Linking Cross-Sectional and Aggregate Expected Returns

Serhiy Kozak† Shrihari Santosh‡

January 27, 2016

Abstract

We argue that in a general equilibrium model, increases in equity risk premia represent “bad” news to investors because they reveal increases in marginal utility. We employ a new empirical methodology and find that the price of “discount rate” risk is negative in the data, contrary to prior research. Our approach relies on using future realized market returns to consistently estimate covariances of asset returns with the market risk premium. Covariances drive observed cross-sectional patterns in the expected returns of stocks and bonds. Ignoring the “risk-premium” factor causes drastic underestimation of the equilibrium price of “level” risk in bond returns.

---

*We thank John Cochrane, Eugene Fama, Stefano Giglio, Valentin Haddad, Lars Peter Hansen, John Heaton, Bryan Kelly, Ralph Koijen, Stefan Nagel, Diogo Palhares, Jay Shanken, and seminar participants at Chicago, Emory, Maryland, Michigan, UIUC, Wisconsin for helpful comments and suggestions.

†Ross School of Business, University of Michigan. Email: sekozak@umich.edu.

‡R.H. Smith School of Business, University of Maryland. Email: shrihari@umd.edu.
1 Introduction

The investment opportunity set is obviously stochastic. Risk-free rates, risk premia, variances, and other moments of return distributions are volatile and predictable. The logic of Merton (1973) suggests there exists a representation for the stochastic discount factor which is a linear combination of the return on the aggregate wealth portfolio and shocks to these return moments. This analysis is, however, incomplete. Without specifying the primitive sources of time-variation, the ICAPM cannot predict the sign of the price of risk related to a particular shock. We propose and test a general equilibrium model in which shocks to risk aversion generate time-variation in the equity risk premium and the price of risk associated with the “risk premium” factor is unambiguously negative. We resolve the issue of unobservability of expected returns using a novel approach and obtain results that contrast with previous research. Our estimates lend support to the model’s prediction that such shocks represent “bad” news for the representative investor and manifest in the cross-section of expected asset returns.

Much of the related research following Merton (1973) takes a “reduced form” approach when applying the ICAPM to empirical work. Brennan et al. (2004) directly specify an SDF and exogenous processes for the riskless rate and the Sharpe ratio of the capital market line. Therefore their model is silent regarding the economic interpretation of these shocks. Bansal et al. (2014), Campbell et al. (2015) rely on the equilibrium form of the SDF given in Epstein and Zin (1989) based on recursive utility and stochastic volatility, but ignore market clearing restrictions. Their models predict that the expected excess return and conditional variance of the market portfolio are perfectly correlated, but this restriction is not imposed during estimation. The prediction comes directly from market clearing: if variance is time-varying, the market risk premium must move in lockstep so the representative agent always chooses to hold the market portfolio. Ignoring this restriction leads to essentially exogenous variation in discount rates. Investors view these shocks favorably since their expected investment returns increase with no offsetting “cost”.

---

1 In an SDF representation which also includes the market return.

2 This condition obtains in a special case of Bansal et al. (2014) without labor income. There is ample empirical evidence against this prediction (Bollerslev et al., 2009, Campbell et al., 2015, Greenwald et al., 2014).

3 Campbell (1996), Campbell and Vuolteenaho (2004) share this feature as well.
In contrast to these partial equilibrium approaches, our method generates *endogenous* discount rate variation and imposes market clearing. Market clearing requires that something must attenuate investors’ desire for risky assets during times of high risk premia. Time-varying risk aversion does just that in our model: equilibrium risk premia move proportionately with risk aversion, providing exactly the correct incentive for investors to always hold the market portfolio. Positive shocks to risk aversion raise the marginal utility of wealth and thus are viewed as undesirable by a representative investor. The model implies that the cross-section of expected stock returns can be explained with only two factors: excess return on the market portfolio (CAPM factor) and expected future excess market returns (“risk premium” factor). Empirically, this single additional factor is sufficient to rationalize a large cross-section of expected returns, including size, book-to-market ratio, recent performance (momentum), and the anomaly portfolios studied in Novy-Marx and Velikov (2014).

