two markets at the same time results in a greater number of potential counterparties and, hence, increased probability of trade. Because investors (a) discount and (b) risk getting hit by a valuation shock, they rather realize gains from trade sooner than later. Higher probability of trade (hence, faster realization of gains) thereby improves the investors’ expected utility. Second, the ease of finding another counterparty increases the investor’s bargaining power and, hence, the surplus she extracts from her counterparty.\(^{20}\) Put together, a long investor who searches in both markets has a higher expected utility than an investor who

---

\(^{18}\)For low-type and average-type agents, selling CDS would be more painful because their valuation of CDS payments would be instead \( \eta J + x + y \) and \( \eta J + x \), respectively.

\(^{19}\)The ability to search simultaneously affects long investors only because other investors (bond sellers and CDS buyers) can establish the positions they seek only in one of the markets.

\(^{20}\)Recall that once a match is found, each side has a bargaining power against the other side. This is because it takes time to find a counterparty and by waiting investors (a) forego hedging benefits a trade would yield and (b) risk getting hit by a valuation shock and lose entirely the gains from trade. Prices negotiated between counterparties—and hence the gains from trade each side realizes—balance these incentives to reach a deal. Prices, at the same time, compensate the counterparties for their foregone trading opportunities.
searches in one market. This effect helps with the intuition for the main result of the paper that I show next.

The next proposition gives the main result of the paper. It shows that shorting bonds through naked CDS purchases increases bond market liquidity.

**Proposition 6** (The Spillover Effect). In the equilibrium of Proposition 2, the illiquidity discount \((d_b)\) is smaller and the bond trading volume \((M_b)\) is larger than in the environment without the CDS market.

The additional trading opportunity affects not only the market choices of those who have already entered but also the participation incentives of long investors who had previously remained on the sidelines. The benefit of entering and trading as a long investor, \(V_{h[0,0]}\), now exceeds the cost: \(V_{h[0,0]} > O_h\) (because of the increase in the probability of trade and the bargaining power). A larger number of long investors, as a result, enter and do so until the increase in \(V_{h[0,0]}\) is reversed and the marginal entrant is again indifferent: \(V_{h[0,0]} = O_h\).

The increase in the number of long investors, importantly, implies an increase in the mass of bond buyers and thereby an increase in the bond price and bond volume. Long investors, because of search frictions in the CDS market, wait to get matched in the CDS market. While they wait, they simultaneously search for a bond seller in the bond market and hence trade as a bond buyer. If they get matched first in the bond market, they just buy bonds. Bond sellers, in turn, benefit from the additional buyers: They unload their bonds faster to an investor with a higher valuation. Given the ease of finding another bond buyer, they demand a higher price. Bond buyers, in turn, are willing to pay the higher price because they know they can sell quickly when it is their turn to sell. Put together, the bond price increases. Trading volume also increases due to the increase in the number of bond buyers.

### 3.1 Key Ingredients

The liquidity spillover effect relies on three key ingredients. The first is endogenous entry. The spillover effect arises because long investors react to the introduction of short investors and increase their participation rate. If the entry rate and hence the mass of long investors is fixed, naked CDS buyers attract long investors away from the bond market. Investors who would have otherwise bought bonds now sell CDS. Bond sellers, as a result, face

\(^{21}\)The increase in the number of long investors reverses the two effects that increased long investors’ utility in the first place. First, the number of potential matches starts to decrease as more of them get matched with other long investors. Second, with more other long investors to compete with, the bargaining power of each long investor declines also.
greater search costs. Thus, introducing CDS while keeping the entry rate of long investors fixed reverses the spillover effect: Bond market liquidity deteriorates.

The entry rate being fixed is equivalent to parameter conditions where, in response to short positions, the entry rate increases but quickly hits the corner value, \( \nu_h = 1 \), and thereafter remains fixed. In this case, naked CDS positions would also decrease bond market liquidity. Thus, for the spillover effect to arise, a sufficient number of long investors has to exist on the sideline that can then enter and absorb the short interest. If not, the demand for short positions is too large relative to the available supply of long capital, and short positions just crowd out bond sellers. The demand for short positions is large if short investors do not have good outside option (\( O_l \) is small), CDS market efficiency \( \lambda_c \) is large, and the holding cost \( y \) is small (a small holding cost implies a large gains from CDS trade, hence a larger flow of short investors).\(^{22}\)

**Proposition 7.** If either the bond or the CDS market is frictionless (\( \lambda_b \to \infty \) or \( \lambda_c \to \infty \)), \( d'_b = \hat{d}_b \) and \( M_b = \hat{M}_b \).

As Proposition 7 shows, the second key ingredient for the spillover effect is search frictions in both the CDS and the bond market. If the CDS market is frictionless (\( \lambda_c \to \infty \)), CDS attracts additional long investors as before, but the additional entrants sell CDS immediately upon entry and, hence, do not at the same time search in the bond market.\(^{23}\) The increase in the aggregate mass of long investors, as a result, does not translate to an increase in the mass of bond buyers. In the absence of CDS market search frictions then, CDS contracts are redundant: They do not affect bond market liquidity. If the bond market itself is frictionless (\( \lambda_b \to \infty \)), the CDS market is again redundant. The illiquidity discount in the bond price is zero and the bond volume is the highest possible already without the CDS market.

**Proposition 8.** If long investors cannot search simultaneously but have to choose one of the two markets to search in, then \( d_b = \hat{d}_b \) and \( M_b = \hat{M}_b \).

The third key ingredient is investors’ ability to search simultaneously in both markets. Proposition 8 shows that if we segment bond and CDS markets by shutting down the ability to search in both markets, the introduction of CDS is again redundant. Recall that the ability to also trade with short

\(^{22}\)The short demand can be measured by how much \( V_{b[0,0]} \) increases before long investors’ entry rate responds.

\(^{23}\)In particular, the increase in the equilibrium number of high-valuation investors, \( (\nu_h - \nu_h) \frac{F}{\hat{F}} \), equals the total demand for CDS (the measure of all low-valuation investors, including those who have purchased CDS: \( \frac{F}{\hat{F}} = \mu_{[0,0]} + \mu_{[0,-1]} \)). The intuition is new high-valuation entrants replace one-to-one the bond buyers that migrate to the CDS market and sell CDS instead.
investors increased the probability of trade and the bargaining power of long
investors. Removing the ability to search simultaneously—by removing the
substitutability between bond and CDS trades—cancels these effects and,
with them, the reasons long investors increased their entry rate in the first
place and thereby the spillover effect. The same number of long investors
enter the bond market as without the CDS market.

3.2 Explaining the Reactions to CDS Bans

How does the model explain the bond market reactions after the permanent
CDS ban? A permanent CDS ban reverses the liquidity spillover effect.\textsuperscript{24} Investors can no longer sell CDS because their counterparties, the naked
CDS buyers, are banned from buying CDS. Long investors exit the CDS
market, but by exiting the CDS market they pull out from the bond market
also. The result is a decrease in bond market liquidity, consistent with the
reaction after the permanent ban. We can interpret the number of long
investors as resources allocated to credit markets. This interpretation implies
that financial institutions respond to a permanent CDS ban by scaling back
their credit market operations. They fire their CDS and bond traders, shut
down their credit trading desks, and switch into equity or currency markets.

The reaction after the temporary ban I explain in the context of fixed
entry rate. Going back to the interpretation of long investors as resources
allocated to credit markets, when investors expect the ban to be temporary,
they do not downsize their resources just to scale it all back up after the ban
is lifted. They instead reshuffle resources locally within the credit market,
specifically, from the CDS into the bond market. The result is a tempo-
rary increase in bond market liquidity, consistent with the reaction after the
temporary German ban.

Thus, the opposite bond market reactions arise from different intensive
versus extensive margin effects. In the long run or with a permanent change
in one of the markets, investors reallocate their resources in and out of credit
markets at the extensive margin. In the short run or with a temporary
change, the total capital allocated to credit markets is sticky, and investors
reshuffle at the intensive margin between assets that are close substitutes,
such as bonds and CDS.

\textsuperscript{24} A permanent CDS ban can be thought of setting the CDS matching efficiency ($\lambda_c$)
to zero or, alternatively, as decreasing the flow of low-valuation investors ($F_l$) to zero
because, except for bond owners, the ban made entering and buying CDS infinitely costly.
Moreover, the actual bans prevented CDS purchases for both speculating and hedging
long positions correlated with the sovereign. In the model, consistent with the actual
bans, both would be considered naked CDS purchases because the CDS buyer does not
hold the underlying bonds.
4 Additional Results

Section 4.1 models bond shorting and compares the effect of shorting through CDS contracts with shorting bonds directly. Section 4.2 shows that covered CDS purchases do not arise in the main environment. Section 4.3 shows that the liquidity spillover effect still arises if investors adjust both their entry rate and the intensities with which they search in the two markets.

4.1 Bond Shorting

I model bond shorting following Vayanos and Weill (2008). After purchasing the bond, long investors now lend the bond in a repo (i.e. a lending) market and, as a result, earn a lending fee. On the other side of the repo transaction, short investors \([l[0, 0]]\) borrow the bond to sell it in the spot market. Meetings in both the spot and repo markets occur through search. I denote with \(\lambda_r\) the exogenous search intensity in the repo market. Parties negotiate over the bond price in spot transactions and over the lending fee in repo transactions. An investor unwinds a short position by first buying the bond in the spot market and then delivering it back to the bond lender. To unwind a bond loan, if the counterparty has not yet sold the bond, a lender recalls the bond, sells it, and exits. If the counterparty has already sold the bond, the lender walks away with the collateral originally posted by the short seller. The full model and the proof of equilibrium existence are in the Online Appendix.\(^{25}\)

**Proposition 9.** The bond price in the presence of a bond lending market is given by:

\[
p_b = \left( \delta - \eta J \right) + x - y - (r + \gamma_d)O_b - (r + 2\gamma_d)\frac{1}{2}\omega_b + q_{rb}\frac{1}{2}\omega_r. \tag{21}\]

Shorting bonds has two effects on the bond market. The first is a cash flow effect and is captured by the third term in (21). Bond owners, who now lend the bond to short investors, earn an extra cash flow from the lending fee. The prospect of an additional cash flow raises long investors’ reservation value for the bond and, as a result, the price negotiated between bond buyers and sellers.

Thus, shorting bonds both directly and through CDS contracts increase the bond price. The former, however, does so by changing the cash flow the bond yields its holders. The latter does so by changing its liquidity. To highlight this difference in the underlying mechanism, Proposition 10 part (a)

\(^{25}\)Below results on how a repo market affects bond liquidity are new relative to Vayanos and Well (2008). First, they keep the entry of high and low-valuation investors fixed, while I endogenize them. Second, they compare bond market liquidity in the cross section across multiple bonds. In contrast, I focus on how liquidity of a single bond differs with vs without a repo market.
shows that the lending fee effect arises even if the underlying bond market is frictionless. Whereas for the CDS’s spillover effect to arise, it requires illiquidity of the bond market. The lending fee effect of shorting has been shown before and is the effect of bond shorting we are most familiar with (see, for example, Duffie, Garleanu, Pedersen (2002) and Vayanos & Weill (2008)).

**Proposition 10.** (a) If the spot market is Walrasian ($\lambda_b \to \infty$), then $p_b > \hat{p}_b$ and $M_b > \hat{M}_b$. (b) If the repo market is Walrasian ($\lambda_r \to \infty$), then $p_b > \hat{p}_b$ and $M_b > \hat{M}_b$.

The second effect of shorting bonds is a liquidity effect. It is the analogue of the spillover effect of CDS. Allowing short positions introduces into the spot market an additional mass of sellers—the short sellers. It also introduces an additional mass of buyers: the investors looking to buy back the bond to unwind their short position. Additionally, the increase in the mass of bond sellers attracts additional long investors into the economy as bond buyers. Both the number of bond buyers and sellers, as a result, increase. The increase in the bond market depth manifests in the bond price as a decrease in the illiquidity discount, the second term in (21). As in the case of CDS, this effect relies on endogenous entry and frictions in the bond market.

Even though both ways to short increase spot market liquidity, the effects differ in the ingredients they require. Recall that the spillover effect of CDS requires frictions in the CDS market (in addition to frictions in the bond market and the endogenous entry of long investors). Direct shorting, in contrast, affects the spot market even if the repo market is frictionless.

The difference in the necessary ingredients highlights the derivative nature of CDS contracts. Directly shorting requires trading in the spot market. Shorting activity and spot market frictions, as a result, are inherently interdependent: A change in shorting activity, by construction, changes trading activity in the spot market. In contrast, shorting through CDSs does not involve trading the bond itself. CDS contracts, as a result, are redundant if the CDS market is frictionless.

### 4.2 Covered CDS

A CDS purchase is called covered if a bondholder purchases CDS: $\{\theta_b, \theta_c\} = \{1, -1\}$. Lemma 2 shows that covered CDS positions do not arise in equilibrium. Among the two types of bond owners (high- and average-valuation), average-valuation bondholders are the only potential CDS buyers (hence,

\footnote{Above with the lending fee effect, bond borrowers in the repo market attract additional long investors into the economy (since they eventually trade as bond lenders). Although similar, here it is instead additional bond sellers in the spot market (the short sellers) that attract long investors as bond buyers.}
covered CDS buyers). If an average-type investor buys CDS from a high-
valuation investor, the gains from trade is proportional to $x - 2y - (r + 7d)O_h$. 
But, by Assumption 1, this is negative. The intuition is: When high and av-
erage types enter a CDS contract, the holding cost both sides incur plus the 
entry cost outweigh the total hedging benefit. Thus, bondholders do not buy 
CDS, and only low-valuation investors buy CDS.

**Lemma 2.** In the equilibrium of Proposition 2, the mass of agents with 
covered CDS positions, $\{\theta_b, \theta_c\} = \{1, -1\}$, is zero.

Enriching the environment so that covered CDS positions do arise would 
change the benchmark environment. Whether bondholders buy CDS or not 
in the benchmark, the marginal effect of naked CDS positions—which is what 
the paper is about—relative to the benchmark should be the same.

### 4.3 Endogenous Search Intensities

In the main environment, search intensities in the bond and the CDS market, 
$\lambda_b$ and $\lambda_c$, are fixed. In this section, I endogenize them and show that the 
liquidity spillover effect still arises when investors react to the introduction 
of CDS by adjusting both their entry rate and their search efforts in the two 
markets.\textsuperscript{27}

I endogenize investors’ search intensities as follows. A high-valuation 
investor searches for a counterparty in market $m \in \{c, b\}$ with search effort 
$\lambda_m$. As a result, she meets a counterparty at Poisson arrival times with an 
intensity equal to her search effort times the mass of potential counterparties. 
For example, in the bond market, a bond buyer searching with intensity 
$\lambda_b$ finds a seller among the mass $\mu_{b,s}$ of sellers with a total intensity $\lambda_b \mu_{b,s}$. Since 
a total mass $\mu_{b,b}$ of bond buyers does the same thing, the total volume of 
matches are:

$$M_b = \lambda_b \mu_{b,s} \mu_{b,b}.$$  \hspace{1cm} (22)

Analogously, in the CDS market, CDS buyers and sellers meet at a total rate:

$$M_c = \lambda_c \mu_{c,s} \mu_{c,b}.$$ 

For simplicity, I endogenize the search effort of long investors only and set 
the search effort of investors on the short side (i.e. of bond sellers and CDS 
buyers) to zero.\textsuperscript{28}

\textsuperscript{27}The endogenous search, segmented, and the main environments can be interpreted as environments with endogenous search effort but with different feasible regions. The feasible region is: $\{\lambda_b, \lambda_c\} \in [0, R] \times [0, R]$ in the endogenous search environment, 
$\{\lambda_b, \lambda_c\} \in \{\lambda_b, 0\} \times \{0, \lambda_c\}$ in the segmented environment (and search effort is indivisible), 
and $\{\lambda_b, \lambda_c\} = \{\lambda_b, \lambda_c\}$ in the baseline environment.

\textsuperscript{28}In the appendix, I endogenize search efforts of all investors who want to rebalance their asset position.
An investor incurs a flow search cost
\[ c(\lambda_b, \lambda_c) \equiv c_0 (\lambda_b)^2 + c_0 (\lambda_c)^2 \] (23)
as a function of her search efforts \( \{\lambda_b, \lambda_c\} \), where \( c_0 > 0 \) is a constant.\(^{29}\)
Thus, if an investor searches in two markets simultaneously, she internalizes the cost of search in both markets. If an investor searches in just one market \( m \), her total search cost is the cost of search in that market only, \( c_0 (\lambda_m)^2 \).

The investor chooses \( \{\lambda_b, \lambda_c\} \) to maximize her expected utility. The first order condition with respect to her search effort in the bond market, \( \lambda_b \), is
\[ r\alpha \frac{\partial c(\lambda_b, \lambda_c)}{\partial \lambda_b} = \mu_{[1,0]} \left( 1 - e^{-r\alpha \left( -p_b + V_h(1,0) - V_h(0,0) \right)} \right), \] (24)
while the first order condition with respect to her search effort in the CDS market, \( \lambda_c \), is
\[ r\alpha \frac{\partial c(\lambda_b, \lambda_c)}{\partial \lambda_c} = \mu_{[0,1]} \left( 1 - e^{-r\alpha \left( V_h(0,1) - V_h(0,0) \right)} \right). \] (25)

In (24) and (25), the optimal search effort equates the marginal cost (the left-hand side) with the marginal benefit (the right-hand side) of an additional unit of search effort. The marginal benefit is the product of the mass of potential matches and the gains from trade upon a match. The steady state equilibrium includes the optimal search efforts as additional control variables that satisfy the first order conditions.

Lemma 3. The characterization of the bond price and the illiquidity discount are the same as (18) and (19).

Proposition 11. With the introduction of CDS, long investors search with less effort in the bond market \( (\lambda_b < \hat{\lambda}_b) \). Nevertheless, bond market liquidity increases \( (d_b < \hat{d}_b \) and \( M_b > \hat{M}_b \)).

Here is the intuition for why long investors lower their search effort in the

\(^{29}\)The convex cost specification implies that if an investor splits her search effort and searches with, say, 50 and 50 units of effort in each market, her total search cost is smaller than if she searches in one market with 100 units. I abstract from microfoundations that would result in such a functional form. The specification captures, for example, complementarities of trading in multiple related markets. The complementarity could be in the form of, for example, information (e.g., resources expended on pricing individual bonds help an investor price CDS relatively quickly and vice versa). Or it could be in the form of trading relationships (e.g., networks formed in the bond market help form trading relationships in the CDS market).

We can specify the total search cost more generally as \( c(\lambda_b, \lambda_c) = (c_0 (\lambda_b)^g + c_0 (\lambda_c)^g)^a \) for some positive constants \( g \) and \( a \). It is impossible to prove the liquidity spillover effect for a general functional form, but numerical results suggest that the main effect arises as long as the cost function is a convex function of \( \lambda_b \) and \( \lambda_c \), and investors can search simultaneously in both markets.
bond market. The value function of a long investor is given by:

\[
(r + \gamma_d)V_{h[0,0]} = \left( \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b - c_0(\lambda_b)^2 \right) + \left( \lambda_c \mu_{l[0,0]} \frac{1}{2} \omega_c - c_0(\lambda_c)^2 \right). \tag{26}
\]

The right-hand side is the sum of the expected gains from trade in the bond and the CDS market minus the search cost in each market. From the first order conditions \(24)-(25)\), \(\lambda_b = \frac{\mu_{a[1,0]} \, \frac{1}{2} \omega_b}{2c_0}\) and \(\lambda_c = \frac{\mu_{l[0,0]} \, \frac{1}{2} \omega_c}{2c_0}\). Substituting in the optimal search efforts, \(26\) becomes:

\[
(r + \gamma_d)V_{h[0,0]} = \frac{\left( \mu_{a[1,0]} \omega_b \right)^2}{16c_0} + \frac{\left( \mu_{l[0,0]} \omega_c \right)^2}{16c_0}.
\]

Thus, for a long investor, the expected gains from a bond transaction net of the search cost is proportional to \(\mu_{a[1,0]} \omega_b\). For the marginal entrant to be indifferent between entering \((V_{h[0,0]}\) and their outside option \((O_b)\), \(\mu_{a[1,0]} \omega_b\) in the equilibrium with CDS has to be smaller than in the equilibrium without CDS. Smaller \(\mu_{a[1,0]} \omega_b\) implies that the marginal benefit of searching for a bond decreases, and that long investors shift their search effort from the bond into the CDS market.

Despite the decrease in long investors’ search effort, the effect of CDS on bond market liquidity is the same as when search intensities are exogenous: CDS increases bond market liquidity. The intuition is also identical. A decrease in \(\mu_{a[1,0]} \omega_b\) requires an increase the mass of bond buyers, \(\mu_{h[0,0]}\), because, intuitively, as more of them compete with each other, the rents each extracts decreases. Thus, as in the environment with exogenous search intensities, short investors attract additional long investors into the economy. Long investors, in turn, search simultaneously in both markets, creating the spillover of bond buyers and, thereby, liquidity into the bond market.

### 5 Conclusion

I build a continuous time, dynamic search model of bond and CDS markets. I show that shorting the underlying bonds through CDS contracts increases liquidity of the underlying bond market. This result contrasts with the existing literature on derivatives, which shows that derivatives instead attract liquidity away from the underlying market. My results help explain how sovereign bond markets reacted during bans on naked CDS purchases.

My results imply that permanently banning naked CDS trading decreased bond market liquidity, reduced bond prices, and, thereby, increased sovereigns’ borrowing cost exactly when governments were trying to achieve the opposite and quell a sovereign debt crisis.
Figure 1: A Snapshot of Transitions Between Agent Types

The figure shows the transitions between agent types. Flows of $\nu_{Fh}$ and $\nu_{Fl}$ agents enter the economy as new high- and low-valuation investors. High- and low-valuation agents switch to an average-valuation with intensities $\gamma_d$ and $\gamma_u$, respectively. A trader seeking a long position ($h[0,0]$) finds a counterparty in the bond and CDS markets with intensities $\lambda_b \mu_a[1,0]$ and $\lambda_c \mu_l[0,0]$, respectively. A bond seller ($a[1,0]$) finds a buyer with intensity $\lambda_b \mu_h[0,0]$. A trader seeking a short position by buying CDS ($l[0,0]$) finds a counterparty with intensity $\lambda_c \mu_h[0,0]$.