Theory alone cannot resolve the issue of whether discount rate shocks are “good” or “bad” (decrease or increase marginal utility of wealth). Analyzing empirical estimates of risk prices (SDF coefficients) and their signs provides a solution; however, another issue remains: expected returns are not observable. Others have tried to address this problem through the use of predictive regressions, usually in the form of a vector auto-regression, or VAR (see Section 1.1). The conclusions drawn from this approach rest heavily on the assumption that the econometrician and investors use the same conditioning information. This is unlikely given that academics themselves cannot even agree on whether stock returns are forecastable, let alone which predictive variables to use4. As shown in Chen and Zhao (2009a), the VAR estimates are highly sensitive to small changes in specification.

We take a different approach and use future realized returns to proxy for expected future returns (see Section 2.4). This method provides consistent estimates of factor loadings and risk prices without ever directly estimating the level of the “risk premium” factor. We rely on investors having rational expectations, whereas the VAR method additionally assumes correct specification of the state vector by the econometrician. Our methodology produces very different cross-sectional patterns in factor loadings compared to the predictive/VAR approach in Bansal et al. (2014), Brennan et al. (2004), Campbell and Vuolteenaho (2004), Campbell et al. (2015). We find that growth stocks and large firm stocks outperform value

---

4Welch and Goyal (2008) note “the profession has yet to find some variable that has meaningful and robust empirical equity premium forecasting power, both in sample and out of sample.”
stocks and small firm stocks, respectively, in response to an increase in expected market returns.

The pattern in loadings we find results in an opposite conclusion about the compensation an investor requires for bearing the risk of time-varying expected returns. Consistent with our theory, the estimated price of risk corresponding to the “risk premium” factor is negative. We conclude that an increase in the expected market return corresponds to a drop in the investor’s utility and hence an increase in her marginal utility of wealth. This implies the investor is willing to pay in order to eliminate this risk from her portfolio. Further evidence of the relation between marginal utility and the market risk premium comes from bonds. Long-term Treasury bonds have higher covariance with innovations to market discount rates than do short-term bonds. Overall, these patterns are consistent with a “flight-to-quality” interpretation where effective investor risk aversion rises, stock prices fall, bond prices rise, and “good” companies outperform “bad” companies.

Our second contribution is decomposing the average return differential between long and short term government bonds into a large positive differential due to loadings on “level risk” (interest-rate risk) and a large negative spread due to loadings on our “risk premium” factor. These net to a slightly upward sloping term structure of expected bond returns. Our results suggest that analyzing fixed income securities in isolation can lead to erroneous conclusions about bond risks and risk premia.
1.1 Literature Review

Our work is related to a large literature motivated by Merton (1973). Brennan et al. (2004), for example, directly specify an SDF and exogenous processes for the riskless rate and the Sharpe ratio of the capital market line. The latter process drives all risk premia in the model, including bond and stock risk premia. In the model, the Sharpe ratio is spanned by real bond yields. They extract this real risk premium by applying a Kalman filter to nominal bond yields and realized inflation. Their approach to recover the unobserved risk premium is thus conceptually similar to estimating the time-series regression of average bond excess returns on all yields and as such delivers estimates similar to those of Cochrane and Piazzesi (2005). In their cross-sectional tests of stock returns, the authors re-estimate prices of risk using the Fama-MacBeth two-step estimation procedure. They find that the price of “bond” discount-rate risk is positive when using size and book-to-market sorted portfolios, but turns negative using industry portfolios. Brennan et al. (2004) differs from our paper on two important dimensions. First, their model is reduced form and delivers no predictions about the sign of the price of “risk premium” risk. Second, their identification uncovers “bond” risk premia and delivers mixed results about the price of “bond” discount rate risk. We instead focus on risk premium estimates recovered from equity markets and our estimate of the price of the “risk premium” factor is unambiguously negative.

Campbell (1996), Campbell and Vuolteenaho (2004) rely on the equilibrium form of the SDF given in Epstein and Zin (1989) based on a recursive utility specification, but assume that all second moments are constant. They log-linearize the model and show the CAPM should be supplemented by one additional factor – news to stock market discount rates. Similar to Brennan et al. (2004), however, variation in discount rates in their model is fully exogenous; no equilibrium conditions are imposed to link variation in discount rates to other equilibrium quantities. Consequently, their models cannot explain why a representative investor always holds the market portfolio and does not vary his equity exposure with the level of the equity premium. Empirically, Campbell and Vuolteenaho (2004) find a positive price of discount-rate risk in their unrestricted model. We find contrasting results due to a fundamental difference in our empirical approaches. Campbell and Vuolteenaho (2004) estimate shocks to discount rates using a VAR. We believe this approach is often unreliable.

---

5 They use U.S. Treasury bonds with maturities of 3 and 6 months, and 1, 2, 3, 4, 5, and 10 years.
Different sample periods (Chen and Zhao, 2009b) and alternative definitions of the state vector (Bansal et al., 2014, Campbell and Vuolteenaho, 2004, Campbell et al., 2015, Chen and Zhao, 2009b) deliver conflicting evidence on the price of discount-rate risk. To circumvent this issue, we use future realized returns to proxy for expected future returns (see Section 2.4).

Campbell et al. (2015) continue in the spirit of Campbell (1996), Campbell and Vuolteenaho (2004), taking a microeconomic “portfolio choice” approach to analyze time varying risk premia and volatility. They relax the assumption of constant second moments and allow a single scalar process to drive the conditional variance of all state variables in their model. The direct implication of this assumption is that news to discount rates are perfectly collinear with news to volatility. This makes sense from an equilibrium (market clearing) perspective. If variance is time varying, the market risk premium must move in lockstep so the representative agent always chooses to hold the market portfolio. However, in their empirical approach, Campbell et al. (2015) ignore this restriction and find that estimated shocks to risk premia and conditional variances are nearly uncorrelated. Hence, they must attribute discount rate variation to exogenous shocks unexplained in equilibrium. Therefore their model is not one of a representative agent and general equilibrium. Indeed, they note “our model does not explain why such an agent would not vary equity exposure with the level of the equity premium.”

Bansal et al. (2014) pursue a very similar approach to that of Campbell et al. (2015), but assume that volatility shocks are homoskedastic. The estimates in Bansal et al. (2014) are essentially consistent with a long-run risks model featuring stochastic volatility of growth rate shocks. Indeed, their estimated discount rate and volatility series have 75% correlation\(^6\) though they do not directly impose market clearing. This consistency, however, raises some concern about their findings. A large literature has found that conditional return volatility has weak predictive power for excess market returns, far outweighed by other economically motivated predictive variables. For example, Bollerslev (2010) finds that high-frequency estimates of variance have negative adjusted \(R^2\) when forecasting excess returns. In contrast, the variance risk premium produces 15% \(R^2\) at annual horizon, with the likely explanation “that the premium proxies for time-varying risk aversion.” Greenwald et al. (2014) find significant time variation in the equity risk premium which is nearly uncorrelated with “traditional macroeconomic fundamentals such as dividends, earnings, consumption volatility or

\(^6\)This result holds for estimated five-year holding period expected return and variance.
broad-based macroeconomic uncertainty, and none of these other variables forecast equity premia.” The authors interpret their results as evidence of time varying risk aversion, which drives the “vast majority of short- and medium-term stock market fluctuations.”