<table>
<thead>
<tr>
<th>Asset Positions</th>
<th>Long Credit Risk</th>
<th>Long Credit Risk</th>
<th>Short Credit Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_h, \theta_l$:</td>
<td>$[1,0]$</td>
<td>$[0,0]$</td>
<td>$[0,1]$</td>
</tr>
</tbody>
</table>

High (h) | Bond owner | Long investor | Sold CDS | Terminates CDS & exits |
---|---|---|---|---|
$\gamma_d$ | $\lambda_b \mu_a[1,0]$ | $\lambda_c \mu_l[0,0]$ | $\gamma_u$ | $\lambda_b \mu_h[0,0]$ |

Avg (a) | Bond seller | | Terminates CDS & exits | |
---|---|---|---|---|
$\gamma_u$ | $\lambda_b \mu_a[1,0]$ | $\lambda_c \mu_l[0,0]$ | $\gamma_d$ | $\lambda_b \mu_h[0,0]$ |

Low (l) | Naked CDS buyer | CDS holder | |
---|---|---|---|
$\gamma_u$ | $\nu_{Fh}$ | $\lambda_c \mu_l[0,0]$ | |

<table>
<thead>
<tr>
<th>Points of Interest</th>
</tr>
</thead>
</table>
| $\lambda_b \mu_a[1,0]$ and $\lambda_c \mu_l[0,0]$:
| Bond and CDS markets |
| $\lambda_b \mu_h[0,0]$: Bond seller finds a buyer |
| $\lambda_c \mu_h[0,0]$: Trader seeking a short position |

22
A Appendix

The counting process $\hat{N}_i(\tau)$ embeds changes in an agent’s type due to both a valuation shock and an endogenous change in an asset position. Consider, for example, agent type $\tau = h[0,0]$. The dimension of $\hat{N}_i(\tau)$ is $K(\tau) = 3$, where the three possible events are: (1) the agent’s valuation changes, (2) the agent finds a counterparty in the bond market, (3) the agent finds a counterparty in the CDS market. Intensities of these events are $\gamma(1,\tau) = \gamma_d$, $\gamma(2,\tau) = \lambda_b\mu_{a[1,0]}$, and $\gamma(3,\tau) = \lambda_c\mu_{l[0,0]}$, respectively. Similarly, consider agent $\tau = h[0,1]$ (an agent who has sold CDS). Then, $K(\tau) = 2$, and the two possible events are (1) the agent himself gets a valuation shock or (2) his counterparty’s valuation changes. The intensities are: $\gamma(1,\tau) = \gamma_d$ and $\gamma(2,\tau) = \gamma_u$. It is analogous for other agent types.

The instantaneous payoff from a transition from $\tau$ to $\tau'$ is given by:

$$P(\tau, \tau') = \begin{cases} 
-p_b & \text{if } \tau = i[0, \theta_c] \text{ and } \tau' = i[1, \theta_c] \\
 p_b & \text{if } \tau = i[1, \theta_c] \text{ and } \tau' = i[0, \theta_c] \\
 0 & \text{else.} \end{cases}
$$

(A.27)

The value functions simplify to

$$rV_{i[0,0]} = \gamma_u(0 - V_{i[0,0]}) + \frac{M_c}{\mu_{l[0,0]}} \frac{1}{2}\omega_c
$$

(A.28)

$$rV_{i[1,0]} = \gamma_d(0 - V_{i[1,0]}) + \frac{M_b}{\mu_{h[0,0]}} \frac{1}{2}\omega_b + \frac{M_c}{\mu_{h[0,0]}} \frac{1}{2}\omega_c
$$

(A.29)

$$rV_{i[0,1]} = (\delta - \eta J) + x - y + \gamma_d(V_{a[1,0]} - V_{h[1,0]})
$$

(A.30)

$$rV_{i[1,1]} = (\delta - \eta J) - y + \frac{M_b}{\mu_{a[1,0]}} \frac{1}{2}\omega_b
$$

(A.31)

$$rV_{h[0,0]} = p_c - (\eta J - x) - y + \gamma_d(-T_b - V_{h[0,1]})
$$

(A.32)

$$rV_{h[0,1]} = -p_c + (\eta J + x) - y + \gamma_u(-T_b - V_{i[0,1]})
$$

(A.33)

Here, I explain further why I need three types. In an environment with just two valuation types and fixed aggregate number of investors, introducing naked CDS positions deteriorates bond market liquidity. The insight of the paper, however, is to show that the aggregate number of investors responds to the introduction of short positions. Thus, endogenous aggregate number of investors (through endogenous entry) is a key ingredient for my mechanism. Modeling entry and exit per se does not require three types as long as one is modeling just bonds and not CDS. Modeling short positions also does not require three types as long as there is no entry and exit. But modeling both entry and exit and short positions requires three valuation types. This is because with just two types, when a short position is introduced, the terminal optimal position for the lower type is a short position $\{0,-1\}$, not $\{0,0\}$, implying that investors never exit. Three different valuations allow for one type whose optimal position is no position and who wants to exit and two other types whose optimal positions are short and long positions, respectively, and who
do not want to exit.

The intuition for Assumption 1 is as follows. The gains from CDS trade between high and average types is proportional to \(x - 2y - (r + \gamma_d)O_h\), which is negative by Assumption 1. Intuitively, the difference in their valuations—hence, the total hedging benefit \((x - 0)\)—is too small relative to the holding cost both sides incur \((2y)\) and the implicit entry cost, \((r + \gamma_d)O_h\). The absence of the gains from trade ensures that (a) average types do not buy CDS from a high type, and (b) once a CDS buyer (initially, a low type) switches to an average type, she prefers to unwind her short position she has with a high-type than to remain a CDS buyer.\(^{30}\)

The gains from CDS trade exists only between high- and low-valuation investors: \(2x - 2y - (r + \gamma_d)O_h - (r + \gamma_u)O_l > 0\). Their valuations are far apart enough that the total hedging benefit, \(2x = x - (-x)\), outweighs the holding cost, \(2y\), and the costs of entry, \((r + \gamma_d)O_h + (r + \gamma_u)O_l\). On the bond side, the gains from trade between high- and average types is proportional to the difference in valuations, \(x - 0\), minus the opportunity cost of entry: \(x - (r + \gamma_d)O_h\).\(^{31}\) This is positive by Assumption 1. Thus, a bond owner who switches to an average type prefers to unwind and sell her bond.

Consider how these parameter conditions relate to the original parameters. For intuition purposes, let us ignore \(O_h\) and \(O_l\).\(^{32}\) Then, Assumption 1 is \(2x > 2y > x\) or \(x > y > \frac{1}{2}x\). Using the definitions of \(x\) and \(y\),

\[
2\rho\sigma_e \eta J > 2 \frac{r\alpha}{\eta} J^2 > r\rho \sigma_e \eta J
\]

Canceling terms:

\[
2\rho\sigma_e > J > \rho\sigma_e.
\]

Thus, Assumption 1 bounds the default size, \(J\), between 1 and 2 units of \(\rho\sigma_e\), which is the part of the endowment risk that can be hedged via bond or CDS.

Consider fees CDS counterparties pay each other to terminate the contract. If a buyer terminates, the seller goes from being a \(h[0, 1]\) type to \(h[0, 0]\), and the seller’s utility decreases by \((V_{h[0,1]} - V_{h[0,0]})\). To make the seller indifferent then, the buyer has to pay a fee equal to the decrease in the seller’s utility:

\[
T_b = V_{h[0,1]} - V_{h[0,0]}, \tag{A.34}
\]

Analogously, a CDS seller (the long side) has to pay the short side:

\[
T_s = V_{l[0,-1]} - V_{l[0,0]}, \tag{A.35}
\]

The right-hand sides coincide with the gains from trade to each side; hence, both equal \(2\rho\sigma_e \eta J\).

\(^{30}\)It is analogous between average and low types. The gains from CDS trade between them is proportional to \(x - 2y - (r + \gamma_d)O_l\), which is negative. This ensures that (a) average types do not sell CDS to a low type, and (b) once a CDS seller (a high type) switches to an average, she prefers to unwind her long position than to remain a CDS seller.

\(^{31}\)When high and average types trade a bond, the total holding cost does not change because, unlike CDS transactions, investors do not create new positions, only the bond changes just ownership.

\(^{32}\)If, for example, \(x - y > 0\), \(x - 2y < 0\), and \((x - 2y) + [x - (r + \gamma_d)O_h - (r + \gamma_u)O_l] > 0\), then Assumption 1 holds.
$\frac{1}{2} \omega_c$.

In the steady state equilibrium, the masses of agent types are constant: the flow of agents turning into a particular type (Inflow) has to equal the flow of agents switching out of that type (Outflow). Thus, the inflow-outflow equations are:

<table>
<thead>
<tr>
<th>Agent Type</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>long investor $h[0,0]$</td>
<td>$\nu_h F_h + \gamma_u \mu_{h[0,1]} = \gamma_d \mu_{h[0,0]} + M_b + M_c$ (A.36)</td>
</tr>
<tr>
<td>naked CDS buyer $l[0,0]$</td>
<td>$\nu_l F_l + \gamma_d \mu_{l[0,-1]} = \gamma_u \mu_{l[0,0]} + M_c$ (A.37)</td>
</tr>
<tr>
<td>bond owner $h[1,0]$</td>
<td>$M_b = \gamma_d \mu_{h[1,0]}$ (A.38)</td>
</tr>
<tr>
<td>bond seller $a[1,0]$</td>
<td>$\gamma_d \mu_{a[1,0]} = M_b$ (A.39)</td>
</tr>
<tr>
<td>sold CDS $h[0,1]$</td>
<td>$M_c = \gamma_d \mu_{h[0,1]} + \gamma_u \mu_{h[0,1]}$ (A.40)</td>
</tr>
<tr>
<td>bought CDS $l[0,-1]$</td>
<td>$M_c = (\gamma_u + \gamma_d) \mu_{l[0,-1]}$ (A.41)</td>
</tr>
</tbody>
</table>

Because proofs of Propositions 1 and 2 are long, I relegate them to the Online Appendix. In the proof of Proposition 2, first, I show that, taking the entry rates as given, population masses, value functions, gains from trade, and prices are uniquely determined and positive. Second, I characterize the solution for the entry rates. Lastly, I show that the conjectured optimal trading strategies are indeed optimal.

**Proof of Proposition 3.** I assumed so far that the correlation of an agent’s endowment with the bond (in terms of magnitude) is the same for high and low-valuation agents. Going forward, I distinguish between high and low types’ correlations so that $x_h = r \alpha p_h \sigma_e \eta J$ and $x_l = r \alpha p_l \sigma_e \eta J$ instead of assuming that the magnitudes of their correlations and, hence, their hedging benefits are the same. My results do not depend on them being different. The distinction just helps keep track of where the effects come from.

Let us, first, derive the bond price with frictions. Substituting the value functions of $h[1,0]$, $h[0,0]$ and $a[1,0]$ into the bond price, (15), and simplifying:

$$rp_b = \delta - \eta J + \frac{1}{2} x_h - y - \frac{1}{2} \gamma_d \omega_b - \frac{1}{2} \left( \lambda_h \mu_{a[1,0]} \frac{1}{2} \omega_b + \lambda_e \mu_{l[0,0]} \frac{1}{2} \omega_c - \lambda_h \mu_{h[0,0]} \frac{1}{2} \omega_b \right).$$

Using (B.64) and (B.65),

$$rp_b = \delta - \eta J + \frac{1}{2} x_h - y - \frac{1}{2} \gamma_d \omega_b - \frac{1}{2} ((r + \gamma_d) O_h - x_h + (r + \gamma_d) O_h + (r + \gamma_d) \omega_b)$$
$$= \delta - \eta J + x_h - y - (r + \gamma_d) O_h - \frac{1}{2} (r + 2 \gamma_d) \omega_b.$$

$$p_b = \frac{\delta - \eta J + x_h - y - (r + \gamma_d) O_h}{r} - \frac{(r + 2 \gamma_d)}{2r} \omega_b.$$ (A.42)

Later, the proof of Proposition 7 shows that as $\lambda_h \to \infty$, $\omega_b \to 0$. Hence, the bond price absent search frictions in the bond market is given by the first term in (A.42).

**Proof of Proposition 4.** The derivation is shown in Proposition 3 proof.
Proof of Proposition 5. Combining (A.29), (A.32), and the termination fees, we get:

\[(r + \gamma_d) \left( V_{h[0,1]} - V_{h[0,0]} \right) = p_c - (\eta J - x_h) - y - \gamma_d \frac{1}{2} \omega_c - \frac{M_b}{\mu_{h[0,0]}} \frac{1}{2} \omega_b - \frac{M_c}{\mu_{h[0,0]}} \frac{1}{2} \omega_c.\]

Using the fact that \( V_{h[0,1]} - V_{h[0,0]} = \frac{1}{2} \omega_c \), the CDS price is given by:

\[p_c = (\eta J - x_h) + y + \frac{1}{2} (r + 2\gamma_d) \omega_c + \frac{M_b}{\mu_{h[0,0]}} \frac{1}{2} \omega_b + \frac{M_c}{\mu_{h[0,0]}} \frac{1}{2} \omega_c = \eta J - x_h + y + \frac{1}{2} (r + 2\gamma_d) \omega_c + (r + \gamma_d) O_h.\]

where the second equality uses (B.65).

Proof of Proposition 6. Consider the set of three equations and three unknowns \( \{\mu_{a[1,0]}, \mu_{h[0,0]}, \omega_b\} \):

\[\lambda_b \mu_{a[1,0]} \mu_{h[0,0]} = \gamma_d \left( S - \mu_{a[1,0]} \right) \quad (A.43)\]

\[(r + \gamma_d) \omega_b = x_h - \lambda_b \left[ \mu_{a[1,0]} + \mu_{h[0,0]} \right] \frac{1}{2} \omega_b - A \quad (A.44)\]

\[(r + \gamma_d) O_h = \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b + A. \quad (A.45)\]

where \( A \equiv \lambda_c \mu_{[0,0]} \frac{1}{2} \omega_c \) is the expected rents a long investor extracts from selling CDS.

The first equation comes from combining (13) and (A.38). The second equation (A.44) comes from combining the value functions for \( h[0,0], h[1,0] \), and \( a[1,0] \), substituting in \( M_b \) and \( M_c \), and simplifying. Substituting \( M_b \) into (A.29) and using the entry condition, (A.29) gives the third equation, (A.45).

For an interior solution, the expected rents a long investor extracts from trading in the bond market, \( \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b \), has to be smaller in the equilibrium with CDS than in the equilibrium without CDS. To see this, the value function of a long investor is given by

\[(r + \gamma_d) V_{h[0,0]} = \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b + \lambda_c \mu_{[0,0]} \frac{1}{2} \omega_c. \quad (A.46)\]

The first term is the expected rents a long investor extracts from trading in the bond market: The probability of finding a counterparty in the bond market times the gains from trade from a bond transaction. The second term is the analogous expected gains from trade in the CDS market. Using the entry condition, (A.46) with and without CDS is

\[(r + \gamma_d) O_h = \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b + \lambda_c \mu_{[0,0]} \frac{1}{2} \omega_c\]

\[(r + \gamma_d) O_h = \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b,\]

respectively. Since \( \lambda_c \mu_{[0,0]} \frac{1}{2} \omega_c > 0 \), and the left hand sides are the same, it has to be that: \( \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b > \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b \).

Add the third equation to the second equation:
\[(r+\gamma_d)\omega_b = (x_h - (r+\gamma_d)O_h) - \lambda_b\mu_{h[0,0]} \frac{1}{2} \omega_b. \quad (A.47)\]

Then, from (A.43) and (A.47), \(\mu_{h[0,0]}\) and \(\omega_b\) are implicit functions of \(\mu_{a[1,0]}\). Consider how they change with \(\mu_{a[1,0]}\):

\[
\frac{\partial \mu_{h[0,0]}}{\partial \mu_{a[1,0]}} = -\frac{\gamma_d + \lambda_b \mu_{h[0,0]}}{\lambda_b \mu_{a[1,0]}},
\]

\[
\frac{\partial \omega_b}{\partial \mu_{a[1,0]}} = -\frac{(\gamma_d + \lambda_b \mu_{h[0,0]}) \frac{1}{2} \omega_b}{\mu_{a[1,0]}(r + \gamma_d + \lambda_b \mu_{h[0,0]} \frac{1}{2})}.
\]

Thus, \(\mu_{h[0,0]}\) decreases in \(\mu_{a[1,0]}\), while \(\omega_b\) increases in \(\mu_{a[1,0]}\). Then, the expected rents a long investor extracts from trading in the bond market, \(\lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b\), as an implicit function of \(\mu_{a[1,0]}\); increases in \(\mu_{a[1,0]}\), As a result, \(\mu_{a[1,0]}\) has to be smaller in the equilibrium with CDS than in the equilibrium without CDS: \(\mu_{a[1,0]} < \hat{\mu}_{a[1,0]}\). Since \(\omega_b\) increases in \(\mu_{a[1,0]}\), it has to be that: \(\omega_b < \hat{\omega}_b\). Since, by definition, the illiquidity discount \(d_b\) just depends on \(\omega_b\), we have: \(d_b < \hat{d}_b\). From (A.43), a decrease in \(\mu_{a[1,0]}\) implies an increase in bond volume: \(M_b > \hat{M}_b\). \(\square\)
Online Appendix: A Theory of Liquidity Spillover Between Bond and CDS Markets

B Proofs

Proof of Proposition 1. I first derive the Hamilton-Jacobi-Bellman (HJB) equations. Eq. (5) can be written recursively as:33

\[
U(W_t, \tau_t) = \max_{C_t \in \mathbb{R}, \tau_t + \Delta t \in T(\tau_t, k)} u(C_t)\Delta t + (1 - \beta \Delta t) \mathbb{E}U(W_{t + \Delta t}, \tau_{t + \Delta t}). \tag{B.48}
\]

Subtract \((1 - \beta \Delta t) U(W_t, \tau_t)\) from both sides and divide by \(\Delta t\):

\[
\beta U(W_t, \tau_t) = \max_{C_t \in \mathbb{R}, \tau_t + \Delta t \in T(\tau_t, k)} u(C_t) + (1 - \beta \Delta t) \mathbb{E} \left[ \frac{U(W_{t + \Delta t}, \tau_{t + \Delta t}) - U(W_t, \tau_t)}{\Delta t} \right]. \tag{B.49}
\]

In the limit as \(\Delta t \to 0\), (B.49) becomes

\[
\beta U(W_t, \tau_t) = \max_{C_t \in \mathbb{R}, \tau_t + dt \in T(\tau_t, k)} u(C_t) + \mathbb{E} \left[ \frac{dU(W_t, \tau_t)}{dt} \right]. \tag{B.50}
\]

Consider the expectation of \(dU(W_t, \tau_t)\). Applying a Taylor series expansion to \(U(W_t, \tau_t)\) and taking its expectation, we get:34

\[
\mathbb{E}dU(W_t, \tau_t) = U_W(W_t, \tau_t) \mathbb{E} [dW_t] + \frac{1}{2} U_{WW}(W_t, \tau_t) \mathbb{E} [dW_t^2] \tag{B.51}
\]

\[+ \sum_{k=1}^{K} \gamma_t(\tau_t, k) dt [U(W_t + P(\tau_t, \tau_t + dt), \tau_{t + dt}) - U(W_t, \tau_t)].\]

Consider \(dW_t\) in the first term. Using (1), (2), (3), and (6) and rearranging, we get:

\[
dW_t = (rW_t - C_t + \mu_t + \delta \theta_{b,t} + p_c \theta_{c,t}) dt + (\sigma_e \rho_t + J(\theta_{b,t} + \theta_{c,t})) (-dN_t)
\]

\[+ \sqrt{1 - p_b^2} \sigma_z dZ_t - p_b d\theta_{b,t}.\]

Using \(\mathbb{E}[dN] = \eta dt\),

\[
\mathbb{E}[dW_t] = (rW_t - C_t + [\mu_t - \sigma_e \rho_t \eta] + (\delta - \eta J) \theta b_t + (p_c - \eta J) \theta c_t) dt. \tag{B.52}
\]

33 This comes from observing that over a small time interval \([0, \Delta t]\), (5) can be written as:

\[
U(W_0, \tau_0) = \mathbb{E} \int_0^\infty e^{-\beta t} u(c_t) dt = u(c^*_t) \Delta t + e^{-\beta \Delta t} \mathbb{E} \left[ \int_{\Delta t}^\infty e^{-\beta(t - \Delta t)} u(c^*_t) dt \right]
\]

where \(\{c^*_t\}\) is the optimal consumption path. The term inside the expectations operation is \(U(W_{\Delta t}, \tau_{\Delta t})\), thus \(U(W_0, \tau_0) = \max u(c_0) \Delta t + e^{-\beta \Delta t} \mathbb{E}U(W_{\Delta t}, \tau_{\Delta t})\). Similarly, if we start at \(\{W_t, \tau_t\}\) and approximate \(e^{-\beta \Delta t} \approx 1 - \beta \Delta t\), we get (B.48).

34 \(dU(W_t, \tau_t) = U_W(W_t, \tau_t) dW_t + \frac{1}{2} U_{WW}(W_t, \tau_t) dW_t^2 + U_r(W_t, \tau_t) d\tau_t + \frac{1}{2} U_{r\tau}(W_t, \tau_t) d\tau_t^2\).
Using $E[dN^2] = \eta dt$,
\[
E[dW_t^2] = (\sigma_e \rho_t + J (\theta_{b,t} + \theta_{c,t}))^2 \eta dt + (1 - \rho_t^2) \sigma_x^2 dt
\]
(B.53)
\[
= (J^2 (\theta_{b,t} + \theta_{c,t})^2 \eta + 2\sigma_e \rho_t J (\theta_{b,t} + \theta_{c,t}) \eta + [(\sigma_e \rho_t)^2 \eta + (1 - \rho_t^2) \sigma_x^2]) dt.
\]

Thus, substituting (B.52) and (B.53) back into (B.51), we get
\[
\mathbb{E}dU(W_t, \tau_t) = U_W(W_t, \tau_t) [(rW_t - C_t + [\mu_t - \sigma_e \rho_t \eta] + (\delta - \eta J)\theta_{b,t} + (p_c - \eta J)\theta_{c,t}) dt]
\]
\[
+ \frac{1}{2} U_{WW}(W_t, \tau_t) \left( (J \theta_{b,t} + J \theta_{c,t})^2 \eta + 2\sigma_e \rho_t \eta (J \theta_{b,t} + J \theta_{c,t}) + [(\sigma_e \rho_t)^2 \eta + (1 - \rho_t^2) \sigma_x^2] \right) dt
\]
\[
+ \sum_{k=1}^K \gamma_t(\tau_t, k) dt [U(W_t + P(\tau_t, \tau_t + dt), \tau_t + dt) - U(W_t, \tau_t)].
\]

Substituting (B.54) back into (B.50), the HJB in the steady state is given by
\[
\beta U(W_t, \tau) = \max_{C \in \mathbb{R}, \tau' \in T(\tau, k)} u(C) + U_W(W_t, \tau) [rW - C + [\mu_t - \sigma_e \rho_t \eta] + (\delta - \eta J)\theta_{b} + (p_c - \eta J)\theta_{c}]
\]
\[
+ \frac{1}{2} U_{WW}(W_t, \tau) \left( (J \theta_{b,t} + J \theta_{c,t})^2 \eta + 2\sigma_e \rho_t \eta (J \theta_{b,t} + J \theta_{c,t}) + [(\sigma_e \rho_t)^2 \eta + (1 - \rho_t^2) \sigma_x^2] \right)
\]
\[
+ \sum_{k=1}^K \gamma(\tau, k) [U(W + P(\tau, \tau'), \tau') - U(W, \tau)].
\]

Using the guessed functional form (8) and the FOC of (B.55) with respect to $C$, the optimal consumption rate for agent $\tau$ is: \( ^{35} \) \[ C_{\tau} = -\frac{\log(r)}{\alpha} + r (W + V_t + \bar{a}). \] (B.56)

Inserting (B.56) back into (B.55) and using (8), $U_W = r e^{-\rho(W+V_t+a)}$, and $U_{WW} = -\tau^2 \alpha^2 e^{-\rho(W+V_t+a)}$, we get:
\[
- \beta e^{-\rho(W_t+V_t+a)} = -e^{\log(r) - \rho(W_t+V_t+a)} +
\]
\[
r e^{-\rho(W_t+V_t+a)} \left[ \frac{\log(r)}{\alpha} - r (V_t + \bar{a}) + [\mu_t - \sigma_e \rho_t \eta] + (\delta - \eta J)\theta_{b} + (p_c - \eta J)\theta_{c} \right]
\]
\[
- \frac{1}{2} \tau^2 \alpha^2 e^{-\rho(W_t+V_t+a)} \left[ (J \theta_{b,t} + J \theta_{c,t})^2 \eta + 2\sigma_e \rho_t \eta (J \theta_{b,t} + J \theta_{c,t}) + (\sigma_e \rho_t)^2 \eta + (1 - \rho_t^2) \sigma_x^2 \right]
\]
\[
+ \sum_{k=1}^K \gamma(\tau, k) \max_{\tau' \in T(\tau, k)} [U(W + P(\tau, \tau'), \tau') - U(W, \tau)].
\]

\( ^{35} \)The FOC with respect to $C$ is: $0 = \alpha e^{-\rho C} - U_W(W_t, \tau_t)$. Using $U_W = r e^{-\rho(W+V_t+a)}$, $r e^{-\rho(W+V_t+a)} = e^{-\alpha C}$. Rewrite it as: $e^{\log(r)} e^{-\rho(W+V_t+a)} = e^{-\alpha C}$.
Divide both sides of (B.57) by \(-\frac{1}{r\alpha}e^{-ra(W+V_{\gamma})}\) and rearrange to get:

\[
0 = rV_{\tau} - e^{ra(W+V_{\gamma})} \frac{1}{r\alpha} \sum_{k=1}^{K} \gamma(\tau, k) \max_{\tau' \in T(\tau, k)} [U(W + P(\tau, \tau'), \tau') - U(W, \tau)] + r\bar{a}
\]

\[
- \frac{1}{r} \left[ \frac{\log (r)}{\alpha} - \frac{r - \beta}{r\alpha} - \frac{1}{2} r\alpha \sigma^2_e + [\mu_{\tau} - \sigma_e \rho_{\tau} \eta] - \frac{1}{2} r\alpha \left[ (\sigma_e \rho_{\tau})^2 \eta - \sigma^2_e \rho^2_{\tau} \right] \right]
\]

\[
- \left[ (\delta - \eta J) \theta_b - \frac{1}{r\alpha} ((J\theta_b + J\theta_c)^2 \eta + 2\sigma_e \rho_{\tau} \eta (J\theta_b + J\theta_c)) + (p_e - \eta J) \theta_c \right].
\]

Define

\[
\bar{a} \equiv \frac{1}{r} \left( \frac{\log (r)}{\alpha} - \frac{r - \beta}{r\alpha} - \frac{1}{2} r\alpha \sigma^2_e \right).
\]

Using \(\bar{a}\) defined in (B.58), \(b_{\tau} \equiv \mu_{\tau} - \sigma_e \rho_{\tau} \eta - \frac{1}{2} r\alpha \left[ (\sigma_e \rho_{\tau})^2 \eta - \sigma^2_e \rho^2_{\tau} \right]\), and \(\theta_b \theta_c = 0\), we get:

\[
rV_{\tau} = b_{\tau} + [(\delta - \eta J) - r\alpha \sigma_e \rho_{\tau} \eta J] \theta_b - \frac{1}{2} r\alpha (J^2 \eta (\theta_b)^2 + J^2 \eta (\theta_c)^2)
\]

\[
+ [p_e - (\eta J + r\alpha \sigma_e \rho_{\tau} \eta J)] \theta_c + e^{ra(W+V_{\gamma})} \frac{1}{r\alpha} \sum_{k=1}^{K} \gamma(\tau, k) \max_{\tau' \in T(\tau, k)} [U(W + P(\tau, \tau'), \tau') - U(W, \tau)].
\]

I assume that \(\mu_{\tau} - \sigma_e \rho_{\tau} \eta - \frac{1}{2} r\alpha \left[ (\sigma_e \rho_{\tau})^2 \eta - \sigma^2_e \rho^2_{\tau} \right] = 0\) so that \(b_{\tau} = 0\). Using \(x_{\tau}\) and \(y\) defined in (10) and (11) and the guessed functional form for \(U(W, \tau)\)

\[
rV_{\tau} = ((\delta - \eta J) - x_{\tau}) \theta_b - y \theta_b + [p_e - (\eta J + x_{\tau})] \theta_c - y \theta_c.
\]

(B.59)

\[
+ \frac{1}{r\alpha} \sum_{k=1}^{K} \gamma(\tau, k) \max_{\tau' \in T(\tau, k)} \left[ 1 - e^{-ra(P(\tau, \tau') + V_{\gamma} - V_{\gamma})} \right].
\]

Proof of Proposition 2. I prove in three steps. In step 1, I show that taking the entry rates as given, population masses, value functions, gains from trade, and prices are uniquely determined and positive. Step 2 characterizes the solution for the entry rates. In step 3, I show the conjectured optimal trading strategies are indeed optimal.

Step 1

Proof. From the market clearing conditions, the masses of inactive agents, who have reached their optimal asset position, are:

\[
\mu_{h[1,0]} = S - \mu_{a[1,0]}
\]

(B.60)

\[
\mu_{[0,-1]} = \mu_{h[0,1]}
\]

(B.61)

\[
\mu_{h[0,1]} = \frac{1}{\gamma_d + \gamma_a} M_e.
\]

(B.62)
They depend only on the masses of active searchers.

I simplify the equilibrium conditions, first, into a set of seven equations of seven unknowns, \( \mu_{a[1,0]}, \mu_{l[0,0]}, \mu_{h[0,0]}, \omega_b, \omega_c, V_l[0,0], \nu_l \):

\[
\lambda_b \mu_{a[1,0]} \mu_{h[0,0]} = \gamma_d (S - \mu_{a[1,0]}) \tag{B.63}
\]

\[
(r + \gamma_d) \omega_b = x_h - \lambda_b \left[ \mu_{a[1,0]} + \mu_{h[0,0]} \right] \frac{1}{2} \omega_b - \lambda_c \mu_{l[0,0]} \frac{1}{2} \omega_c \tag{B.64}
\]

\[
(r + \gamma_d) O_h = \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b + \lambda_c \mu_{l[0,0]} \frac{1}{2} \omega_c \tag{B.65}
\]

\[
(r + \gamma_d + \gamma_u) \omega_c = (x_h + x_l - 2y) - \lambda_b \mu_{a[1,0]} \frac{1}{2} \omega_b - \lambda_c \left( \mu_{l[0,0]} + \mu_{h[0,0]} \right) \frac{1}{2} \omega_c \tag{B.66}
\]

\[
(r + \gamma_u) V_l[0,0] = \lambda_c \mu_{h[0,0]} \frac{1}{2} \omega_c \tag{B.67}
\]

\[
\frac{\nu_l F_l}{\gamma_u} = \mu_{l[0,0]} + \frac{1}{\gamma_d + \gamma_u} M_c \tag{B.68}
\]

\[
\nu_l = \begin{cases} 
1 & V_l[0,0] > O_l \\
[0, 1] & V_l[0,0] = O_l \\
0 & V_l[0,0] < O_l.
\end{cases} \tag{B.69}
\]

The first equation comes from combining (13) and (A.38). The second equation (B.64) comes from combining the value functions for \( h[0,0], h[1,0], \) and \( a[1,0], \) substituting in \( M_b \) and \( M_c, \) and simplifying. Substituting \( M_b \) into (A.29), (A.29) gives the third equation, (B.65). Combining the value functions for \( h[0,0], h[1,0], l[0,0], \) and \( l[0,-1] \) and substituting in \( M_b \) and \( M_c \) yields the fourth equation, (B.66). Substituting \( M_b \) into (A.28), (A.28) gives the fifth equation: (B.67). The sixth equation (B.68) comes from (A.37), and the seventh is the entry condition for low types, given in (4).

Next, I simplify the seven equations further into a set of three equations of three unknowns. Combining (B.64) and (B.65),

\[
\omega_b = \frac{x_h - (r + \gamma_d) O_h}{r + \gamma_d + \lambda_b \mu_{h[0,0]} \frac{1}{2}}. \tag{B.70}
\]

Combining (B.65) and (B.66),

\[
\omega_c = \frac{x_h + x_l - 2y - (r + \gamma_d) O_h}{r + \gamma_d + \gamma_u + \lambda_c \mu_{h[0,0]} \frac{1}{2}}. \tag{B.71}
\]

From (B.63), solve for \( \mu_{a[1,0]} \):

\[
\mu_{a[1,0]} = \frac{\gamma_d S}{(\lambda_b \mu_{h[0,0]} + \gamma_d)}. \tag{B.72}
\]
From (B.68), solve for $\mu_{l[0,0]}$:

$$
\mu_{l[0,0]} = \frac{\nu_t F_t}{\frac{1}{\gamma_d + \gamma_u} \lambda_c \mu_{h[0,0]}}.
$$

(B.73)

Plugging these expressions back into (B.64)-(B.69) gives three equations of three unknowns $\{\mu_{h[0,0]}, V_{l[0,0]}, \nu_t\}$:

$$(r + \gamma_d)O_h - \lambda_h \frac{\gamma_d S}{(\gamma_d + \lambda_b \mu_{h[0,0]})} \frac{1}{2} x_h - (r + \gamma_d)O_h
\quad - \lambda_c \frac{\gamma_d + \gamma_u}{\gamma_d + \gamma_u + \lambda_c \mu_{h[0,0]}} \frac{1}{2} \left( x_h + x_l - 2y - (r + \gamma_d)O_h \right)
= 0.
$$

(B.74)

$$
V_{l[0,0]} = \frac{1}{r + \gamma_u} \lambda_c \mu_{h[0,0]} \frac{1}{2} \left( x_h + x_l - 2y - (r + \gamma_d)O_h \right).
$$

(B.75)

$$
\nu_t = \begin{cases} 
1 & V_{l[0,0]} > O_l \\
[0,1] & V_{l[0,0]} = O_l \\
0 & V_{l[0,0]} < O_l.
\end{cases}
$$

(B.76)

Consider the solution for $\mu_{h[0,0]}$. Applying the Implicit Function Theorem to (B.74), $\mu_{h[0,0]}$ strictly increases in $\nu_t$. The right-hand-side of (B.75) increases in $\mu_{h[0,0]}$ and, hence, in $\nu_t$. Thus, $V_{l[0,0]}$ also strictly increases in $\nu_t$. The latter ensures that, as elaborated further in Step 2, that an interior solution for $\nu_t$ exists. Solving (B.74)-(B.76) for the case where $\nu_t$ is an interior solution (i.e. $V_{l[0,0]} = O_l$), $\mu_{h[0,0]}$ is uniquely given by:

$$
\mu_{h[0,0]} = \frac{(r + \gamma_d + \gamma_a)(r + \gamma_u)O_l}{\frac{1}{2} \lambda_c (x_h + x_l - 2y - (r + \gamma_d)O_h - (r + \gamma_u)O_l)}.
$$

(B.77)

This is positive by Assumption 1. Moreover, since $\mu_{h[0,0]}$ strictly increases in $\nu_t$, for a corner solution $\nu_t = 1$, $\mu_{h[0,0]}$ is also positive and unique. Hence, for any positive solution for $\nu_t$, (B.74) has a unique positive solution in $\mu_{h[0,0]}$.

Equations (B.60)-(B.62), (B.72), (B.73), and (B.74) uniquely determine the masses of agent types. In turn, $\omega_b$ and $\omega_c$ are uniquely given by (B.70) and (B.71). Given the agent masses, (A.28)-(A.33) and (15)-(16) uniquely determine the value functions and prices.

The solution, moreover, is positive. Using Assumption 1 and the fact that $\mu_{h[0,0]} > 0$, $\omega_b > 0$. Using Assumption 1 and $\mu_{h[0,0]} > 0$, the numerator in (B.71) is positive; thus, $\omega_c > 0$. In turn, $\mu_{h[0,0]} > 0$ implies that $\mu_{a[1,0]} > 0$ and $\mu_{l[0,0]} > 0$. Nonsearcher masses are also positive.  

\hfill \Box

**Step 2**

*Proof.* Consider the entry rates.
Using the fact that \( V_{[0,0]} \) increases in \( \nu_t \), Table 1 characterizes solutions for \( \nu_t \) depending on the parameter conditions. For example, if the parameter values are such that

\[
V_{[0,0]}(\nu_t = 0) < O_t < V_{[0,0]}(\nu_t = 1),
\]

where \( V_{[0,0]} \) and \( \mu_{h[0,0]} \) are a solution to (B.74) and (B.75), then a unique interior

\[
\nu_t = \frac{(r + \gamma_d)O_h - \lambda_b \frac{\gamma_d S}{\lambda_b \mu_{h[0,0]} + \gamma_d} \frac{1}{2} x_h - (r + \gamma_d)O_h}{\lambda_c \frac{(\gamma_d + \gamma_u) \frac{V_{[0,0]}}{\nu_t}}{2} x_h + x_l - 2y - (r + \gamma_d)O_h} \frac{1}{2} r + \gamma_d + \gamma_u + \lambda_c \mu_{h[0,0]} \frac{\nu_t}{2}.
\]

and two corners solutions (\( \nu_t = 0 \) and \( \nu_t = 1 \)) exist. Thus, even if it is feasible to trade CDS, in the case of a corner solution \( \nu_t \neq 0 \), investors may not trade CDS in equilibrium. Since the paper is about the effect of CDS, I contrast the equilibria with CDS (i.e. \( \nu_t > 0 \) be it an interior or a corner solution, \( \nu_t = 1 \)) to the environment without CDS. The last parameter condition in Table 1 says that if low type investors’ outside option is too good, then in equilibrium none of them enters. I rule out such condition to ensure that the CDS market can exist in equilibrium.

<table>
<thead>
<tr>
<th>Parameter condition</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_t &lt; V_{[0,0]}(\nu_t = 0) )</td>
<td>( \nu_t = 1 )</td>
</tr>
<tr>
<td>( O_t = V_{[0,0]}(\nu_t = 0) )</td>
<td>( \nu_t = 0, \nu_t = 1 )</td>
</tr>
<tr>
<td>( V_{[0,0]}(\nu_t = 0) &lt; O_t &lt; V_{[0,0]}(\nu_t = 1) )</td>
<td>( \nu_t = 0, \nu_t \in (0,1), \nu_t = 1 )</td>
</tr>
<tr>
<td>( O_t = V_{[0,0]}(\nu_t = 1) )</td>
<td>( \nu_t = 0, \nu_t = 1 )</td>
</tr>
<tr>
<td>( V_{[0,0]}(\nu_t = 1) &lt; O_t )</td>
<td>( \nu_t = 0 )</td>
</tr>
</tbody>
</table>

Consider the entry rate of high-valuation investors, \( \nu_h \). \( V_{h[0,0]} \) is a solution to:

\[
(r + \gamma_d)V_{h[0,0]} - \lambda_b \frac{\gamma_d S}{\lambda_b \mu_{h[0,0]} + \gamma_d} \frac{1}{2} x_h - (r + \gamma_d)V_{h[0,0]} \frac{1}{2} r + \gamma_d + \lambda_b \mu_{h[0,0]} \frac{\nu_h}{2} \]

\[
- \lambda_c \frac{(\gamma_d + \gamma_u) \frac{V_{[0,0]}}{\nu_h}}{2} x_h + x_l - 2y - (r + \gamma_d)V_{h[0,0]} = 0.
\]

For a given level of \( \nu_t \), applying the Implicit Function Theorem to (B.79), we have that \( \frac{\partial V_{h[0,0]}}{\partial \mu_{h[0,0]}} < 0 \). Moreover, using (A.36)

\[
\nu_h F_h = \gamma_d \mu_{h[0,0]} + \gamma_d \frac{\lambda_b \mu_{h[0,0]} S}{\lambda_b \mu_{h[0,0]} + \gamma_d} + \gamma_d \lambda_c \mu_{h[0,0]} \frac{\nu_t F_t}{\gamma_u + \gamma_d + \lambda_c \mu_{h[0,0]}}.
\]

From (B.80), for a given level of \( \nu_t \), \( \mu_{h[0,0]} \) increases in \( \nu_h \); thus, \( \frac{\partial V_{h[0,0]}}{\partial \nu_h} < 0 \). As a result, \( \nu_h \) is uniquely determined for a given level of \( \nu_t \). For simplicity, I focus on the parameter values such that \( \nu_h \), characterized by (B.80), is an interior solution when evaluated at
\( \nu_l = 1: \)

\[
0 < \frac{\gamma_d \mu_h[0,0]S \lambda_c \mu_h[0,0]}{\lambda C \mu_h[0,0] + \gamma_d} \frac{\Gamma_l}{\Gamma_h} \frac{\Gamma_l}{\Gamma_h} < 1 \quad \text{(B.81)}
\]

where is \( \mu_h[0,0] \) is given by (B.74) evaluated at \( \nu_l = 1 \). If \( \nu_h \) is an interior solution at \( \nu_l = 1 \), then it will also be an interior solution for an interior solution of \( \nu_l \). An easy way to satisfy (B.81) is if \( F_h \) is sufficiently large.

**Step 3**

*Proof.* What remains is to verify that the conjectured trading strategies are in fact optimal. The gains from a bond transaction is positive, \( \omega_b > 0 \). The bond buyer and seller’s respective trading strategies, as a result, are optimal. This also shows that an average-valuation investor prefers to not hold a bond. Analogously, since the total gains from a CDS transaction is positive: \( \omega_c > 0 \), it is optimal for \( l[0,0] \) type agent to buy CDS, and for \( h[0,0] \) type agent to sell CDS as conjectured.

Consider an average-valuation investor who was previously a high type and had sold CDS. Upon reverting to an average-type, if she pays the termination fee and terminates the CDS contract, her utility changes by \(-T_s + 0 - V_h[0,1]\). Whereas, if she remains a CDS seller, the change is \( V_a[0,1] - V_h[0,1] \), where

\[
rV_a[0,1] = p_c - \eta J - y + \gamma_u(T_n + 0 - V_a[0,1]).
\]

Thus, she prefers to terminate if \( V_a[0,1] < -T_s \), where \( T_s = \frac{1}{2}\omega_c \). Consider the difference:

\[
V_a[0,1] + T_s = V_a[0,1] + \frac{1}{2}\omega_c
= \frac{p_c - \eta J - y + \gamma_u T_n}{r + \gamma_u} + \frac{1}{2}\omega_c
= \frac{(r + \gamma_u + \gamma_d)(x_l - 2y) - (x_h - (r + \gamma_d)O_h)\lambda_c \mu_h[0,0]}{(r + \gamma_u)(r + \gamma_d + \gamma_u + \lambda_c \mu_h[0,0])} \frac{1}{2}. \quad \text{(B.82)}
\]

where the third equality uses (20) and (B.71). For an interior solution of \( \nu_l \) (i.e. using \( \mu_h[0,0] \) given by (B.77)), the numerator simplifies to:

\[
[r + \gamma_d + \gamma_u][2x - 2y - (r + \gamma_d)O_h][x - 2y - (r + \gamma_u)O_l]
2x - 2y - (r + \gamma_d)O_h - (r + \gamma_u)O_l
\]

By Assumption 1, the second squared bracket is positive, the third bracket is negative, and the denominator is positive. The whole term then is negative. For a corner solution \( \nu_l = 1 \), since \( \mu_h[0,0] \) increases in \( \nu_l \) and the numerator of (B.82) decreases in \( \mu_h[0,0] \), the numerator will remain negative. Thus, \( V_a[0,1] < -T_s \): once a CDS seller switches to an average type, she prefers to pay the fee and exit the market than to remain a CDS seller and wait until her counterparty terminates the contract. This shows that an average type
prefers no position than a long position through the CDS market.

Consider an average-valuation investor who was previously a low type and had bought CDS. Upon switching to an average type, if a CDS buyer pays the termination fee and exits, her utility changes by: \(-T_b + 0 - V_{[0,-1]}\). If she remains a CDS buyer, her utility changes by: \(V_{a[0,-1]} - V_{[0,-1]}\), where

\[
rV_{a[0,-1]} = -p_c + \eta J - y + \gamma_d(T_a + 0 - V_{a[0,-1]}).
\]

She prefers to pay the fee if

\[
V_{a[0,-1]} < T_b,
\]

where \(T_b = \frac{1}{2} \omega_c\). Consider the difference:

\[
V_{a[0,-1]} + T_b = \frac{x_h - 2y - (r + \gamma_d)O_h}{r + \gamma_d}.
\]

The left-hand-side is negative by Assumption 1. Thus, a CDS buyer, upon a valuation shock, prefers to pay the fee and exit the market than to remain a CDS buyer (until the counterparty exits). That is, an average type prefers no position than a short position through the CDS market.