The assumption of homoskedasticity of volatility shocks in Bansal et al. (2014) is not innocuous. It implies that cashflow and volatility news are uncorrelated and that volatility news depends only on innovations to state variables which are themselves homoskedastic (Campbell et al., 2015). As a result, Bansal et al. (2014) and Campbell et al. (2015) find contrasting empirical results. Campbell et al. (2015) find that equities have positive volatility betas and therefore volatility risk premia are negative. Bansal et al. (2014) impose a negative volatility risk premium, but estimate that equities have negative volatility betas. On the other hand, Bansal et al. (2014) find that a value-minus-growth bet has a positive beta with volatility news and therefore commands negative volatility premium. Campbell et al. (2015) disagree. Bansal et al. (2014) explain nearly all cross-sectional differences in expected returns solely by cash-flow betas. In fact, their discount-rate and volatility betas, if anything, line up with expected returns the “wrong” way. Campbell et al. (2015) find a larger role for volatility risk in explaining returns on book-to-market sorted portfolios but also find that discount-rate betas line up with expected returns the wrong way in their restricted model. In their unrestricted model, the price of discount-rate risk \( g_2 \) flips sign. In contrast to these studies, we find that nearly all cross-sectional variation in expected returns is explained by discount rate betas. Finally, the two papers produce surprisingly conflicting evidence on the correlation between discount-rate and volatility news (shocks): Campbell et al. (2015) finds it is slightly negative (-4%), while Bansal et al. (2014) find it is as high as 90% in their baseline model. Such discrepancies in key estimates suggests that the VAR methodology employed in both papers might be unreliable for the purpose of estimating shocks to volatilities and risk premia.

If Bansal et al. (2014), Campbell et al. (2015) were to impose all equilibrium restrictions, their models would force perfect correlation between news to the market risk premium and volatility. High discount-rate news therefore would mechanically translate into high volatility news and will be viewed as undesirable by a representative investor. The price of discount-

---

7 The pattern in betas exacerbates pricing errors relative to the CAPM. Their model fits the data due to the pattern in cash flow betas.

8 Both papers use essentially the same methodology and have similar state vectors.
rate risk would thus be negative, consistent with our findings. Therefore, an equilibrium model with stochastic volatility and full market clearing (for instance, a long-run-risk model such as Bansal and Yaron, 2004) is observationally equivalent to our specification for the purpose of unconditional asset pricing tests. However, the direct implication of such a model is that volatilities predict returns in the time-series. Empirical evidence suggests they do not (Bollerslev et al., 2009, Campbell et al., 2015). This is the primary reason why we prefer a specification with time-varying risk aversion instead of stochastic volatility.

Our joint estimation of bond and stock returns is related to Koijen et al. (2015). They specify an unconditional pricing model with three priced factors: return on the market portfolio, the “level factor” in interest rates, and the bond return forecasting factor (CP) from Cochrane and Piazzesi (2005). Conceptually, their model and empirical technique are quite similar to Brennan et al. (2004). The model jointly prices bond and stock returns and the estimated price of CP-factor risk is positive. Like we do, they decompose bond risk premia into three components: a positive component due to covariance with the level factor, negative risk premium due to covariance with CP factor, and negligible risk premium due to covariance with the market portfolio. While we obtain a similar decomposition of risk premia, our price of the equity risk premium factor risk (and corresponding betas) is of opposite sign to the price of the bond risk premium factor in Koijen et al. (2015). As with the other papers discussed above, the positive sign is difficult to reconcile with market clearing.

Finally, the idea behind using future realized returns as a proxy for contemporaneous discount rates is related to Kelly and Pruitt (2013). In that paper the authors use the three-pass-regression-filter methodology of Kelly and Pruitt (2015) to construct a predictor of equity returns using the cross-section of dividend-price ratios. The methodology condenses the cross-section of price-dividend ratios according to covariance with the forecast target – future realized returns on the stock market.
2 The Model

2.1 Specification

We specify a model in which variation in discount rates is driven by time-varying risk aversion as in Dew Becker (2011), Kozak (2015). Investors have Epstein-Zin preferences, unitary elasticity of intertemporal substitution and time varying coefficient of risk aversion. Time-variability in risk aversion leads to endogenously time-varying discount rates in the model.

Our choice of modeling time variation in risk aversion is motivated by empirical evidence, namely, the lack of predictability of excess market returns using volatility. Since the “quantity of risk” doesn’t forecast returns, which are nevertheless predictable, the “price of risk” must be time-varying. Greenwald et al. (2014) provide additional evidence that time-variation in risk premia are due to time-varying risk aversion rather than stochastic volatility. An alternative specification with time-varying volatility of consumption growth leads to time-varying quantity of risk and, as a result, also time-varying discount rates. Although the mechanisms in these two specifications are different, both models generate negative price of “discount rate” risk, because in either of the models high discount rates reveal high marginal utility of consumption, either due to high risk aversion or high volatility of consumption growth.

Consider a dynamic problem of an investor. Denote his consumption and wealth at time \( t \) as \( C_t \) and \( W_t \), respectively. Let \( \alpha_t = 1 - \gamma_t \), where \( \gamma_t \) is the risk-aversion coefficient. The investor’s value function \( J(W_t, \alpha_t) \) solves

\[
J(W_t, \alpha_t) = \max_{\{C_t, \delta_t\}} \left\{ C_t^{1 - \delta} \left( E_t \left[ J(W_{t+1}, \alpha_{t+1})^{\alpha_t} \right] \right)^{\frac{\delta}{\alpha_t}} \right\}
\]  

subject to the budget constraint

\[
W_{t+1} = (W_t - C_t) \theta_t R_{t+1}
\]  

Their approach is motivated by Campbell and Cochrane (1999).

As a robustness check, we forecast future market excess returns with monthly realized variance and use the residual as our risk premium factor. Estimated covariances, prices of risk, and cross-sectional \( R^2 \) are essentially unchanged from our primary specification.

\[\text{Eq. 1 is the limit of } J(W_t, \alpha_t) = \max_{\{C_t, \alpha_t\}} \lim_{\rho \to 0} \left\{ (1 - \delta) C_t^{\rho} + \delta (E_t \left[ J(W_{t+1}, \alpha_{t+1})^{\alpha_t} \right])^{\frac{\delta}{\alpha_t}} \right\}^{\frac{1}{\rho}} \text{ where } \rho \equiv 1 - \frac{1}{\psi} \to 0 \text{ and } \psi \text{ is the EIS coefficient.}\]
where $\delta$ is the rate of time discounting, $\theta_t$ is a vector of weights allocated to each asset in his portfolio, and $R_{t+1}$ denotes a vector of returns on investment assets. We assume that log consumption growth $\Delta c_{t+1} \equiv \ln \left( \frac{C_{t+1}}{C_t} \right)$ is i.i.d.,

$$\Delta c_{t+1} = \mu_c + \sigma_c \varepsilon_{t+1}^c,$$

and that the risk aversion parameter follows an AR(1) process,

$$\gamma_{t+1} = \mu_\gamma + \phi \gamma_t + \sigma_\gamma \varepsilon_{t+1}^\gamma.$$ 

(4)

Based on the evidence in Greenwald et al. (2014), we assume the two shocks are uncorrelated and jointly normally distributed. Finally, market clearing requires $C_t = D_t$, $\theta_t = 1$, and $W_t = P_t$.

### 2.2 Solution

We solve the model in Appendix A and summarize its implications for asset prices in the following theorem:

**Theorem 1.** The two-factor conditional model for excess returns holds:

$$\mathbb{E}_t \left( r_{x_{t+1}}^i \right) + \frac{1}{2} \text{var}_t \left( r_{x_{t+1}}^i \right) = \gamma_t \text{cov}_t \left( r_{x_{t+1}}^i, r_{x_{t+1}^M} \right) + a_1 (\gamma_t - 1) \text{cov}_t \left( r_{x_{t+1}}^i, \gamma_{t+1} \right)$$

$$= \delta_t^M \text{cov}_t \left( r_{x_{t+1}}^i, r_{x_{t+1}^M} \right) + \delta_t^\lambda \text{cov}_t \left( r_{x_{t+1}}^i, \lambda_{t+1} \right)$$