**Proof of Lemma 1.** Consider the market choice problem. I denote with \(V_{m[0,0]}\) a long investor’s expected utility associated with market choice \(m \in \{b, c, bc\}\) and with \(\nu_h^m\) the fraction of long investors that choose \(m\), where \(b, c, \text{and } bc\) stand for entering just the bond market, just the CDS market, and both markets, respectively.\(^{36}\) The equilibrium entry rates \(\{\nu_h^m\}_{m \in \{b, c, bc\}}\) solve:

\[
\nu_h^m = \begin{cases} 
1 & V_{h[0,0]}^m > \left\{ V_{h[0,0]}^b, V_{h[0,0]}^c, V_{h[0,0]}^{bc}, O_h \right\} / V_{h[0,0]}^m \\
0 & V_{h[0,0]}^m < \left\{ V_{h[0,0]}^b, V_{h[0,0]}^c, V_{h[0,0]}^{bc}, O_h \right\} / V_{h[0,0]}^m 
\end{cases}
\] (B.83)

Where the distinction is necessary, I denote in superscripts the investor’s market choice: \(m \in \{b, c, bc\}\). Similarly, the superscripts on prices and fees denote the market choice of the involved parties. Value functions are:

\(^{36}\)The ability to search simultaneously affects long investors only because other investors (bond sellers and CDS buyers) can establish positions only in one of the markets.

35
\begin{align*}
    rV_{h[1,0]} &= \delta (1 - \eta) + x - y + \gamma_d \left( V_{a[1,0]} - V_{h[1,0]} \right) \\
    rV_{a[1,0]} &= \delta (1 - \eta) - y + \lambda_b \mu_{b[0,0]} \frac{1}{2} \left( V_{h[1,0]} - V_{a[1,0]} - V_{h[0,0]}^{bc} \right) \\
    &\quad + \lambda_b \mu_{h[0,0]}^{bc} \frac{1}{2} \left( V_{h[1,0]} - V_{a[1,0]} - V_{h[0,0]}^{bc} \right) \\
    rV_{h[0,0]}^{bc} &= \gamma_d \left( 0 - V_{h[0,0]}^{bc} \right) + \lambda_b \mu_{a[1,0]} \frac{1}{2} \left( V_{h[1,0]} - V_{a[1,0]} - V_{h[0,0]}^{bc} \right) \\
    rV_{h[0,0]}^{bc} &= rV_{h[0,0]}^{bc} - \gamma_d V_{h[0,0]}^{bc} + \lambda_b \mu_{a[1,0]} \frac{1}{2} \left( V_{h[1,0]} - V_{a[1,0]} - V_{h[0,0]}^{bc} \right) + \lambda_c \mu_{[0,0]} \left( V_{h[0,1]}^{bc} - V_{h[0,0]}^{bc} \right) \\
    rV_{l[0,1]}^{bc} &= p_c^{bc} - \delta y + x - y + \gamma_d \left( -T_{l[0,1]}^{bc} - V_{h[0,1]}^{bc} \right) \\
    rV_{l[0,1]}^{bc} &= -p_c^{bc} + \delta y + x - y + \gamma_u \left( -T_{l[0,1]}^{bc} - V_{l[0,1]}^{bc} \right) \\
    rV_{l[0,0]}^{bc} &= \gamma_u \left( 0 - V_{l[0,0]}^{bc} \right) + \lambda_c \mu_{l[0,0]}^{bc} \left( V_{l[0,1]}^{bc} - V_{l[0,0]}^{bc} \right) + \lambda_c \mu_{l[0,0]}^{bc} \left( V_{l[0,1]}^{bc} - V_{l[0,0]}^{bc} \right) \\
    rV_{l[0,0]}^{bc} &= \gamma_d \left( 0 - V_{l[0,0]}^{bc} \right) + \lambda_c \mu_{l[0,0]}^{bc} \left( V_{l[0,1]}^{bc} - V_{l[0,0]}^{bc} \right) \\
    rV_{l[0,1]}^{bc} &= p_c^{bc} - \delta y + x - y + \gamma_d \left( -T_{c} - V_{l[0,1]}^{bc} \right) \\
    rV_{l[0,1]}^{bc} &= -p_c^{bc} + \delta y + x - y + \gamma_u \left( -T_{c} - V_{l[0,1]}^{bc} \right)
\end{align*}

A CDS buyer and a long investor negotiate a CDS price characterized by

\begin{equation}
    V_{h[0,1]}^{bc} - V_{h[0,0]}^{bc} = \frac{1}{2} \left( V_{h[0,1]}^{bc} + V_{l[0,1]}^{bc} - V_{h[0,0]}^{bc} - V_{l[0,0]} \right)
\end{equation}

when the long investor searches in both markets and by

\begin{equation}
    V_{h[0,0]}^{bc} - V_{l[0,0]}^{bc} = \frac{1}{2} \left( V_{h[0,0]}^{bc} + V_{l[0,1]}^{bc} - V_{h[0,0]}^{bc} - V_{l[0,0]} \right)
\end{equation}

when the long investor searches in just the CDS market. Bond prices are given by

\begin{equation}
    p_b^{bc} = \frac{1}{2} \left( V_{h[1,0]}^{bc} - V_{h[0,0]}^{bc} \right) + \frac{1}{2} V_{a[1,0]}
\end{equation}

when the buyer searches in just the bond market and by

\begin{equation}
    p_b^{bc} = \frac{1}{2} \left( V_{h[1,0]}^{bc} - V_{h[0,0]}^{bc} \right) + \frac{1}{2} V_{a[1,0]}
\end{equation}

when the buyer searches in both markets simultaneously. Termination fees are given by

\begin{equation}
    T^{bc} = \frac{1}{2} \left( V_{h[0,1]}^{bc} + V_{l[0,1]}^{bc} - V_{h[0,0]}^{bc} - V_{l[0,0]} \right)
\end{equation}

\begin{equation}
    T^{bc} = \frac{1}{2} \left( V_{h[0,0]}^{bc} + V_{l[0,1]}^{bc} - V_{h[0,0]}^{bc} - V_{l[0,0]} \right)
\end{equation}

We can write the value function of a long investor searching in both markets as
\[ V_{h[0,0]}^{bc} = \frac{(r + \gamma_d)}{r + \gamma_d + \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} 0 + \frac{\lambda_b \mu_{a[1,0]}}{r + \gamma_d + \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} (V_{h[1,0]} - P_{bc}) \]

\[ + \frac{\lambda_c \mu_{l[0,0]}}{r + \gamma_d + \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} V_{h[0,1]}^{bc} \]

Thus, it is a weighted average between three different outcomes: (1) getting a valuation shock and exiting in which case her utility is zero, (2) finding a counterparty in the bond market in which case her utility is the utility of a bond owner minus the cost of becoming a bond owner, and (3) finding a counterparty in the CDS market in which case her utility is that of a CDS seller. The weights capture the probabilities of these outcomes.

To compare \( V_{h[0,0]}^{bc} \) with \( V_{h[0,0]}^{b} \), we can express \( V_{h[0,0]}^{bc} \) as:

\[ V_{h[0,0]}^{bc} = \frac{(r + \gamma_d)}{r + \gamma_d + \frac{1}{2} \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} 0 + \frac{\frac{1}{2} \lambda_b \mu_{a[1,0]}}{r + \gamma_d + \frac{1}{2} \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} (V_{h[1,0]} - V_{a[1,0]}) \]

\[ + \frac{\lambda_c \mu_{l[0,0]}}{r + \gamma_d + \frac{1}{2} \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} V_{h[0,1]}^{bc} \]

The value function of a long \((h[0,0])\) investor searching in only the bond market can be expressed as:

\[ V_{h[0,0]}^{b} = \frac{(r + \gamma_d)}{r + \gamma_d + \frac{1}{2} \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} 0 + \frac{\frac{1}{2} \lambda_b \mu_{a[1,0]}}{r + \gamma_d + \frac{1}{2} \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} (V_{h[1,0]} - V_{a[1,0]}) \]

Combining the two, we get:

\[ V_{h[0,0]}^{bc} = \frac{r + \gamma_d + \frac{1}{2} \lambda_b \mu_{a[1,0]}}{r + \gamma_d + \frac{1}{2} \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} V_{h[0,0]}^{b} + \frac{\lambda_c \mu_{l[0,0]}}{r + \gamma_d + \frac{1}{2} \lambda_b \mu_{a[1,0]} + \lambda_c \mu_{l[0,0]}} V_{h[0,1]}^{bc} \] (B.95)

Thus, \( V_{h[0,0]}^{bc} \) can be expressed as a weighted average between the utility of an investor who searches in just the bond market and the utility of a CDS seller. The weight on \( V_{h[0,0]}^{b} \) reflects both the probability of getting a valuation shock and the probability finding a bond seller. Thus, the utility of a long investor searching in both markets imbeds the utility of the counterfactual market choice of searching in just the bond market, \( V_{h[0,0]}^{b} \). This is because the outcomes that affect the investor searching in just the bond market—getting a valuation shock and finding a bond seller—also affect the investor searching simultaneously. Then, from (B.95) a long investor prefers to search in both markets than just the bond market (i.e. \( V_{h[0,0]}^{bc} > V_{h[0,0]}^{b} \)) if \( V_{h[0,0]}^{bc} > V_{h[0,0]}^{b} \) or, equivalently, if \( V_{h[0,1]}^{bc} > V_{h[0,1]}^{bc} \). In turn, \( V_{h[0,1]}^{bc} > V_{h[0,0]}^{bc} \) as long as there is gains from trade from selling CDS. The argument is analogous for an investor who searches in just the CDS market: As long as, there is gains from a bond transaction, she will prefer to search in both markets, not just the CDS market. Then, in an equilibrium where some investors search simultaneously \( (\nu^{bc} > 0) \), it has to be that \( \nu^{b} = 0 \) and \( \nu^{c} = 0 \) (that is, it cannot be the case that some choose to search in just one of the markets). Then, the possible
equilibrium entry rates are \{\nu^{bc} > 0, \nu^b = 0, \nu^c = 0\} and \{\nu^{bc} = 0, \nu^b \geq 0, \nu^c \geq 0\}.

Next, I show that \{\nu^{bc} > 0, \nu^b = 0, \nu^c = 0\} is the only equilibrium and that \{\nu^{bc} = 0, 
\nu^b \geq 0, \nu^c \geq 0\} cannot be an equilibrium. Showing this is equivalent to showing that gains from a bond and a CDS transaction are positive for those searching simultaneously. I show by contraction. Suppose \{\nu^{bc} = 0, \nu^b > 0, \nu^c > 0\} is the set of equilibrium entry rates: No investors choose to search simultaneously. It is straightforward that the mass of investors searching simultaneously is zero: \(\mu_{h[0,0]}^{bc} = 0\). Combining (B.84), (B.85), (B.87), the entry condition \(V_{h[0,0]}^b = O_h\), and \(M_b = \gamma_d(S - \mu_{a[1,0]}\)) we get:

\[
\omega_b^h = \frac{O_h r_d (-2r_d + \gamma_d) + \sqrt{O_h r_d (O_h r_d (-2r_d + \gamma_d)^2 + 4s (x - O_h r_d \gamma_d \lambda_b)}}{s \gamma_d \lambda_b} \tag{B.96}
\]

\[
\mu_{a[1,0]}^b = \frac{O_h r_d (2r_d - \gamma_d) + \sqrt{O_h r_d (O_h r_d (-2r_d + \gamma_d)^2 + 4s (x - O_h r_d \gamma_d \lambda_b)}}{2 (x - O_h r_d \lambda_b} \tag{B.97}
\]

\[
\mu_{h[0,0]}^b = \frac{-O_h r_d (2r_d + \gamma_d) + \sqrt{O_h r_d (O_h r_d (-2r_d + \gamma_d)^2 + 4s (x - O_h r_d \gamma_d \lambda_b)}}{2O_h r_d \lambda_b} \tag{B.98}
\]

Combining (B.93), (B.92), (B.94), (B.91), termination fees, and the entry condition \(V_{h[0,0]}^c = O_h\), we get:

\[
\omega_c^e = \frac{2r - 2y - O_h r_d - O_t r_u}{r + \gamma_d + \gamma_u} \tag{B.99}
\]

\[
\mu_{h[0,0]}^c = \frac{2O_t r_u (r_d + \gamma_u)}{\lambda_c (2x - 2y - O_h r_d - O_t r_u)} \tag{B.100}
\]

\[
\mu_{[0,0]}^c = \frac{2O_h r_d (r_d + \gamma_u)}{\lambda_c (2x - 2y - O_h r_d - O_t r_u)} \tag{B.101}
\]

The investor who searches in just the CDS market has an incentive to also search in the bond market if, for her, the gains from a bond transaction, \(\omega_b^{bc}\), is positive. Combining (B.84), (B.85), (B.88), and setting \(\mu_{h[0,0]}^{bc} = 0\), \(\omega_b^{bc}\) is

\[
\omega_b^{bc} = \frac{4x (r_d + \gamma_u) - 2(r_d + \gamma_u) \lambda_b \mu_{h[0,0]}^{bc} + \lambda_c \mu_{a[1,0]}^b + \lambda_c \mu_{h[0,0]}^c \omega_c^e}{2 (r + \gamma_d + \gamma_u) (2r_d + \lambda_b \mu_{a[1,0]}^b) + 2r_d \lambda_c \mu_{h[0,0]}^c} \tag{B.102}
\]

The sign depends on the sign of the numerator. Substituting in (B.96), (B.99), (B.98), (B.100), and (B.101), the numerator is:

\[
\frac{4 (2x - 2y - O_t r_u) (r + \gamma_d) (r + \gamma_d + \gamma_u) \left[O_h (r + \gamma_d) (2r + \gamma_d) + \sqrt{A}\right]}{s \gamma_d (2x - 2y - O_h r_d - O_t r_u) \lambda_b} \tag{B.102}
\]

where \(A \equiv O_h r_d (4s x \gamma_d \lambda_b + O_h r_d ((2r + \gamma_d)^2 - 4s \gamma_d \lambda_b))\). The term in square brackets

\[37\]In the proof of equilibrium existence, I showed that these gains are positive, but I did so by assuming \(\nu^b_h\) and \(\nu^c_h\) are zero. For simplicity, let us focus on the interior solutions for \(\nu^b\) and \(\nu^c\) so that \(V_{h[0,0]}^b = O_h\) and \(V_{[0,0]} = O_t\).
is positive because \( x - r_dO_h > 0 \) by Assumption 1, which, in turn, implies that \( A > (O_h r_d (2r + \gamma_d))^2 \). The denominator in (B.102) is positive by Assumption 1. This also implies that \( 2x - 2y - O_1 r_u > 0 \). Thus, \( \omega^b_{bc} > 0 \).

Analogously, the investor who searches in just the bond market has an incentive to also search in the CDS market. To see this, combine (B.89), (B.88), (B.90), and (B.91), and get:

\[
\omega^b_{bc} = \frac{8r_d(x - y) + (2x - 4y)\lambda_b \mu_{a[1,0]} + \lambda_b^2 \mu_{a[1,0]}^2 \mu^b_{bc} + \lambda_c (2r_d + \lambda_b \mu_{a[1,0]}) \mu_{bc}^b - \lambda_c (2r_d + \lambda_b \mu_{a[1,0]}) \mu_{bc}^b - \lambda_c (2r_d + \lambda_b \mu_{a[1,0]}) \mu_{bc}^b}{2 (r + \gamma_d + \gamma_u) (2 (r + \gamma_d) + \lambda_b \mu_{a[1,0]}) + 2 (r + \gamma_d) \lambda_c \mu_{[0,0]}}
\]

Substituting in (B.96), (B.99), (B.98), (B.100), and (B.97), the sign of \( \omega^b_{bc} \) depends on:

\[
\frac{(2x - 2y - r_d O_h - r_u O_l) \left( r_d (2 (x - O_h r_d) + x + (x - O_h \gamma_d)) + \sqrt{A} \right)}{x - O_h r_d}
\]

By analogous arguments, \( \omega^b_{bc} > 0 \). Thus, when \( \nu^b = 0 \), an investor searching in just one market wants search simultaneously in the other market, which contradicts that \( \{ \nu^b = 0, \nu^b \geq 0, \nu^c \geq 0 \} \) is the equilibrium entry rates. Thus, \( \{ \nu^b = 0, \nu^b \geq 0, \nu^c \geq 0 \} \) cannot be an equilibrium. Instead, in equilibrium, all long investors who enter search simultaneously in both markets: \( \{ \nu^b > 0, \nu^b = 0, \nu^c = 0 \} \).

**Proof of Proposition 7.** Consider, first, \( \lambda_b \to \infty \).

I consider three possible limits of \( \nu_l \) as \( \lambda_b \to \infty \), and, under each, I analyze what \( \mu_{h[0,0]} \) limits to. First, suppose \( \nu_l \) limits to a corner solution: \( \lim_{\lambda_b \to \infty} \nu_l = 0 \), then (B.74) evaluated at \( \nu_l = 0 \) is

\[
(r + \gamma_d) O_h - \lambda_b \frac{\gamma_d S}{(r + \gamma_d + \lambda_b \mu_{h[0,0]})} \frac{x_h - (r + \gamma_d) O_h}{2 r + \gamma_d + \lambda_b \mu_{h[0,0]}} = 0.
\]  
(B.103)

If \( \lambda_b \mu_{h[0,0]} < \infty \), the left-hand side is negative and, hence, a contradiction. It has to be that \( \infty > \lim_{\lambda_b \to \infty} \mu_{h[0,0]} > 0 \), and, hence, \( \lim_{\lambda_b \to \infty} \lambda_b \mu_{h[0,0]} = \infty \).

Next, suppose \( \nu_l \) limits to an interior position solution. Then, (B.75) and (B.76) together imply

\[
O_l = \frac{1}{r + \gamma_u} \lambda_c \mu_{h[0,0]} \frac{1}{2} \frac{x_h + x_l - 2y - (r + \gamma_d) O_h}{r + \gamma_d + \gamma_u + \lambda_c \mu_{h[0,0]}}
\]  
(B.104)

If \( \mu_{h[0,0]} \) limits to zero, (B.104) does not hold and, hence, a contradiction. It has to be that \( \infty > \lim_{\lambda_b \to \infty} \mu_{h[0,0]} > 0 \).

Suppose \( \lim_{\lambda_b \to \infty} \nu_l = 1 \), then (B.75) and (B.76) together imply

\[
\frac{1}{r + \gamma_u} \lambda_c \mu_{h[0,0]} \frac{1}{2} \frac{x_h + x_l - 2y - (r + \gamma_d) O_h}{r + \gamma_d + \gamma_u + \lambda_c \mu_{h[0,0]}} > O_l.
\]  
(B.105)

Again from (B.105), it cannot be that \( \lim_{\lambda_b \to \infty} \mu_{h[0,0]} = 0 \), but it has to be that \( \infty > \lim_{\lambda_b \to \infty} \mu_{h[0,0]} > 0 \).
Thus, regardless of what the entry rate of low types converges to, $\mu_{h[0,0]}$ has a finite limit: $\infty > \lim_{\lambda_b \to \infty} \mu_{h[0,0]} > 0$. Using (B.70), the latter implies $\lim_{\lambda_b \to \infty} \omega_b = 0$ and that introducing CDS market does not affect the bond price. From (B.72), $\mu_{a[1,0]} \to 0$ as $\lambda_b \to \infty$. Thus, bond volume, given in (B.63), approaches $\gamma_d S$ and is unaffected by the introduction of the CDS market.

Now consider $\lambda_c \to \infty$. Suppose the limit of $b$ is given by a positive solution (interior or corner): $\nu_l > 0$.

\[(r + \gamma_d)O_h - \lambda_b \frac{\gamma_d S}{(\gamma_d + \lambda_b \mu_{h[0,0]})} \frac{1}{2} x_h - (r + \gamma_d)O_h \quad \text{(B.106)}\]
\[- \lambda_c \frac{(\gamma_d + \gamma_a) \mu_{F1}}{\mu} \frac{1}{2} x_h + x_l - 2y - (r + \gamma_d)O_h \quad \text{for} \quad \frac{1}{2} r + \gamma_d + \gamma_u + \lambda_c \mu_{h[0,0]}\]

If $\mu_{h[0,0]}$ limits to zero, the left-hand side limits to $-\infty$, which is a contradiction. Thus, it has to be that $0 < \lim_{\lambda_c \to \infty} \mu_{h[0,0]} < \infty$, and, hence, $\lim_{\lambda_c \to \infty} \mu_{c[0,0]} = \infty$. Then, (B.106) at the limit is

\[(r + \gamma_d)O_h - \lambda_b \frac{\gamma_d S}{(\gamma_d + \lambda_b \mu_{h[0,0]})} \frac{1}{2} x_h - (r + \gamma_d)O_h = 0. \quad \text{(B.107)}\]

In the absence of CDS, $\mu_{h[0,0]}$ is characterized by the same equation as (B.107). Mass $\mu_{h[0,0]}$, as a result, is the same with or without CDS. In turn, the trading surplus ($\omega_b$), the bond illiquidity discount ($d_b$), the mass of sellers ($\mu_{a[1,0]}$), and bond volume are the same with or without CDS.

If the entry rate of low types limits to zero, then the environment with CDS converges to the environment without CDS. Hence, the presence of CDS again does not affect either the bond illiquidity discount or bond volume. \qed

**Proof of Proposition 8.** The value functions of investors participating in the bond market are characterized by:

\[
r V^b_{h[0,0]} = \gamma_d (0 - V^b_{h[0,0]}) + \lambda \mu_{a[1,0]} \frac{1}{2} \omega_b \quad \text{(B.108)}
\]
\[
r V^b_{h[1,0]} = (\delta - \eta J) + x_h - y + \gamma_d (V^b_{a[1,0]} - V^b_{h[1,0]}) \quad \text{(B.109)}
\]
\[
r V^b_{a[1,0]} = (\delta - \eta J) - y + \frac{M_b}{\mu_{a[1,0]}} \frac{1}{2} \omega_b. \quad \text{(B.110)}
\]

Population masses $\{\mu_{h[1,0]}, \mu^b_{h[0,0]}, \mu_{a[1,0]}\}$ are given by

\[
\nu^b_h F_h = \gamma_d \mu^b_{h[0,0]} + M_b
\]
\[
M_b = \gamma_d \mu_{h[1,0]}
\]
\[
\mu_{h[1,0]} + \mu_{a[1,0]} = S.
\]

And the entry rate into the bond market, $\nu^b_h$, solves (B.83).
The value functions of investors in the CDS market are characterized by:

\[
\begin{align*}
rV_{h[0,0]}^c &= \gamma_d(0 - V_{h[0,0]}^c) + \frac{M_c}{\mu_{h[0,0]}} \frac{1}{2} \omega_c \\
rV_{f[0,0]} &= \gamma_a(0 - V_{f[0,0]}^c) + \frac{M_c}{\mu_{f[0,0]}} \frac{1}{2} \omega_c \\
rV_{h[0,1]} &= p_c - (\eta J - x_h) - y + \gamma_d(-T_h - V_{h[0,1]}) \\
rV_{f[0,-1]} &= -p_c + (\eta J + x_i) - y + \gamma_a(-T_h - V_{f[0,-1]}) \cdot 
\end{align*}
\] (B.112) (B.113) (B.114) (B.115)

Population masses \( \{\mu_{h[0,1]}, \mu_h^c_{[0,0]}, \mu_{f[0,-1]}, \mu_{f[0,0]}\} \) of investors in the CDS market are characterized by:

\[
\begin{align*}
\nu_h^c F_h + \gamma_a \mu_{h[0,1]} &= \gamma_d \mu_h^c_{[0,0]} + M_c \\
\nu_f F_f + \gamma_d \mu_{f[0,-1]} &= \gamma_a \mu_{f[0,0]} + M_c \\
M_c &= \gamma_d \mu_{h[0,1]} + \gamma_a \mu_{h[0,1]} \\
\mu_{h[0,1]} &= \mu_{f[0,-1]}.
\end{align*}
\] (B.116)

And the entry rates into the CDS market, \( \nu_h^c \) and \( \nu_f \), solve (B.83) and (4), respectively.

Combining the equations characterizing population masses, we get

\[
\frac{\nu_h^b F_h}{\gamma_d} = \mu_h^b_{[0,0]} + \frac{\lambda_d \mu_h^b_{[0,0]} + \gamma_d}{\left(\lambda_d \mu_h^b_{[0,0]} + \gamma_d\right)} S.
\]

Applying the Implicit Function Theorem to it, \( \mu_h^b_{[0,0]} \) increases with \( \nu_h^b \).

I describe next how bond market liquidity depends on the mass and, hence, the entry rate of high valuation investors. The characterization of the bond price and illiquidity discount are the same as in the benchmark environment: (18) and (19). Hence, they only depend on \( \omega_b \). In turn, combining the value functions for \( h[0,0], h[1,0], a[1,0] \) and (B.111),

\[
\omega_b = \frac{x_h \left(\lambda_d \mu_{h[0,0]} + \gamma_d\right)}{r_d \left(\lambda_d \mu_{h[0,0]} + \gamma_d\right) + \lambda_b \gamma_d S \frac{1}{2} + \lambda_b \mu_{h[0,0]} \left(\lambda_b \mu_{h[0,0]} + \gamma_d\right) \frac{1}{2}}. 
\] (B.117)

Thereby, \( \omega_b \) and the bond illiquidity discount are decreasing in \( \mu_{h[0,0]} \). Combining the equations characterizing population masses,

\[
\begin{align*}
M_b &= \gamma_d \mu_{h[1,0]} \\
&= \gamma_d \left( S - \mu_{a[1,0]} \right) \\
&= \gamma_d \left( S - \frac{\gamma_d S}{\left(\lambda_b \mu_{h[0,0]} + \gamma_d\right)} \right) \\
&= S \frac{\lambda_b \mu_{h[0,0]} \gamma_d}{\lambda_b \mu_{h[0,0]} + \gamma_d}. 
\end{align*}
\] (B.118) (B.119) (B.120) (B.121)

Bond volume, as a result, increases in \( \mu_{h[0,0]} \). Thus, the bond market is more liquid if
high-valuation investors enter at a higher rate.

Combining the value functions and population masses of the bond market investors:

$$(r + \gamma_d) V_{h[0]}^b = \frac{\lambda_b \gamma_d S}{\lambda_b \gamma_d S + 2(r_d + \lambda_b \mu_b^{h[0]} \frac{1}{2}) (\lambda_b h_{h[0]}^b + \gamma_d)} x_h.$$ 

Thus, $V_{h[0]}^b$ is decreasing in $\mu_b^{h[0]}$ and, hence, in $\nu_h^b$. To simplify the analysis I assume that, in the absence of the CDS market, the entry rate into the bond market, $\nu_h^b$, is given by an interior solution.

When CDS contracts are feasible, we can rule out $V_{h[0]}^b < O_h$ as an equilibrium outcome. This is because $V_{h[0]}^b < O_h$ implies $\nu_h^b = 0$, but $V_{h[0]}^b$ is larger with smaller $\nu_h^b$. If the entry rate without CDS is an interior solution ($V_{h[0]}^b = O_h$), then a decrease in $\nu_h^b$ due to CDS implies $V_{h[0]}^b > O_h$, which is a contradiction. Thus, in equilibrium when CDS is feasible: $V_{h[0]}^b \geq O_h$.

<table>
<thead>
<tr>
<th></th>
<th>Combined</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$V_{h[0]}^c &gt; O_h$</td>
</tr>
<tr>
<td>2</td>
<td>$V_{h[0]}^b &gt; V_{h[0]}^c$</td>
</tr>
<tr>
<td>3</td>
<td>$V_{h[0]}^b &gt; V_{h[0]}^c$</td>
</tr>
<tr>
<td>4</td>
<td>$V_{h[0]}^b &gt; V_{h[0]}^b$</td>
</tr>
<tr>
<td>5</td>
<td>$V_{h[0]}^b &gt; V_{h[0]}^b$</td>
</tr>
<tr>
<td>6</td>
<td>$V_{h[0]}^b &gt; V_{h[0]}^b$</td>
</tr>
<tr>
<td>7</td>
<td>$V_{h[0]}^b = O_h$</td>
</tr>
<tr>
<td>8</td>
<td>$V_{h[0]}^b &gt; V_{h[0]}^b$</td>
</tr>
</tbody>
</table>

To figure out the equilibrium outcomes, Table 2 lists possible orderings of $V_{h[0]}^b$, $V_{h[0]}^c$, and $O_h$. The first row cannot be an equilibrium. $V_{h[0]}^b > V_{h[0]}^c$ would mean that $\nu_h^b = 1$ and $\nu_h^c = 0$. At $\nu_h^b = 0$, however, low types do not enter either ($\nu_l = 0$) and hence $V_{h[0]}^c = 0$. This contradicts $V_{h[0]}^c > O_h$. The fourth and fifth rows cannot be an equilibrium. $V_{h[0]}^b > O_h \geq V_{h[0]}^c$ implies that $\nu_h^c = 0$ and $\nu_h^b = 1$, but since $V_{h[0]}^b$ is decreasing in $\nu_h^b$, $V_{h[0]}^b$ evaluated at $\nu_h^b = 1$ should be lower than $O_h$. Hence, a contradiction. The sixth row is not an equilibrium. $V_{h[0]}^c > V_{h[0]}^b = O_h$ implies that $\nu_h^b = 1$ and $\nu_h^c = 0$. However, $V_{h[0]}^b$ evaluated at $\nu_h^b = 0$ does not equal $O_h$.

Four types of equilibria exist. In two of them, the entry rate into the bond market, $\nu_h^b$, stays the same as in the environment without CDS. In one of these, the entry into the CDS market is positive: $\nu_h^c + \nu_h^b \leq 1$ and $V_{h[0]}^c = V_{h[0]}^b = O_h$, while in the other, none of the high type investors enter the CDS market, and $V_{h[0]}^c = O_h > V_{h[0]}^c = 0$. In the other two equilibria, the entry rate into the bond market, $\nu_h^b$, decreases. In one of these, shown in row 3, $V_{h[0]}^c > V_{h[0]}^b > O_h$ implying that $\nu_h^b = 0$ and $\nu_h^c = 1$. This means all
high type investors enter the CDS, and trading activity in the bond market disappears. In the other, shown in row 2, $V_{bh[0,0]}^b = V_{bh[0,0]}^c > O_h$ implying that the entry rate into the bond market decreases (so that $V_{bh[0,0]}^b$ is higher).\footnote{The entry rate could decrease to another, smaller, rate $\nu^b_h > 0$, or all the way to zero, $\nu^b_h = 0$.}

Thus, with the introduction of the CDS market, high valuation investors enter the bond market either at the same or at a lower rate. If the entry rate remains the same, the illiquidity discount and bond volume also remain the same as without the CDS market. If the entry rate decreases, bond market liquidity deteriorates: Bond volume decreases, and the illiquidity discount increases.

\section*{C Bond Shorting}

I use the bond shorting framework of Vayanos and Weill (2008). In contrast to Vayanos and Weill (2008), I endogenize the entry rate of long and short investors. To be self-contained, I describe the main features and mechanism of the Vayanos and Weill (2008) framework, but for more details I refer the reader to their paper.

Agents' asset positions, $[\theta_b, \theta_r]$, denoted in subscripts on population masses and value functions, are: $\theta_b = 1$ if the investor holds the bond, $\theta_b = 0$ if the investor does not hold the bond, $\theta_r = 1$ if the investor has the lent the bond, and $\theta_r = -1$ if the investor has borrowed the bond. Combined, we have five asset positions: $[1,0]$ is a bond owner, $[0,1]$ is an investor who has lent out her bond, $[1,-1]$ is an investor who has borrowed a bond but has not yet sold short it, $[0,-1]$ is an investor who has borrowed a bond and has short sold it, and $[0,0]$ is an investor with no asset position.

The conjectured equilibrium agent types and their optimal trading strategies are as follows. In equilibrium, a bond owner, $[1,0]$, can be either a high- or average-valuation type. The high-valuation type bond owner searches for a bond borrower in the repo market (i.e. for a short investor) to lend her bond to. The average type bond owner prefers neither a long nor a short position and, hence, searches for a bond buyer in the bond market to sell her bond to. An investor who has lent her bond, $[0,1]$, is a high type investor. If she reverts to an average-valuation while her counterparty has not yet sold the bond, she demands back the bond and becomes a bond seller ($a[1,0]$). If she reverts to an average-valuation after her counterparty has (short-) sold the bond, she seizes the collateral and exits the market. An investor who has borrowed but has not yet sold the bond, $[1,-1]$, is a low type investor. She is the short seller who searches for a bond buyer in the bond market to sell her bond to. If she reverts to an average valuation (that is, before she could short sell the bond), she immediately delivers back the bond and exits the market. An investor with the asset position $[0,-1]$ can be either low or average type. A low type investor with $[0,-1]$ asset position is the short investor who has reached her optimal asset position. An average type with $[0,-1]$ position is an investor who was previously a short investor that reverted to an average valuation. She searches to buy back the bond in order to deliver back the bond and unwind her short position.
with no position can be high or low type: If they are a high type, they seek a long position by buying a bond. If they are a low type, they seek a short position by borrowing a bond.

Define \( \mu_s \equiv \mu_l[1,-1] + \mu_a[1,0] \) the total mass of bond sellers where \( l[1,-1] \) are the short sellers and \( a[1,0] \) are the regular bond sellers (that is, those with a long position looking to unwind their long position). Analogously, \( \mu_a \equiv \mu_h[0,0] + \mu_a[0,-1] \) is the total mass of bond buyers: \( h[0,0] \) are the long investors seeking a long position by buying a bond, and \( a[0,-1] \) are the investors looking to buy back the bond in order to deliver it to its repo counterparty and unwind their short position. In the repo market, the active searchers are the bond borrowers, \( l[0,0] \), and bond lenders, \( h[1,0] \).

Define with \( q \)'s meeting intensities: \( q_{sa} \equiv \lambda_s \mu_a \), \( q_{sb} \equiv \lambda_s \mu_s \), \( q_{lb} \equiv \lambda_l \mu_h[1,0] \), and \( q_{bo} \equiv \lambda_r \mu_h[0,0] \). I also define: \( r_d \equiv r + \gamma_d \) and \( r_u \equiv r + \gamma_u \) to simplify notation. To distinguish environments with and without bond shorting, I denote the environment without bond shorting with hats.

The inflow-outflow equations are:

\[
\nu_h F_h = \gamma_d \mu_h[0,0] + q_{sa} \mu_h[0,0] \quad \text{(C.122)}
\]
\[
q_{sb} \mu_h[0,0] + \gamma_u \mu_l[1,-1] + q_{lb} \mu_l[0,-1] = q_{bo} \mu_h[1,0] + \gamma_d \mu_l[1,0] \quad \text{(C.123)}
\]
\[
q_{bo} \mu_h[1,0] + \gamma_u \mu_l[1,-1] + q_{ls} \mu_s[0,-1] = q_{bo} \mu_h[1,0] + \gamma_d \mu_s[0,-1] \quad \text{(C.124)}
\]
\[
\nu_l F_l + \gamma_d \mu_l[1,-1] + \gamma_d \mu_l[0,-1] = \gamma_u \mu_l[0,0] + q_{la} \mu_l[0,0] \quad \text{(C.125)}
\]
\[
q_{la} \mu_l[0,0] = \gamma_a \mu_l[1,1] + q_{lb} \mu_l[1,1] + \gamma_d \mu_l[1,1] \quad \text{(C.126)}
\]
\[
q_{la} \mu_l[1,1] = (\gamma_u + \gamma_d) \mu_l[0,-1] \quad \text{(C.127)}
\]
\[
\gamma_u \mu_l[0,-1] = q_{lb} \mu_l[0,-1] + \gamma_d \mu_l[0,-1] \quad \text{(C.128)}
\]
\[
\gamma_d \mu_h[1,0] + \gamma_d \mu_l[1,-1] = q_{bo} \mu_a[1,0] \quad \text{(C.129)}
\]

Market clearing conditions are:

\[
\mu_h[1,0] + \mu_a[1,0] + \mu_l[1,-1] = S \quad \text{(C.130)}
\]
\[
\mu_h[0,1] = \mu_l[1,-1] + \mu_l[0,-1] + \mu_a[0,-1] \quad \text{(C.131)}
\]

To characterize the value functions of agents who have lent their bond, \( h[0,1] \), we track their counterparty. The subscript on the value functions of an \( h[0,1] \) investor, as a result, denotes both the agent’s own type and the type of its counterparty. For example, \( V_{h[0,1][1,-1]} \) denotes the value function of an investor who has lent her bond \( (h[0,1]) \) whose counterparty is \( l[1,-1] \) type.

Value functions are:

\[
r V_h[0,0] = \gamma_d (0 - V_h[0,0]) + q_{sb} (V_h[1,0] - V_h[0,0] - p_b) \quad \text{(C.132)}
\]
\[
r V_h[1,0] = \delta + x_h - y + q_{bo} (V_{h[0,1][1,-1]} - V_h[1,0]) + \gamma_d (V_a[1,0] - V_h[1,0]) \quad \text{(C.133)}
\]
Lemma 4. The inflow-outflow and market clearing equations boil down to a set of four equations of four variables $\mu_h[0,0]$, $\mu_t[0,0]$, $\mu_s$, and $\mu_a$:

$$\nu_h F_h = \gamma_d \mu_h[0,0] + q_s \mu_h[0,0]$$ (C.144)
$$
\nu_l F_l = \frac{\gamma_u}{\gamma_u + \gamma_d} \lambda_r (S - \mu_s) \mu_{l[0,0]} + \gamma_a \mu_{l[0,0]} \quad (C.145)
$$

$$
\mu_a = \mu_{h[0,0]} + \frac{\gamma_u}{\lambda_b \mu_u + \gamma_d \gamma_u + \gamma_d} \lambda_b \mu_{u[0,0]} + \frac{\lambda_r \mu_{l[0,0]} (S - \mu_s)}{\gamma_u + \gamma_d + \lambda_b \mu_u} \quad (C.146)
$$

$$
\mu_a = \frac{\gamma_d s}{\gamma_d + \lambda_b \mu_u} + \frac{\lambda_r (S - \mu_s) \mu_{l[0,0]}}{\gamma_u + \gamma_d + \lambda_b \mu_u} \quad (C.147)
$$

**Proof.** The first equation (C.144) is given by (C.122) and characterizes $\mu_{h[0,0]}$.

The second equation (C.145) characterizes $\mu_{l[0,0]}$. Substituting $q_u \mu_{l[1,1]}$ from (C.127) into (C.126) and solving for $\mu_{l[1,1]} + \mu_{l[0,0]}$, we get

$$
\mu_{l[1,1]} + \mu_{l[0,0]} = \frac{q_{lE} \mu_{l[0,0]}}{\gamma_u + \gamma_d} \quad (C.148)
$$

Substituting it into (C.125) and simplifying yields:

$$
\nu_l F_l + \gamma_d q_{lE} \mu_{l[0,0]} = \gamma_u \mu_{l[0,0]} + q_{lE} \mu_{l[0,0]} \quad (C.149)
$$

Then, using $q_{lE} = \lambda_r \mu_{h[1,0]}$ and the bond market clearing condition, we get:

$$
\mu_{l[1,1]} = \frac{\gamma_u}{\gamma_u + \gamma_d + \lambda_b \mu_u} \lambda_r (S - \mu_s) \mu_{l[0,0]} + \gamma_a \mu_{l[0,0]} \quad (C.150)
$$

The third equation (C.146) characterizes $\mu_a$. Solving for $\mu_{l[1,1]}$ from (C.126) and using $\mu_{h[1,0]} = s - \mu_s$:

$$
\mu_{l[1,1]} = \frac{\gamma_u}{\gamma_u + \gamma_d + \lambda_b \mu_u} \lambda_r (S - \mu_s) \mu_{l[0,0]} \quad (C.151)
$$

Substituting it into (C.127) and using $q_u = \lambda_b \mu_u$, we get

$$
\mu_{l[0,0]} = \frac{\lambda_b \mu_u}{\gamma_u + \gamma_d + \lambda_b \mu_u} + \frac{\lambda_r (S - \mu_s) \mu_{l[0,0]}}{\gamma_u + \gamma_d + \lambda_b \mu_u} \quad (C.152)
$$

Plugging (C.152) into (C.128) gives

$$
\mu_{a[0,1]} = \frac{\gamma_u}{\gamma_u + \gamma_d + \lambda_b \mu_u} \lambda_r (S - \mu_s) \mu_{l[0,0]} \quad (C.153)
$$

Plugging (C.153) into $\mu_a = \mu_{h[0,0]} + \mu_{a[0,1]}$, we get

$$
\mu_a = \mu_{h[0,0]} + \frac{\gamma_u}{\lambda_b \mu_u + \gamma_d \gamma_u + \gamma_d} \lambda_b \mu_{u[0,0]} + \frac{\lambda_r \mu_{l[0,0]} (S - \mu_s)}{\gamma_u + \gamma_d + \lambda_b \mu_u} \quad (C.154)
$$

The fourth equation (C.147) characterizes $\mu_s$. Solve for $\mu_{a[1,0]}$ from (C.129):

$$
\mu_{a[1,0]} = \frac{\gamma_d s}{\gamma_d + \lambda_b \mu_u} \quad (C.155)
$$

Adding $\mu_{l[1,1]}$, derived in (C.151), to $\mu_{a[1,0]}$ and using the definition $\mu_s = \mu_{l[1,1]} + \mu_{a[1,0]}$, we get
we get:

\[ \mu_s = \frac{\gamma_d S}{\gamma_d + \lambda_b \mu_n} + \frac{\lambda_r (S - \mu_s) \mu_{l[0,0]}}{\gamma_u + \gamma_d + \lambda_b \mu_n}. \]  

(C.156)

The next lemma shows that, taking the entry rate \( \nu' \)'s as given, the four equations characterizing \( \mu_{h[0,0]}, \mu_{l[0,0]}, \mu_s, \) and \( \mu_n \) have a unique positive solution. In doing so, it shows that \( S - \mu_s > 0 \), which I use later for the proof of Proposition 10.

**Lemma 5.** For any given \( \nu_h \) and \( \nu_l \), a unique positive solution exists for \( \mu \)'s.

**Proof.** Make the following change of variables: \( q_{ho} = \lambda_r \mu_{h[0,0]} \) and \( q_s = \lambda_b \mu_s \), \( e_b = \frac{1}{\lambda_b} \), and \( e_r = \frac{1}{\lambda_r} \) in (C.144)-(C.147):

\[ \nu_h F_h = \gamma_d \mu_{h[0,0]} + q_s \mu_{h[0,0]} \]  

(C.157)

\[ \nu_l F_l = \frac{\gamma_u}{\gamma_u + \gamma_d} q_{ho} (S - \epsilon_b q_s) + \gamma_u e_r q_{ho} \]  

(C.158)

\[ \mu_h = \mu_{h[0,0]} + \frac{\gamma_u}{q_s + \gamma_d} \frac{\mu_n}{\gamma_u} \frac{q_{ho} (S - \epsilon_b q_s)}{\gamma_u + \gamma_d e_b (\gamma_u + \gamma_d) + \mu_n} \]  

(C.159)

\[ q_h = \frac{\gamma_d S}{e_b \gamma_d + \mu_n} + \frac{q_{ho} (S - \epsilon_b q_s)}{e_b (\gamma_u + \gamma_d) + \mu_n}. \]  

(C.160)

From (C.157), solve for \( \mu_{h[0,0]} \) as a function of \( q_h \):

\[ \mu_{h[0,0]} = \frac{\nu_h F_h}{\gamma_d + q_s}. \]  

(C.161)

Thus, \( \mu_{h[0,0]} \) decreases in \( q_s \).

Next, I show that \( \mu_n \) decreases in \( q_s \). From (C.158),

\[ q_{ho} = \frac{\nu_l F_l}{\gamma_u e_r + \frac{\gamma_u}{\gamma_u + \gamma_d} (S - \epsilon_b q_s)}. \]  

(C.162)

Plugging it into (C.159) and rearranging gives:

\[ \left( \mu_n - \mu_{h[0,0]} \right) = \frac{1}{(e_b (\gamma_u + \gamma_d) + \mu_n) (q_s + \gamma_d)} \mu_n (e_r (\gamma_u + \gamma_d) + (S - \epsilon_b q_s)) \nu_l F_l (S - \epsilon_b q_s). \]  

(C.163)

Rearrange it and get:

\[ 1 = \frac{\mu_{h[0,0]}}{\mu_n} + \frac{1}{(e_b (\gamma_u + \gamma_d) + \mu_n) (\gamma_d + q_s)} (e_r (\gamma_u + \gamma_d) + (S - \epsilon_b q_s)) \nu_l F_l (S - \epsilon_b q_s). \]  

(C.164)

Since \( \mu_{h[0,0]} \) is an implicit function of \( q_s \), (C.164) characterizes \( \mu_n \) as an implicit function of \( q_s \). Applying the Implicit Function Theorem to (C.164), \( \mu_n \) decreases in \( q_s \).
I simplify (C.144)-(C.147) into a single equation of one unknown, \( q_s \). From (C.159),
\[
\frac{q_{ho} \left(S - e_b q_s\right)}{e_b \left(e_b \gamma_u + \gamma_d\right) + \mu_b} = \left(\mu_b - \mu_{h[0,0]}\right) \frac{\left(g_u + g_d\right) \left(q_s + g_d\right)}{\gamma_u \gamma_d}.
\] (C.165)
Plug (C.165) into (C.160) and multiply both sides by \( \mu_b \):
\[
q_s \mu_b = \frac{\gamma_d S \mu_b}{e_b \gamma_d + \mu_b} + \left(\mu_b - \mu_{h[0,0]}\right) \frac{\left(g_u + g_d\right) \left(q_s + g_d\right)}{\gamma_u}.
\] (C.166)
Add \( q_s \mu_{h[0,0]} \) to both sides
\[
q_s \mu_b + q_s \mu_{h[0,0]} = \frac{\gamma_d S \mu_b}{e_b \gamma_d + \mu_b} + \left(\mu_b - \mu_{h[0,0]}\right) \frac{\left(g_u + g_d\right) \left(q_s + g_d\right)}{\gamma_u} + q_s \mu_{h[0,0]}.
\] (C.166)
Subtract \( q_s \mu_b \) from both sides, simplify, and plug the expression for \( \mu_{h[0,0]} \) into the left-hand side of (C.166):
\[
\frac{\nu_h F_h}{\gamma_d + q_s} q_s = \frac{\gamma_d S \mu_b}{e_b \gamma_d + \mu_b} + \left(\mu_b - \mu_{h[0,0]}\right) \frac{\gamma_d}{\gamma_u} \left(\gamma_u + g_d + q_s\right).
\] (C.167)
The left-hand side increases in \( q_s \). The first term on the right-hand side, \( \frac{\gamma_d S \mu_b}{e_b \gamma_d + \mu_b} \), increases in \( \mu_b \) and, hence, decreases in \( q_s \). Consider the second term on the right-hand side. From (C.163),
\[
\left(\mu_b - \mu_{h[0,0]}\right) \left(\gamma_u + g_d + q_s\right) = \frac{1}{\left(e_b \left(\gamma_u + g_d\right) + \mu_b\right) \left(q_s + g_d\right) \left(e_r \left(\gamma_u + g_d\right) + (S - e_b q_s)\right)} \nu_l F_l \left(S - e_b q_s\right).
\] (C.168)
The right-hand side of (C.168) decreases in \( q_s \) and increases in \( \mu_b \). Since \( \mu_b \) decreases in \( q_s \), the right-hand side of (C.168), as a result, decreases in \( q_s \). Put together, left-hand side-right-hand side of (C.167) (as a function of one unknown, \( q_s \)) strictly increases in \( q_s \).
At \( q_s = 0 \), the left-hand side of (C.167) is 0, the right-hand side is positive, and, hence, left-hand side - right-hand side \( < 0 \). For \( q_s = \frac{S}{e_b} \), the left-hand side is equal to \( \frac{S}{e_b} \mu_{h[0,0]} \). Consider the right-hand side. From (C.163), \( \mu_b - \mu_{h[0,0]} = 0 \). Thus, \( RHS = \frac{\gamma_d S \mu_b}{e_b \gamma_d + \mu_b} \), which is less than \( \frac{S}{e_b} \mu_{h[0,0]} \). Thereby, at \( q_s = \frac{S}{e_b} \), left-hand side - right-hand side \( > 0 \). Thus, (C.167) has a unique solution in \( \left(0, \frac{S}{e_b}\right) \).

The other unknowns, \( \mu_{h[0,0]} \), \( \mu_b \), and \( q_{ho} \) are uniquely given as functions of \( q_s \). Thus, taking as given \( \nu_h \) and \( \nu_l \) in \([0, 1]\), the system of four equations and four unknowns (C.144)-(C.147) has a unique positive solution. This proof for an existence of a unique positive solution holds for any \( e_r \) including \( e_r = 0 \). Importantly, the solution is such that \( q_s < \frac{S}{e_b} \) implying that \( \mu_{h[1,0]} = S - e_b q_s > 0 \).

**Proof of Proposition 9.** Take the difference between (C.133) and (C.132) and get
\[
r \left(V_{h[1,0]} - V_{h[0,0]}\right) = \tilde{\delta} + x_h - y + q_{ho} \left(V_{h[0,1]}[1,1] - V_{h[1,0]}\right) - \gamma_d \omega_b - q_s \frac{1}{2} \omega_b.
\] (C.169)
Combine this with (C.141) and solve for $\omega_b$:

$$
\omega_b = \frac{x_h + q_{ho} \frac{1}{2} \omega_r}{(r + \gamma_d + q_{s} \frac{1}{2} + q_{u} \frac{1}{2})}.
$$

(C.170)

Combine (C.141) and (C.169) and simplify:

$$
rp_b = \delta - y + \frac{1}{2} x_h + \frac{1}{2} \left( -\gamma_d \omega_b - r_d V_{h[0,0]} + q_{a} \frac{1}{2} \omega_b \right) + \frac{1}{4} q_{ho} \omega_r.
$$

(C.171)

Substitute it in $q_{u} \frac{1}{2} \omega_b = x_h + q_{ho} \frac{1}{2} \omega_r - \omega_b \left( r + \gamma_d + q_{s} \frac{1}{2} \right)$ from (C.170) and $V_{h[0,0]} = O_h$ and simplify to get

$$
rp_b = \delta + x_h - y - (r + \gamma_d) O_h - \left( \frac{r + 2 \gamma_d}{2} \right) \omega_b + q_{ho} \frac{1}{2} \omega_r.
$$

(C.172)

The bond price without shorting is

$$
rp_b = \delta + x_h - y - (r + \gamma_d) O_h - \left( \frac{r + 2 \gamma_d}{2} \right) \omega_b.
$$

(C.173)

\[ \square \]

**Lemma 6.** The gains from trade from a repo transaction, $\omega_r$, is characterized by:

$$
\left( r_d + \gamma_u + q_{le} \frac{1}{2} + q_{ho} \frac{1}{2} \left( 1 + \frac{q_{a} \gamma_u}{(r_d + q_{a} + \gamma_u) (r_d + q_{s})} \right) \right) \omega_r
$$

$$
= \frac{q_{a}}{(r_d + \gamma_u + q_{a})} \cdot \left[ x_l + \frac{r_d + \gamma_u + q_{s}}{r_d + q_{a}} \left[ -2y + \frac{r_d + q_{a}}{(r_d + q_{s} \frac{1}{2} + q_{u} \frac{1}{2})} \frac{1}{2} \left( x_h + q_{ho} \frac{1}{2} \omega_r \right) \right] \right].
$$

(C.174)

**Proof.** Take the difference between (C.138) and (C.137):

$$
r V_{l[1,1]} - r V_{l[0,0]} = -f_{ee} - \gamma_u (V_{l[1,1]} - V_{l[0,0]}) + q_{a} (V_{l[0,1]} - V_{l[1,1]} + p_{b}) + \gamma_d (V_{l[0,0]} - V_{l[1,1]}) - q_{le} (V_{l[1,1]} - V_{l[0,0]}).
$$

(C.175)

Take the difference between (C.134) and (C.133):

$$
r V_{h[0,1][1,1]} - r V_{h[0,1]} = f_{ee} + \gamma_d (V_{a[1,0]} - V_{h[0,1][1,1]}) - \gamma_u (V_{h[0,1][1,1]} - V_{h[1,0]})
$$

$$
+ q_{a} (V_{h[0,1][0,1]} - V_{h[0,1][1,1]}) - \gamma_d (V_{a[0,0]} - V_{h[0,0]})
$$

$$
- q_{ho} (V_{h[0,1][1,1]} - V_{h[1,0]}).
$$

(C.176)

Then, adding the two differences and simplifying yields:

$$
\left( r_d + \gamma_u + q_{ho} \frac{1}{2} + q_{le} \frac{1}{2} \right) \omega_r = q_{a} (V_{h[0,1][0,1]} - V_{h[0,1][1,1]} + V_{l[0,1]} - V_{l[1,1]} + p_{b}).
$$

(C.177)
Next, I derive $V_{h[0,1][0,-1]} - V_{h[0,1][1,-1]} + V_{l[0,-1]} - V_{l[1,-1]} + p_b$. Take the difference between (C.139) and (C.138):

$$rV_{[0,-1]} - rV_{[1,-1]} = -(\bar{\delta} - x_t) - y + \gamma_u (V_{a[0,-1]} - V_{l[0,-1]}) + \gamma_d (-c - V_{l[0,-1]})$$
$$- (\gamma_u (0 - V_{l[1,-1]}) + q_a (V_{l[0,-1]} - V_{l[1,-1]} + p_b) + \gamma_d (-V_{l[1,-1]})).$$

Take the difference between (C.135) and (C.134):

$$rV_{h[0,1][0,-1]} - rV_{h[0,1][1,-1]} = -\gamma_d (V_{h[0,1][0,-1]} - V_{h[0,1][1,-1]}) + \gamma_u (V_{h[0,1][0,-1]} - V_{h[0,1][1,-1]})$$
$$- q_a (V_{h[0,1][0,-1]} - V_{h[0,1][1,-1]}) - \gamma_u (V_{h[1,0]} - V_{h[0,1][1,-1]}).$$

Add the two differences and subtracting $V_{l[0,-1]}$, we get

$$V_{h[0,1][0,-1]} + V_{a[0,-1]} + p_b - V_{h[1,0]}.$$ Adding together (C.136), (C.140), $rp_b$, and subtracting (C.133), we get

$$(r_d + q_a) (V_{l[0,-1]} - V_{l[1,-1]} + V_{h[0,1][0,-1]} - V_{h[0,1][1,-1]} + p_b) = \gamma_d (rp_b - \bar{\delta} - y + \gamma_d p_b) - q_{bo} \frac{1}{2} \omega_r. \quad (C.178)$$

Next, I derive $V_{h[0,1][a[0,-1]} + V_{a[0,-1]} + p_b - V_{h[1,0]}$. Adding together (C.136), (C.140), $rp_b$, and subtracting (C.133), we get

$$r_d + q_a) (V_{h[0,1][a[0,-1]} + V_{a[0,-1]} + p_b - V_{h[1,0]}) = \gamma_d (rp_b - \bar{\delta} - y + \gamma_d p_b - V_{a[0,-1]}) - q_{bo} \frac{1}{2} \omega_r. \quad (C.179)$$

Plug (C.178) and (C.179) back into (C.177) and get:

$$\left(\frac{r_d + q_a + q_{bo}}{2} \left(1 + \frac{q_{bo} \gamma_u}{(r_d + q_a + \gamma_u)(r_d + q_b)}\right) + q_{bo} \frac{1}{2}\right) \omega_r. \quad (C.180)$$

The expression characterizes $\omega_r$ but still has $p_b$ and $\omega_b$. Thus, I derive now, $rp_b - \bar{\delta} - y + \gamma_d \frac{1}{2} \omega_b$. Re-arrange the bond price equation (C.172) and use $(r + \gamma_d) V_{h[0,0]} = q_{bo} \frac{1}{2} \omega_b$,

$$rp_b - \bar{\delta} + \gamma_d \frac{1}{2} \omega_b = \frac{r + \gamma_d + q_a}{(r + \gamma_d + q_a \frac{1}{2} + q_{bo} \frac{1}{2}) \frac{1}{2}} \left(x_b + q_{bo} \frac{1}{2} \omega_r\right). \quad (C.181)$$

Add (-2y) to both sides:

$$rp_b - \bar{\delta} - y + \gamma_d \frac{1}{2} \omega_b = -2y + \frac{r + \gamma_d + q_a}{(r + \gamma_d + q_a \frac{1}{2} + q_{bo} \frac{1}{2}) \frac{1}{2}} \left(x_b + q_{bo} \frac{1}{2} \omega_r\right). \quad (C.182)$$
Plugging this back into (C.180) gives (C.174).

Proposition C.1 shows the existence of an equilibrium when \( \lambda_r \to \infty \). The existence of equilibrium when \( \lambda_b \to \infty \) can be shown analogously and is equally tedious.

**Proposition C.1.** Define

\[
V_{l[0,0]}(\nu_l) \equiv \frac{1}{r + \gamma_u} \frac{q_u}{(r_d + \gamma_u + q_u)} \left( x_l + \frac{r_d + \gamma_u + q_u}{r_d + q_u} \left( -2y + \frac{r_d + q_u}{r_d + q_u \frac{1}{2} + q_u \frac{1}{2}} x_h \right) \right)
\]

where \( q_s \) and \( q_u \) are the solution to:

\[
q_s = \frac{\gamma_d S \lambda_b}{\gamma_d + q_u} + \frac{\nu_l \ell}{\gamma_u} \frac{(\gamma_u + \gamma_d) \lambda_b}{(\gamma_u + \gamma_d) + q_u} x_h
\]

\[
(r + \gamma_d)O_h = q_s \frac{x_h}{(r + \gamma_d + q_u \frac{1}{2} + q_u \frac{1}{2})}.
\]

Suppose

\[
V_{l[0,0]}(\nu_l) > 0
\]

for \( \nu_l \in [0, 1] \),

\[
V_{l[0,0]}(1) \geq O_l, \tag{C.184}
\]

and

\[
x_l + \frac{r_d + \gamma_u + q_u}{r_d + q_u} \left( \frac{\gamma_u(q_u - q_s)}{r_d + \gamma_u + q_u} \frac{1}{2} x_h - 2y \right) > 0. \tag{C.185}
\]

Then, in the limit as \( \lambda_r \to \infty \), an equilibrium with bond shorting exists. In particular,

1. \( \lim_{\lambda_r \to \infty} q_{ho} < \infty, \lim_{\lambda_r \to \infty} q_{he} = \infty, \lim_{\lambda_r \to \infty} \mu_u < \infty, \lim_{\lambda_r \to \infty} \mu_{h[0,0]} < \infty, \lim_{\lambda_r \to \infty} \omega_r = 0, \) and \( \lim_{\lambda_r \to \infty} \text{fee} = 0. \)

2. The entry rate of low-valuation investors is positive: \( \nu_l > 0. \)

3. The gains from a repo transaction, \( \omega_r \), approaches zero from above: \( \omega_r > 0. \)

4. The gains from a bond transaction, \( \omega_b \), is positive.

5. \( -V_{a[0,-1]} > V_{h[1,0]} - V_{h[0,0]} \).

6. \( V_{a[1,0]} > V_{l[1,-1]} - V_{l[0,-1]} \).

7. Unlike \( h[1,0] \) type agent, \( a[1,0] \) type agent prefers to sell her bond rather than to lend it out.

8. A short seller upon switching to an average-valuation (and becoming \( a[0,-1] \) type agent) prefers to buy back the bond and deliver the bond than to keep the short position.
Proposition C.1.3 shows that it is optimal for \( h[1,0] \) to lend her bond and for \( l[0,0] \) to borrow it as conjectured. Proposition C.1.4 shows that, as conjectured, it is optimal for \( h[0,0] \)-type investor to buy the bond, and for \( a[1,0] \) to sell her bond and unwind her long position. In Proposition C.1.5, \(-V_{a[0,-1]}\) is \( a[0,-1] \)-type investor’s reservation value for the bond, while \( V_{l[1,0]} - V_{h[0,0]}\) is \( h[0,0] \)-type investor’s reservation value for the bond. The inequality ensures that it is optimal for \( a[0,-1] \) investor to buy the bond at the price characterized by (C.172). In Proposition C.1.6, \( V_{a[1,0]} \) is \( a[1,0] \)'s reservation value for the bond, while \( V_{l[1,-1]} - V_{l[0,1]} \) is short seller’s \((l[1,0])\) reservation value for the bond. The inequality ensures that short sellers find it optimal to (short) sell the bond at the price characterized by (C.172). Proposition C.1.7 implies that an average valuation investor prefers to unwind her long position than to keep the long position. Thus, parts 7 and 8 show that an average valuation investor prefers neither a long nor a short position.

**Proof of Proposition C.1.1.** In this part of the proposition, I derive the limits of \( q_{bo}, \ q_{ls}, \ \mu_{u}, \ \mu_{h[0,0]}, \ \omega_{r}, \) and fee as \( \lambda_{r} \to \infty \) for any \( \nu_{t} \geq 0 \). Then using the limits, I show the other parts of the proposition.

From (C.158):

\[
q_{bo} = \frac{\nu_{t} F_{l}}{\gamma_{u}} (S - e_{b} q_{s}) + e_{r}.
\]

(C.186)

Lemma 5 showed that \( S - e_{b} q_{s} > 0 \). Thus, \( \lim_{\lambda_{r} \to \infty} q_{bo} < \infty \), and \( \lim_{\lambda_{r} \to \infty} e_{r} q_{bo} = \lim_{\lambda_{r} \to \infty} \mu_{l[0,0]} = 0 \).

Consider the limits of \( \mu_{u} \) and \( q_{s} \). Set \( e_{r} = 0 \) in (C.157)-(C.160):

\[
\nu_{h} F_{h} = \gamma_{d} \mu_{h[0,0]} + q_{h} \mu_{h[0,0]}
\]

(C.187)

\[
\nu_{l} F_{l} = \frac{\gamma_{u}}{\gamma_{u} + \gamma_{d}} q_{bo} (S - e_{b} q_{s})
\]

(C.188)

\[
\mu_{u} = \mu_{h[0,0]} + \frac{\gamma_{u}}{q_{s} + \gamma_{d}} \mu_{u} q_{bo} (S - e_{b} q_{s}) + \mu_{u}
\]

(C.189)

\[
q_{s} = \frac{\gamma_{d} S}{e_{b} \gamma_{d} + \mu_{u}} + \frac{q_{bo} (S - e_{b} q_{s})}{e_{b} (\gamma_{u} + \gamma_{d}) + \mu_{u}}.
\]

(C.190)

Solving for \( q_{bo} \) and \( \mu_{h[0,0]} \) from the first two and substituting them into the other two, we get

\[
\mu_{u} = \frac{\nu_{h} F_{h}}{\gamma_{d} + q_{s}} + \frac{\mu_{u}}{q_{s} + \gamma_{d} e_{b} (\gamma_{u} + \gamma_{d}) + \mu_{u}},
\]

(C.191)

\[
q_{s} = \frac{\gamma_{d} S}{e_{b} \gamma_{d} + \mu_{u}} + \frac{\nu_{l} F_{l} (\gamma_{u} + \gamma_{d})}{\gamma_{u} (e_{b} (\gamma_{u} + \gamma_{d}) + \mu_{u})}.
\]

(C.192)

Solving for \( q_{s} \) from (C.191), we get

\[
q_{s} = \frac{\nu_{h} F_{h}}{\mu_{u}} + \frac{\nu_{l} F_{l} (\gamma_{u} + \gamma_{d})}{e_{b} (\gamma_{u} + \gamma_{d}) + \mu_{u}} - \gamma_{d}.
\]

(C.193)
Substituting this into (C.192) and simplifying, we get
\[
\frac{\nu_h F_h}{\gamma_d} - \mu_n \left( 1 + \frac{S}{\mu_n + e_b \gamma_d} + \frac{\nu_l F_l}{\gamma_a (\mu_n + e_b (\gamma_d + \gamma_a))} \right) = 0. \tag{C.194}
\]

For \( \mu_n = 0 \), the left-hand side is positive, while for a large \( \mu_n \), the left-hand side is negative. The left-hand side strictly decreases in \( \mu_n \). Thus, it has a unique, positive, and finite solution in \( \mu_n \). In turn, from (C.192), \( 0 < \lim q_b < \infty \).

Consider now the limits of \( q_{LE} \) and \( \omega_r \).

\[
\lim_{\lambda_r \to \infty} q_{LE} = \lim_{\lambda_r \to \infty} \lambda_r \mu_{h[1,0]} = \lim_{\lambda_r \to \infty} \lambda_r (S - \mu_a) = \infty. \tag{C.195}
\]

Moreover, \( \lim q_{bo} < \infty \) and \( \lim q_{LE} = \infty \) imply that \( \lim \omega_r = 0 \) from (C.174).

I now characterize the lending fee and its limit. Combining (C.134) and (C.133) yields:
\[
(r + \gamma_d + \gamma_u + q_{bo}) (V_{h[0,1][1,-1]} - V_{h[1,0]}) = \text{fee} + q_n (V_{h[0,1][0,1]} - V_{h[0,1][1,-1]}). \tag{C.196}
\]

Consider \( V_{h[0,1][0,1]} - V_{h[0,1][1,-1]} \) by combining (C.135) and (C.134):
\[
V_{h[0,1][0,1]} - V_{h[0,1][1,-1]} = \frac{\gamma_u (V_{h[0,1][0,1]} - V_{h[0,1][1,-1]}) + \gamma_u (V_{h[0,1][0,1]} - V_{h[1,0]})}{r + \gamma_d + q_n}. \tag{C.197}
\]

Combining (C.136) and (C.135), we get:
\[
V_{h[0,1][0,1]} - V_{h[0,1][1,-1]} = \frac{q_n}{r + \gamma_d + \gamma_u} (V_{h[1,0]} - V_{h[0,1][0,1]}). \tag{C.198}
\]

Combine (C.133) and (C.136) and simplify:
\[
V_{h[1,0]} - V_{h[0,1][0,1]} = \frac{q_{bo} (V_{h[0,1][1,-1]} - V_{h[1,0]}) - \text{fee}}{r + q_n + \gamma_d}. \tag{C.199}
\]

Incrementally substitute (C.197)-(C.199) into (C.196), use \( V_{h[0,1][1,-1]} - V_{h[1,0]} = \frac{1}{2} \omega_r \), and solve for the fee as:
\[
\text{fee} = \frac{r_d + q_{bo} + \frac{r_d \gamma_u}{r_d + q_n} - \frac{\gamma_u q_n q_{bo}}{(r_d + q_n)(r_d + \gamma_u)(r_d + q_n)}}{1 - \frac{\gamma_u q_n q_{bo}}{(r_d + q_n)(r_d + \gamma_u)(r_d + q_n)}} \frac{1}{2} \omega_r.
\]

Since \( q_{bo} \), \( q_n \), and \( q_s \) have finite limits, and \( \lim_{\lambda_r \to \infty} \omega_r = 0 \), we have: \( \lim_{\lambda_r \to \infty} \text{fee} = 0 \). \( \square \)

**Proof of Proposition C.1.2**: The solution for the entry rate is such that \( \nu_l > 0 \).

Consider the entry rate of high-valuation investors. Combining (C.189) and (C.190), we get an expression for \( \mu_{h[0,0]} \) in terms of just \( q_n \) and \( q_s \):
\[
\mu_{h[0,0]} = \frac{1}{\lambda_b} \left( q_n - \frac{\gamma_u}{q_s + \gamma_d \gamma_u + \gamma_d} \left( q_n - \frac{\gamma_d S \lambda_b}{\gamma_d + q_n} \right) \right). \tag{C.200}
\]

Combining (C.170), (C.132), the entry condition, and the limits for \( q_{bo} \) and \( \omega_r \), we get an
expression for \( q_s \) in terms of just \( q_b \)
\[
q_s = \frac{O_h r_d (r_d + \frac{1}{2} q_b)}{x_h - \frac{1}{2} O_h r_d} \tag{C.201}
\]

Plug (C.200) and (C.201) into (C.187), and define
\[
\nu_h(\nu_l) \equiv \frac{\gamma_d}{F_h \lambda_b} \left( 1 + \frac{1}{\gamma_u + \gamma_d} \frac{O_h r_d (r_d + \frac{1}{2} q_b)}{x_h - \frac{1}{2} O_h r_d} + \frac{\gamma_u + \gamma_d + q_b}{\gamma_u + \gamma_d + q_b} \right) q_u, \tag{C.202}
\]
where \( q_u \) is an implicit function of \( \nu_l \) and is characterized by:
\[
\nu_l F_l = \frac{1}{\lambda_b} \frac{\gamma_u}{\gamma_u + \gamma_d} \left( \frac{O_h r_d (r_d + \frac{1}{2} q_b)}{x_h - \frac{1}{2} O_h r_d} - \frac{\gamma_d S \lambda_b}{\gamma_d + q_b} \right) \left( \gamma_u + \gamma_d + q_b \right). \tag{C.203}
\]

Eq. (C.203) comes from combining (C.188), (C.189), the limit \( \mu_{[0,0]} \to 0 \), and (C.201). I focus on the interior solution for \( \nu_h \) by assuming that:
\[
\nu_h : [0, 1] \to (0, 1].
\]

This is satisfied if, for example, \( F_h \) is large.

I now show that the entry rate of low-valuation investors is positive. Plug (C.170) into (C.132):
\[
(r + \gamma_d) O_h = q_s \frac{x_h + q_{bo} \frac{1}{2} \omega_r}{(r + \gamma_d + q_s \frac{1}{2} + q_{bo} \frac{1}{2})}. \tag{C.204}
\]

Then (C.204), (C.158), (C.159), (C.160), and (C.174) characterize a set of five equations and five unknowns, \( q_s, \mu_u, \mu_{[0,0]}, q_{bo}, \) and \( \omega_r \) as implicit functions of \( \nu_l \). Set \( \epsilon_r = 0 \) in (C.204), (C.158)-(C.160), and (C.174):
\[
\nu_l F_l = \frac{\gamma_u}{\gamma_u + \gamma_d} q_{bo} (S - e_b q_s) \tag{C.205}
\]

\[
\mu_u = \mu_{[0,0]} + \frac{\gamma_u}{q_s + \gamma_d} \frac{\mu_u}{\gamma_u + \gamma_d} q_{bo} (S - e_b q_s) \tag{C.206}
\]

\[
q_s = \frac{\gamma_d S}{e_b \gamma_d + \mu_u} + \frac{q_{bo} (S - e_b q_s)}{e_b (\gamma_u + \gamma_d) + \mu_u} \tag{C.207}
\]

\[
q_{bo} \frac{1}{2} \omega_r = \frac{q_u}{(r_d + \gamma_d + q_u)} \left( x_l + \frac{r_d + \gamma_u + q_s}{r_d + q_s} \left[ -2 y + \frac{r_d + q_u}{(r_d + q_s \frac{1}{2} + q_{bo} \frac{1}{2})} \frac{1}{2} x_h \right] \right) \tag{C.208}
\]

\[
(r + \gamma_d) O_h = q_s \frac{x_h}{(r + \gamma_d + q_s \frac{1}{2} + q_{bo} \frac{1}{2})}. \tag{C.209}
\]

From (C.205):
\[
\nu_l F_l \left( \gamma_u + \gamma_d \right) = q_{bo} (S - e_b q_s). \tag{C.209}
\]
Substitute it into (C.207). Together with (C.204), we have two equations that characterize \( \mu_u \) and \( q_s \) as implicit functions of \( \nu_l \):

\[
q_s = \frac{\gamma_d S}{e_b \gamma_d + \mu_u} + \frac{\nu F}{\gamma_u \gamma_d + \mu_u} (\gamma_u + \gamma_d) \tag{C.210}
\]

\[
(r + \gamma_d)O_h = q_s \left( \frac{1}{(r + \gamma_d + q_s \frac{1}{2} + q_u \frac{1}{2})} \right). \tag{C.211}
\]

Substitute the expression for \( q_s \) from (C.210) into (C.211) and get:

\[
r_d O_h \left[ r_d + \left( \frac{\gamma_d S}{e_b \gamma_d + \mu_u} + \frac{\nu F}{\gamma_u \gamma_d + \mu_u} (\gamma_u + \gamma_d) \right) \frac{1}{2} + q_s \frac{1}{2} \right] = \left[ \frac{\gamma_d S}{e_b \gamma_d + \mu_u} + \frac{\nu F}{\gamma_u \gamma_d + \mu_u} (\gamma_u + \gamma_d) \right] x_h. \tag{C.212}
\]

Applying the Implicit Function Theorem,

\[
\frac{\partial \mu_u}{\partial \nu_l} = \frac{(\gamma_u + \gamma_u)(r_d + \frac{1}{2} \mu_u \lambda_b)}{(\gamma_u + \gamma_u + \mu_u \lambda_b)} \cdot \left( \frac{S\gamma_d}{r_d + \gamma_u \mu_u \lambda_b} \right) + \frac{\nu}{\gamma_u \gamma_d + \mu_u \lambda_b}
\]

Thus, \( \frac{\partial \mu_u}{\partial \nu_l} > 0 \), and \( \mu_u \) strictly increases in \( \nu_l \).

Now consider how \( V_l[0,0] \) changes with \( \mu_u \) or, equivalently, with \( q_u \). From (C.208),

\[
V_l[0,0] = \frac{1}{r + \gamma_u} q_u \overline{x} \frac{1}{2} \overline{\lambda} r.
\]

\[
= \frac{1}{r + \gamma_u} \left( r_d + \gamma_u + q_u \right) \left( x_l + \frac{r_d + \gamma_u + q_u}{r_d + q_s} \left[ -2y + \frac{r_d + q_u}{r_d + q_s \frac{1}{2} + q_u \frac{1}{2}} \frac{1}{2} x_h \right] \right) \tag{C.213}
\]

where \( q_s \) and \( q_u \) are solution to (C.210) and (C.211). Define

\[ A \equiv x_l + \frac{r_d + \gamma_u + q_u}{r_d + q_s} \left[ -2y + \frac{r_d + q_u}{r_d + q_s \frac{1}{2} + q_u \frac{1}{2}} \frac{1}{2} x_h \right]. \]

Then,

\[
V_l[0,0] = \frac{1}{r + \gamma_u} \frac{q_u}{(r_d + \gamma_u + q_u)} A
\]

and

\[
\frac{\partial V_l[0,0]}{\partial q_u} = \frac{1}{r_u (r_d + \gamma_u + q_u)} A + \frac{1}{r_u (r_d + \gamma_u + q_u)} \frac{\partial A}{\partial q_u}. \tag{C.214}
\]

The sign, as a result, depends on the sign of \( A \) and \( \frac{\partial A}{\partial q_u} \). Using assumption (C.183) (which implies \( A > 0 \)), the first term in (C.214) is positive.

Consider \( \frac{\partial A}{\partial q_u} \). Solving for \( q_s \) from (C.209):

\[
q_s = \frac{(r_d + \frac{1}{2} q_u) r_d O_h}{\frac{1}{2} (x_h - r_d O_h) \lambda_b}. \tag{C.215}
\]
and substituting it in the expression for $A$, we get:

$$A = x_h + x_l - 2y - r_d O_h - \left( \frac{(2y - x_h) \gamma_u}{r_d (O_h (r_d + q_u) + x_h)} + \frac{r_d + \gamma_u}{q_u + 2r_d} \right) (x_h - r_d O_h).$$

Using $x_h - 2y - r_d O_h < 0$ and $x_h - r_d O_h > 0$ from Assumption (1), $A$ increases in $q_u$, and the second term in (C.214) is also positive. Put together, we get

$$\frac{\partial V(0,v_1)}{\partial q_u} > 0.$$  

Since $q_u = \lambda v_1$ also strictly increases in $\nu_1$, we have:  

$$\frac{\partial V(0,v_1)}{\partial q_u} > 0.$$  

Using the above limits in (C.174), the sign of $\omega_r$ depends on the sign of the right-hand side of

$$q_v \left( \frac{\omega_r}{2} \right) = \frac{q_v}{r_d + \gamma_u + q_u} \left[ x_l + \frac{r_d + \gamma_u + q_u}{r_d + q_u} \left( -2y + \frac{r_d + q_u}{r_d + q_u \frac{1}{2} + q_u \frac{1}{2}} \right) \right].$$

(C.216)

The sign depends on the expression inside the square brackets. Since the solution for $\nu_1$ is such that

$$V_{[0,0]}(\nu_1) = \frac{1}{r + \gamma_u (r_d + \gamma_u + q_u)} \left[ x_l + \frac{r_d + \gamma_u + q_u}{r_d + q_u} \left( -2y + \frac{r_d + q_u}{r_d + q_u \frac{1}{2} + q_u \frac{1}{2}} \right) \right] \geq O_1 > 0$$

It has to be that the expression inside the square brackets is positive. 

\[ \text{Proof of Proposition C.1.3: } \lim_{\lambda \to \infty} q_v \omega_r > 0. \]

Proof of Proposition C.1.4: \( \lim_{\lambda \to \infty} \omega_b > 0. \)

Recall that

$$\omega_b = \frac{x_h + q_v \frac{1}{2} \omega_r}{(r + \gamma_d + q_v \frac{1}{2} + q_v \frac{1}{2})},$$

(C.217)

Using the fact that \( \lim_{\lambda \to \infty} \omega_r = 0 \) and \( \lim_{\lambda \to \infty} q_v = 0 \),

$$\lim_{\lambda \to \infty} \omega_b = \frac{x_h}{(r + \gamma_d + q_v \frac{1}{2} + q_v \frac{1}{2})},$$

(C.218)

This is positive.

\[ \text{Proof of Proposition C.1.5} \quad -V_a(0,1) > V_b(1,0) - V_b(0,0). \] I will show

$$-V_a(0,1) - p_b > V_b(1,0) - V_b(0,0) - p_b.$$

(C.219)

Since \( V_b(1,0) - V_b(0,0) - p_b = \frac{1}{2} \omega_b \), showing (C.219) is equivalent to showing:

$$-V_a(0,1) - p_b > \frac{1}{2} \omega_b.$$

(C.220)
Solving \( V_{a[0,-1]} \) from (C.140) and using the fact that \( c = V_{a[1,0]} \),

\[
-V_{a[0,-1]} - p_b = \frac{fe+ y + \tilde{\delta} - (p_b - V_{a[1,0]}) \gamma_d - p_b r}{r + q_s + \gamma_d}.
\]

Thus, showing (C.220) amounts to showing:

\[
\frac{fe+ y + \tilde{\delta} - \frac{1}{2} \omega_b \gamma_d - p_b r}{r + q_s + \gamma_d} > \frac{1}{2} \omega_b. \tag{C.221}
\]

Rearranging (C.221), we get:

\[
rp_b < \tilde{\delta} + y + fee - \frac{1}{2} \omega_b (r + q_s). \tag{C.222}
\]

The left-hand side of (C.222) converges to

\[
rp_b = \tilde{\delta} + x_h - y - r_d V_{h[0,0]} - \left( \frac{r + 2 \gamma_d}{2} \right) \omega_b.
\]

The right-hand side of (C.222) converges to

\[
\tilde{\delta} + y - \gamma_d \omega_b - \frac{1}{2} \omega_b (r + q_s).
\]

Substituting the limits into (C.222) and using the fact that at the limit, \( r_d V_{h[0,0]} = q_s \frac{1}{2} \omega_b \), (C.222) is equivalent to

\[
\frac{x_h - 2y}{r + \gamma_d + q_s} < 0.
\]

Assumption (1) does not necessarily imply that \( x_h - 2y < 0 \). I assume this additional parameter condition holds. Then, \( -V_{a[0,-1]} > V_{h[1,0]} - V_{h[0,0]} \). \( \square \)

**Proof of Proposition C.1.6**

\( V_{a[1,0]} > V_{l[1,-1]} - V_{l[0,-1]} \). Showing

\[
V_{a[1,0]} > V_{l[1,-1]} - V_{l[0,-1]}, \tag{C.223}
\]

ensures that \( l[1,-1] \) type finds it optimal to sell the bond at \( p_b \) given by (C.142). Inequality (C.223) is equivalent to

\[
-V_{a[1,0]} + p_b < V_{l[0,-1]} - V_{l[1,-1]} + p_b. \tag{C.224}
\]

Combining the value functions of \( l[0,-1] \) and \( l[1,-1] \), the right-hand side of (C.224) is

\[
V_{l[0,-1]} - V_{l[1,-1]} + p_b = \frac{rp_b - (\tilde{\delta} - x_l) - y + \gamma_a (p_b + V_{a[0,-1]}) + \gamma_d (p_b - V_{a[1,0]})}{(r + q_s + \gamma_a + \gamma_d)}.
\]
By assumption (C.185), the expression inside the square bracket is positive. Thus,

\[
\frac{rp_b - (\delta - x_l) - y + \gamma_u (p_b + V_{a[0,1]}) + \gamma_d (p_b - V_{a[1,0]})}{(r + q_n + \gamma_u + \gamma_d)} > p_b - V_{a[1,0]}.
\]

That is,

\[
rp_b - (\delta - x_l) - y + \gamma_u (p_b + V_{a[0,1]}) > (p_b - V_{a[1,0]}) (r + q_n + \gamma_u).
\]  \hspace{1cm} (C.225)

Using the fact \( p_b + V_{a[0,1]} = \frac{rp_b - \delta - y - fee + \gamma_d (p_b - V_{a[1,0]})}{r + q_n + \gamma_d} \), (C.225) is equivalent to

\[
rp_b - (\delta - x_l) - y + \gamma_u \left( \frac{rp_b - \delta - y - fee + \gamma_d (p_b - V_{a[1,0]})}{r + q_n + \gamma_d} \right) > \frac{1}{2} \omega_b (r + q_n + \gamma_u).
\]

Multiply both sides by \( r + q_n + \gamma_d \):

\[
(rp_b - (\delta - x_l) - y) (r + q_n + \gamma_d) + \gamma_u \left( \frac{rp_b - \delta - y - fee + \gamma_d \frac{1}{2} \omega_b}{r + q_n + \gamma_d} \right) > \frac{1}{2} \omega_b (r + q_n + \gamma_u) (r + q_n + \gamma_d).
\]  \hspace{1cm} (C.226)

Substituting in the bond price, which is

\[
rp_b = \delta + x_h - y - r_d V_{h[0,1]} - (r + 2 \gamma_d) \frac{1}{2} \omega_b
\]

\[
= \delta + x_h - y - (q_n + r_d + \gamma_d) \frac{1}{2} \omega_b,
\]

(C.226) becomes:

\[
\left( x_h + x_l - 2y - (q_n + r_d + \gamma_d) \frac{1}{2} \omega_b \right) (r_d + q_n) + \gamma_u \left( x_h - 2y - (q_n + r_d + \gamma_d) \frac{1}{2} \omega_b - fee + \gamma_d \frac{1}{2} \omega_b \right) > \frac{1}{2} \omega_b (r + q_n) (r_d + q_n).
\]

Using \( \lim_{\lambda_0 \to \infty} fee = 0 \) and simplifying it yields:

\[
(r_d + q_n) (x_h + x_l - 2y) - (2y - x_h) \gamma_u - \frac{1}{2} (r_d + q_n) (q_n + q_n + r_d + r_u + \gamma_d + \gamma_u) \omega_b > 0.
\]

Substitute in \( \omega_b \) (given in (C.218)) and re-express the inequality as

\[
(r_d + q_n) \left( x_l + \frac{r_d + \gamma_u + q_n}{r_d + q_n} \left( -2y + \frac{\gamma_u (q_n - q_n)}{r_d + \gamma_u + q_n} \omega_b \right) \right) > 0.
\]

By assumption (C.185), the expression inside the square bracket is positive. Thus, \( V_{a[1,0]} > V_{l[1,-1]} - V_{l[0,1]} \).
Proof of Proposition C.1.7 \( p_b - V_{a[1,0]} > V_{a[0,1][1,-1]} - V_{a[1,0]} \). This ensures that an average-valuation investor prefers to unwind her long position than to keep the long position. If \( a[1,0] \) sells the bond, the change in her utility is \( p_b - V_{a[1,0]} \). If \( a[1,0] \) instead lends (a conjectured off-equilibrium strategy), the change in her utility is \( V_{a[0,1][1,-1]} - V_{a[1,0]} \). Consider the value functions associated with the off-equilibrium trading strategy:

\[
rV_{a[1,0]} = \bar{\delta} - y + q_{\text{no}} (V_{a[0,1][1,-1]} - V_{a[1,0]})
\]

(C.227)

\[
rV_{a[0,1][1,-1]} = \bar{\delta} + \text{fee} - y + \gamma_u (V_{a[1,0]} - V_{a[0,1][1,-1]}) + q_{\text{b}} (V_{a[0,1][0,-1]} - V_{a[0,1][1,-1]})
\]

(C.228)

\[
rV_{a[0,1][0,-1]} = \bar{\delta} - y + \text{fee} + \gamma_u (V_{a[1,0]} - V_{a[0,1][0,-1]})
\]

(C.229)

\[
rV_{a[0,1][a,0,-1]} = \bar{\delta} - y + \text{fee} + \gamma_u (V_{a[1,0]} - V_{a[0,1][a,0,-1]})
\]

(C.230)

Taking the difference between (C.228) and (C.227),

\[
V_{a[0,1][1,-1]} - V_{a[1,0]} = \frac{\text{fee} + q_{\text{u}} (V_{a[0,1][0,-1]} - V_{a[0,1][1,-1]})}{r + \gamma_u + q_{\text{no}}}
\]

(C.231)

In turn, consider \( V_{a[0,1][0,-1]} - V_{a[0,1][1,-1]} \) by combining (C.228) and (C.229) and simplifying:

\[
V_{a[0,1][0,-1]} - V_{a[0,1][1,-1]} = \gamma_u \frac{\text{fee} + (r + \gamma_u + q_{\text{no}}) (V_{a[0,1][a,0,-1]} - V_{a[0,1][0,-1]})}{(r + q_{\text{b}}) (r + q_{\text{no}}) + r \gamma_u}
\]

(C.232)

Now consider \( V_{a[0,1][a,0,-1]} - V_{a[0,1][0,-1]} \) using (C.230) and (C.229):

\[
V_{a[0,1][a,0,-1]} - V_{a[0,1][0,-1]} = \frac{q_{\text{a}}}{r + \gamma_u} (V_{a[1,0]} - V_{a[0,1][a,0,-1]})
\]

(C.233)

where, from (C.227) and (C.229), \( V_{a[1,0]} - V_{a[0,1][a,0,-1]} \) is,

\[
V_{a[1,0]} - V_{a[0,1][a,0,-1]} = \frac{q_{\text{no}} (V_{a[0,1][1,-1]} - V_{a[1,0]}) - \text{fee}}{r + q_{\text{a}}}
\]

(C.234)

Substituting (C.232)-(C.234) back into (C.231) and simplifying, we get

\[
V_{a[0,1][1,-1]} - V_{a[1,0]} = \frac{(r + q_{\text{a}}) r_{\text{a}} + q_{\text{b}} (r_{\text{a}} + q_{\text{b}}) \text{fee}}{(r + q_{\text{a}}) r_{\text{a}} + q_{\text{b}} (r_{\text{a}} + q_{\text{b}}) + q_{\text{no}} ((r + q_{\text{no}}) r_{\text{a}} + q_{\text{b}} (r_{\text{a}} + q_{\text{no}}))}
\]

Since the limits of \( q_{\text{a}}, q_{\text{b}}, \) and \( q_{\text{no}} \) are finite, and the fee converges to zero, the right-hand side converges to zero. Thus, \( \lim_{\lambda \to \infty} p_b - V_{a[1,0]} > \lim_{\lambda \to \infty} V_{a[0,1][1,-1]} - V_{a[1,0]} = 0 \).

Proof of Proposition C.1.8 An average type does not prefer a short position.

In particular, I show that an average type investor prefers to buy back and deliver the bond than keep the short position. The change in her utility if she buys back the bond and delivers it is \( 0 - p_b - V_{a[0,1]} \). If she keeps the short position, the change in her utility is zero. Thus, she prefers to buy back and deliver if \( 0 - p_b - V_{a[0,1]} > 0 \). This holds as shown in the proof of Proposition C.1.2.
Proof of Proposition 10.a. From (C.158), \(0 < \lim q_{ho} < \infty\). Since \(\mu_{h[1,0]} < S\), \(q_{LE} = \lambda_r \mu_{h[1,0]} < \infty\) for \(\lambda_r < \infty\). But both \(q_{LE}\) and \(q_{ho}\) are finite.

I now show that both \(\lim_{\lambda_r \to \infty} q_{ho} = \infty\) and \(\lim_{\lambda_r \to \infty} q_{ho} = \infty\). I prove by contradiction. Suppose \(\lim_{\lambda_r \to \infty} q_{ho} = \infty\) and \(\lim_{\lambda_r \to \infty} q_{ho} = \infty\). Then, \(V_{h[0,0]}\) is zero asymptotically, and \(\nu_h = 0\). Since \(\nu_h \in [0,1]\), from (C.157), \(\mu_{h[0,0]} \to 0\). Combine (C.159) and (C.160),

\[
\mu_a = \mu_{h[0,0]} + \frac{\gamma_u}{q_s + \gamma_d} \frac{\mu_a}{\gamma_u + \gamma_d} \left( q_s - \frac{\gamma_d S}{e_b \gamma_d + \mu_a} \right).
\]

(C.235)

Using \(\mu_{h[0,0]} = 0\) in (C.235),

\[
\mu_a = \frac{\gamma_u}{q_s + \gamma_d} \frac{\mu_a}{\gamma_u + \gamma_d} \left( q_s - \frac{\gamma_d S}{e_b \gamma_d + \mu_a} \right).
\]

Using \(q_s < \infty\),

\[
\mu_a \left( q_s + \gamma_d \right) \gamma_u + \mu_a \left( q_s + \gamma_d \right) \gamma_d = \gamma_u \mu_a q_s - \gamma_u \mu_a \frac{\gamma_d S}{e_b \gamma_d + \mu_a}.
\]

\[
\gamma_u \gamma_d + \left( q_s + \gamma_d \right) \gamma_d = -\gamma_u \frac{\gamma_d S}{e_b \gamma_d + \mu_a}.
\]

This is a contradiction as the left-hand size is finite and positive, while the right-hand is negative when \(q_s < \infty\).

Suppose both \(q_s < \infty\) and \(q_a < \infty\). From (C.160),

\[q_s = \frac{\gamma_d S}{e_b \gamma_d + \mu_a} + \frac{q_{ho} \left( S - e_b q_s \right)}{e_b \left( \gamma_u + \gamma_d \right) + \mu_a}.
\]

Set \(e_b = 0\):

\[q_s = \frac{\gamma_d S}{\mu_a} + \frac{q_{ho} \left( S \right)}{\mu_a}.
\]

Since \(\mu_a \to 0\) (from the conjecture that \(\lim_{\lambda_r \to \infty} q_{ho} < \infty\)), \(q_s \to \infty\). This a contradiction. Thus, it has to be that \(q_s \to \infty\).

Consider the limit of \(q_a\). If \(\lim_{\lambda_r \to \infty} q_a < \infty\), from

\[r_d V_{h[0,0]} = \frac{1}{2} r x_h + q_{ho} \frac{1}{2} \omega_r = x_h + q_{ho} \frac{1}{2} \omega_r > r_d O_h + q_{ho} \frac{1}{2} \omega_r.
\]

So \(\lim_{\lambda_r \to \infty} q_a < \infty\) implies that \(\nu_h = 1\). To simplify the derivations, I instead focus on the interior solution. For \(\nu_h\) to be given by an interior solution, \(q_a\) has to converge to \(\infty\) and, in particular, at a rate such that

\[
\frac{q_s \frac{1}{2}}{r + \gamma_d + q_s \frac{1}{2} + q_a \frac{1}{2}} \left( x_h + q_{ho} \frac{1}{2} \omega_r \right) = r_d O_h.
\]

(C.236)
Now, I derive the limit of $\omega_r$. From (C.236),

$$q_u = \frac{-O_h r_d \left( r + \frac{1}{2}q_b + \gamma_d \right) + \frac{1}{2}q_b \left( x_h + \frac{1}{2}q_{bo}\omega_r \right)}{\frac{1}{2}O_h r_d}.$$  

Substitute it into (C.174) and subtract the right-hand side of (C.174) from the left-hand side. Then, taking the limit as $q_s \to \infty$, we get

$$2y + O_h r_d - x_h - x_l + \left( r_d + \gamma_u + \frac{1}{2}q_{le} \right) \omega_r = 0.$$  

Solving for $\omega_r$:

$$\omega_r = \frac{x_h + x_l - 2y - r_d O_h}{r_d + \gamma_u + \frac{1}{2}q_{le}}.$$  

This is positive.

Now, I derive $q_{le}$ and $V_{l[0,0]}$. Solving for $\mu_s$ from (C.147),

$$\mu_s = \frac{S \left( \gamma_d + \frac{q_{uo} q_{uo}}{q_b + \frac{q_{uo} q_{uo}}{\gamma_d + \gamma_u}} \right)}{q_u + \gamma_d}.$$  

Taking its limit, $\mu_s \to 0$. In turn, $\mu_h[1,0] \to S$ and $\lim_{\lambda_u \to \infty} q_{le} = \lambda_r S > 0$. Plug this back into the value function of $l[0,0]$:

$$V_{l[0,0]} = \frac{1}{r_u} \frac{1}{2} \frac{x_h + x_l - 2y - r_d O_h}{r_d + \gamma_u + \frac{1}{2}q_{le}} = \frac{1}{r_u} \frac{1}{2} \frac{x_h + x_l - 2y - r_d O_h}{r_d + \gamma_u + \frac{1}{2}\lambda_r S}.$$  

Thus, as long as

$$\frac{1}{r_u} \frac{1}{2} \frac{x_h + x_l - 2y - r_d O_h}{r_d + \gamma_u + \frac{1}{2}\lambda_r S} \geq O_l,$$

the entry rate of low-valuation investors is positive: $\nu_l > 0$.

Since $\omega_b \to 0$,

$$\lim_{\lambda_u \to \infty} p_h = \frac{\delta + x_b - y - (r + \gamma_d)O_h}{r} + q_{uo} \frac{1}{r} \omega_r.$$  

(C.237)

Without the repo market,

$$\lim_{\lambda_u \to \infty} \hat{p}_b = \frac{\delta + x_b - y - (r + \gamma_d)O_h}{r}.  \tag{C.238}$$  

Above, I showed that $q_{uo} \frac{1}{r} \omega_r > 0$. The bond price, as a result, is higher in the presence of bond shorting: $\lim_{\lambda_u \to \infty} p_b > \lim_{\lambda_u \to \infty} \hat{p}_b$.  

61
Consider now the effect on bond volume. Setting $e_b = 0$ in (C.160),

$$q_b = \frac{\gamma_d S}{\mu_b} + \frac{q_{bo} (S - \mu_s)}{\mu_b}.$$  

Multiply both sides by $\mu_b$ and that $\mu_s \neq 0$,

$$q_b \mu_b = \gamma_d S + q_{bo} S.$$  

Solving for $q_{bo}$ from (C.158) and using $\mu_s \neq 0$,

$$q_{bo} = \frac{\nu_l F_l}{\gamma_u} \frac{\gamma_u + \gamma_d}{s + e_r (\gamma_u + \gamma_d)}.$$  

Combining the bond volume converges to:

$$q_b \mu_b = \gamma_d S + \frac{\nu_l F_l}{\gamma_u} \frac{\gamma_u + \gamma_d}{s + e_r (\gamma_u + \gamma_d)} S.$$  

In the absence of the repo market, the bond volume converges to $\gamma_d S$. The second term is positive. Bond shorting, as a result, increases the trading volume in the bond market. □

**Proof of Proposition 10.b.** As $\lambda_r \to \infty$, the bond price with repo limits to

$$\lim_{\lambda_r \to \infty} r p_b = \bar{\delta} + x_h - y - (r + \gamma_d) O_h - \left(\frac{r + 2\gamma_d}{2}\right) \omega_b.$$  

(C.239)

The bond price without repo is

$$r \hat{p}_b = \bar{\delta} + x_h - y - (r + \gamma_d) O_h - \left(\frac{r + 2\gamma_d}{2}\right) \hat{\omega}_b.$$  

(C.240)

In the limit then, the characterization of the bond price is the same with and without repo.

Recall that

$$\lim_{\lambda_r \to \infty} \omega_b = \frac{x_h}{(r + \gamma_d + q_{b,1}^2 + q_{b,2}^2)}.$$  

(C.241)

Thus, to show that the bond price is higher with repo, it is sufficient to show that both $\mu_b$ (or, equivalently, $q_b$) and $\mu_s$ (or, equivalently, $q_s$) increase with repo so that $\lim_{\lambda_r \to \infty} \omega_b < \hat{\omega}_b$.

As discussed in the proof of Proposition C.1, the variables $\mu_b$ and $q_b$ are implicit functions of $\nu_l$, and their solution evaluated at $\nu_l > 0$ and $\nu_l = 0$ correspond to $\mu_b$ and $q_b$ in the presence and absence of bond shorting, respectively. The proof of Proposition C.1 shows that $\mu_b$ strictly increases in $\nu_l$ for all $\nu_l \geq 0$. In turn, from (C.215), $q_b$ strictly increases in $\mu_b$ and, hence, in $\nu_l$. Thus, both $\mu_s$ and $\mu_b$ increase in $\nu_l$ and, thereby, in the presence of bond shorting.

The bond volume also increases in the presence of bond shorting because the trading volume is the product of $\mu_b$ and $q_b$. □
D  Covered CDS

Proof of Lemma 2. This is a corollary from Proposition 2 proof (in particular, Step 3). □

E  Endogenous Search Efforts

I setup the environment so that all investors who are looking to rebalance their asset position choose their search effort optimally. In particular, in addition to long investors, bond sellers, \(a[1,0]\), and CDS buyers, \(l[0,0]\), also choose their search effort. As a result, the total bond volume is \(M_b = (\lambda_{b,s} + \lambda_{b,b})\mu_{b,s}\mu_{b,b}\), while the CDS volume is \(M_c = (\lambda_{c,s} + \lambda_{c,b})\mu_{c,s}\mu_{c,b}\). Then, only for the proof of Proposition 11, I set the search efforts of the short size to zero: \(\lambda_{l[0,0]} = 0\) and \(\lambda_{a[1,0]} = 0\).

In the paper, I focus on the simpler environment with only the long side choosing search efforts because endogenizing search efforts on both sides of the market complicates derivations significantly. Given the intractability, in Proposition E.1, I only characterize the parameter conditions under which introducing CDS still increases bond market liquidity. Moreover, the additional complication does not seem to come with any added benefit. Numerically, the results when both sides choose search intensities versus when only one side chooses are analogous.

The HJB equations can be derived analogously as in the environment with exogenous search intensities. As a result, the value functions are characterized by:

\[
\begin{align*}
    rV_{\tau} &= -c(\lambda_{b,\tau}, \lambda_{c,\tau}) + ((\delta - \eta J) - x_{\tau})\theta_b - y|\theta_b| + (p_c - (\eta J + x_{\tau}))\theta_c - y|\theta_c| \\
    &+ \sum_{k=1}^{K(\tau)} \gamma(k, \tau) \max_{\tau' \in T(\tau, k) \tau' \alpha} \left( 1 - e^{-\gamma c(V_{\tau'} - V_{\tau} + P(\tau, \tau'))} \right),
\end{align*}
\]

where \(c(\lambda_{b,\tau}, \lambda_{c,\tau})\) is the agent’s total search cost. Simplifying (E.242) further, for the non-searcher agent types, the value functions are identical to (A.30), (A.32), and (A.33). The value functions of the searcher agents include the costs of the search efforts:

\[
\begin{align*}
    rV_{l[0,0]} &= \gamma_a(0 - V_{l[0,0]}) - c(0, \lambda_{l[0,0]}) + \frac{M_c}{\mu_{l[0,0]}} \frac{1}{2} \omega_c \\
    rV_{h[0,0]} &= \gamma_d(0 - V_{h[0,0]}) - c(\lambda_{b,h[0,0]}, \lambda_{c,h[0,0]}) + \frac{M_b}{\mu_{h[0,0]}} \frac{1}{2} \omega_b + \frac{M_c}{\mu_{h[0,0]}} \frac{1}{2} \omega_c. \\
    rV_{a[1,0]} &= (\delta - \eta J) - y - c(\lambda_{a[1,0]}, 0) + \frac{M_b}{\mu_{a[1,0]}} \frac{1}{2} \omega_b.
\end{align*}
\]

The volume of matches in the bond and the CDS market are

\[
\begin{align*}
    M_b &= (\lambda_{b,h[0,0]} + \lambda_{a[1,0]})\mu_{a[1,0]}\mu_{h[0,0]} \\
    M_c &= (\lambda_{c,h[0,0]} + \lambda_{l[0,0]})\mu_{l[0,0]}\mu_{h[0,0]}
\end{align*}
\]

The first order conditions with respect to the search efforts are:
2c_0 \lambda_{b,h[0,0]} = \mu_{a[1,0]} \left( -p_b + V_{h[1,0]} - V_{h[0,0]} \right) \tag{E.246}
\begin{equation}
2c_0 \lambda_{c,h[0,0]} = \mu_{h[0,0]} \left( V_{h[0,1]} - V_{h[0,0]} \right) \tag{E.247}
\end{equation}
\begin{equation}
2c_0 \lambda_{l[0,0]} = \mu_{h[0,0]} \left( V_{l[0,1]} - V_{l[0,0]} \right) \tag{E.248}
\end{equation}
\begin{equation}
2c_0 \lambda_{a[1,0]} = \mu_{h[0,0]} \left( p_b - V_{a[1,0]} \right) \tag{E.249}
\end{equation}

The equilibrium equations are analogous to the baseline environment plus (E.246)-(E.249) that pin down the optimal search efforts.

**Proof of Lemma 3.** Combining (A.30) and (E.244), consider the reservation value of the buyer:

\[ r \left( V_{h[1,0]} - V_{h[0,0]} \right) = (\delta - \eta J + x - y) - \gamma_d \omega_b + c(\lambda_{b,h[0,0]}, \lambda_{c,h[0,0]}) - q_{bs} \frac{1}{2} \omega_b - q_{cn} \frac{1}{2} \omega_c. \]

where \( q_{bs} = \frac{M_b}{\mu_{h[0,0]}} \) and \( q_{cn} = \frac{M_c}{\mu_{h[0,0]}} \). Using (E.244), we can write it as

\[ r \left( V_{h[1,0]} - V_{h[0,0]} \right) = (\delta - \eta J + x - y) - (r + \gamma_d) V_{h[0,0]} - \gamma_d \omega_b. \]

Combining (A.30), (E.244), and (A.31), we get

\[ (r + \gamma_d) \omega_b = x + c(\lambda_{b,h[0,0]}, \lambda_{c,h[0,0]}) + c(\lambda_{a[1,0]}, 0) - q_{bs} \frac{1}{2} \omega_b - q_{bs} \frac{1}{2} \omega_b - q_{cn} \frac{1}{2} \omega_c. \tag{E.250} \]

where \( q_{bs} = \frac{M_b}{\mu_{a[1,0]}} \). Using (E.244), this becomes

\[ (r + \gamma_d) \omega_b = x - (r + \gamma_d) V_{h[0,0]} + c(\lambda_{a[1,0]}, 0) - q_{bs} \frac{1}{2} \omega_b. \tag{E.251} \]

From (E.251),

\[ q_{bs} \frac{1}{2} \omega_b - c(\lambda_{a[1,0]}, 0) = x - (r + \gamma_d) \omega_b - (r + \gamma_d) V_{h[0,0]} \]

Plug it into the reservation value of the seller and get:

\[ r V_{a[1,0]} = \delta - \eta J + x - y - (r + \gamma_d) V_{h[0,0]} - (r + \gamma_d) \omega_b. \]

Thus, combining the reservation values of the bond buyer and the seller, the characterization of the bond price is the same as in the environment with exogenous search efforts. \( \square \)

The above results holds for both the environment where only the long side searches (i.e. \( \lambda_{b,h[0,0]} \) and \( \lambda_{c,h[0,0]} \) are endogenous, and \( \lambda_{a[1,0]} = 0 \) and \( \lambda_{l[0,0]} = 0 \)) and the environment where both sides of the market search (i.e. \( \lambda_{b,h[0,0]}, \lambda_{c,h[0,0]}, \lambda_{a[1,0]}, \lambda_{l[0,0]} \) are endogenous). In Proposition 11, only the long side chooses its search efforts. Then, Proposition E.1 allows both sides to choose their search effort and characterizes the parameter.
Proof of Proposition 11. Consider the set of three equations and three unknowns \( \{ \mu_{a[1,0]}, \mu_{b[0,0]}, \omega_b \} \):

\[
\frac{1}{4c_0} \mu_{h[0,0]} \mu_{a[1,0]}^2 \omega_b = \gamma_d (S - \mu_{a[1,0]}) \tag{E.252}
\]

\[
(r + \gamma_d) \omega_b = \frac{\mu_{a[1,0]} + \mu_{h[0,0]} + 2 \mu_{h[0,0]} \omega_b}{16c_0} - A \tag{E.253}
\]

\[
(r + \gamma_d) O_h = \frac{(\mu_{a[1,0]} \omega_b)^2}{16c_0} + A, \tag{E.254}
\]

where (E.252) comes from combining the inflow-outflow equations with (E.246), (E.253) comes from combining (E.250) with (E.246), (E.254) combines (E.244) with (E.246)- (E.247), and

\[
A = \frac{M_c}{\mu_{h[0,0]}} \frac{1}{2} \omega_c - c_0 (\lambda_{c,h[0,0]})^2.
\]

With and without CDS, \( \{ \mu_{a[1,0]}, \mu_{b[0,0]}, \omega_b \} \) is the solution to (E.252)–(E.254) with \( A > 0 \) and \( A = 0 \), respectively. Applying the Implicit Function Theorem,

\[
\frac{\partial \omega_b}{\partial A} = -\frac{4c_0 (4c_0 \gamma_d + \mu_{a[1,0]} \mu_{h[0,0]} \omega_b)}{\mu_{a[1,0]} \omega_b (2c_0 (2r_d + \gamma_d) + \mu_{a[1,0]} \mu_{h[0,0]} \omega_b)}
\]

\[
\frac{\partial \mu_{a[1,0]}}{\partial A} = -\frac{4c_0 (8c_0 (r + \gamma_d) + \mu_{a[1,0]} \mu_{h[0,0]} \omega_b)}{\mu_{a[1,0]} \omega_b^2 (2c_0 (2(r + \gamma_d) + \gamma_d) + \mu_{a[1,0]} \mu_{h[0,0]} \omega_b)}
\]

Thus, \( \omega_b \) decreases in the presence of the CDS market which means the bond illiquidity discount decreases, and the bond price increases. The bond volume is given by: \( M_b = \gamma_d (S - \mu_{a[1,0]}) \). Since the mass of bond sellers also decreases, it necessarily means that the bond volume increases.

Since both \( \omega_b \) and \( \mu_{a[1,0]} \) decrease with the introduction of CDS, a direct corollary is that, for a long investor, the marginal benefit of searching in the bond market decreases. A long investor, as a result, lowers its search effort in the bond market. \( \square \)

Proposition E.1. Consider the environment where both sides of the market choose search efforts (i.e. \( \lambda_{b,h[0,0]}, \lambda_{c,h[0,0]}, \lambda_{a[1,0]}, \text{ and } \lambda_{h[0,0]} \) are endogenous). Suppose (E.258) holds. Then, \( d_b \leq \hat{d}_b \), and \( M_b > M_b \).

Proof. Now the equations equivalent to (E.252)–(E.254) are

\[
\frac{1}{4c_0} (\mu_{a[1,0]} + \mu_{h[0,0]} \mu_{b[0,0]} \omega_b) = \gamma_d (S - \mu_{a[1,0]}) \tag{E.255}
\]

\[
(r + \gamma_d) \omega_b = \frac{\mu_{a[1,0]} + \mu_{h[0,0]} + 2 \mu_{h[0,0]} \omega_b}{16c_0} - A \tag{E.256}
\]

\[
(r + \gamma_d) O_h = \frac{\mu_{a[1,0]} (\mu_{a[1,0]} + \mu_{h[0,0]} \omega_b)^2}{16c_0} + A \tag{E.257}
\]
where (E.252) comes from combining the inflow-outflow equations with (E.246), (E.253) comes from combining (E.250) with (E.246), (E.254) combines (E.244) with (E.246)-(E.247), and
\[ A = \frac{M_c}{\mu_h[0,0]} \frac{1}{2} \omega_c - c_0(\lambda_{c,h[0,0]})^2. \]

Applying the Implicit Function Theorem:
\[
\begin{align*}
\frac{\partial \mu_{a[1,0]}}{\partial A} &= -\frac{4c_0 \left( 8c_0 \left( \mu_{a[1,0]} + 2\mu_{h[0,0]} \right) r_d + \mu_{h[0,0]} \left( \mu_{a[1,0]}^2 + 3\mu_{a[1,0]} + \mu_{h[0,0]}^2 \right) \omega_b \right)}{B}, \\
\frac{\partial \omega_b}{\partial A} &= -\frac{4c_0 \left( 4c_0 \left( \mu_{a[1,0]} + \mu_{h[0,0]} \right) \gamma_d + \mu_{h[0,0]} \left( \mu_{a[1,0]}^2 + \mu_{a[1,0]} + \mu_{h[0,0]}^2 \right) \omega_b \right)}{B},
\end{align*}
\]

where
\[ B \equiv -16c_0^2 r_d \gamma_d \omega_b + 2c_0 \left( \mu_{a[1,0]}^2 + \mu_{a[1,0]} \mu_{h[0,0]} + \mu_{h[0,0]}^2 \right) \left( 2r_d + \gamma_d \right) \omega_b^2 + \mu_{h[0,0]} \left( \mu_{a[1,0]}^2 + \mu_{a[1,0]} \mu_{h[0,0]} + \mu_{h[0,0]}^2 \right) \omega_b^3. \]

Thus, \( \frac{\partial \omega}{\partial A} < 0 \) and \( \frac{\partial \mu_{a[1,0]}}{\partial A} < 0 \) if \( B > 0 \). (E.258)

\[ \square \]

References


66


Oehmke, Martin, and Adam Zawadowski, 2013, Synthetic or real? The equilibrium effects of credit default swaps on bond markets, Working paper, Columbia University.


Shen, Ji, Bin Wei, and Hongjun Yan, 2015, Financial intermediation chains in a search market, Working paper.


Thompson, James R, 2007, Credit risk transfer: To sell or to insure, Working paper, Queen’s University.