(5)

where $\delta_t^M = \gamma_t$ is the price of the market risk, $\delta_t^\lambda = a_1 \delta_t^M (\gamma_t - 1) = a_1 \frac{\gamma_t}{\bar{R}_M^t} (\gamma_t - 1)$ is the price of the “discount rate” risk, $a_1$ is a negative constant provided in Appendix A, $\gamma$ and $\bar{R}_M^t$ are mean risk aversion and mean market risk premium, $r_{x_{t+1}}^i = r_{t+1}^i - r_t^f$ denotes the excess log return on asset $i$ and

$$\lambda_t \equiv \mathbb{E}_t \left( r_{x_{t+1}^M} \right) = -\frac{1}{2} \sigma_c^2 + \gamma_t \sigma_M^2$$

(6)

denotes the log equity risk premium.

The price of market discount rate risk $\delta_t^\lambda$ is negative when investors’ risk aversion is higher than unity. Investors dislike assets that pay off poorly in states when the market discount rate is high, and require higher risk premium on those assets.
2.3 Unconditional Pricing

The model above is specified conditionally with “shocks” as factors. In Appendix B.1 we derive an unconditional representation in terms of “levels”. Theorem 2 below summarizes the result.

Theorem 2. When conditional covariances and variances are constant, the conditional model in Eq. 5 implies the following linear pricing relation:

\[
E(r_{x_i}^t) + \frac{V_i}{2} = cov(r_{x_i}^t, r_{x_M}^t) \times \hat{\delta}_M + cov(r_{x_i}^t, \lambda_t) \times \hat{\delta}_\lambda
\]  

(7)

where \(V_i = \text{var}(r_{x_i}^{t+1})\), \(r_{x_i}^t\) denotes returns in excess of risk-free rate on the asset \(i\) (or the market portfolio), \(\hat{\delta}_M\) and \(\hat{\delta}_\lambda\) are two constant unconditional prices of market and “discount rate” risk, respectively, and \(\lambda_t = \mathbb{E}_t(r_{x_M}^{t+1})\). Furthermore, the negative unconditional price of “discount rate” risk \(\hat{\delta}_\lambda < 0\) implies negative expected conditional price of “discount rate” risk \(\mathbb{E}[\hat{\delta}_\lambda] < 0\).

Proof. Refer to the proof of a more general theorem with an arbitrary number of factors and risk prices in Section B.1, Theorem 4. Theorem 5 derives the link between conditional and unconditional prices of risk and shows that \(\hat{\delta}_\lambda < 0 \implies \mathbb{E}[\hat{\delta}_\lambda] < 0\).

Note that the assumption of constant second moments is automatically satisfied in our model for the market portfolio. We therefore only assume that systematic components of excess returns on other assets (which are exogenous to the model) are homoskedastic. Idiosyncratic components (orthogonal to \(\varepsilon_i^t\) and \(\varepsilon_i^t\)) are allowed to be heteroskedastic.

Two aspects about the relation in Eq. 7 are worth emphasizing. First, all covariances in the formula are unconditional and hence can be easily estimated. Second, the formula involves covariances with levels of discount rates rather than shocks. The reason we can substitute levels of discount rates \(\lambda_t\) in place of their shocks \((\mathbb{E}_t - \mathbb{E}_{t-1})\lambda_t\) lies in the particular structure of the problem in Eq. 5, namely that: (i) covariances are constant; (ii) discount rates follow an AR(1) process; and, most importantly, (iii) that \(\lambda_t \equiv \mathbb{E}_t(r_{x_M}^{t+1})\), so that the market portfolio is priced by the same model and thus \(\mathbb{E}_t(r_{x_M}^{t+1})\) can be iteratively
substituted back in using Eq. 5. Theorem 4 shows how this structure naturally leads us to Eq. 7 expressed in terms of levels of discount rates rather than shocks and modified prices of risk that are a linear transform of initial expected risk prices.

2.4 New Measure of the Covariance with Discount Rates

Discount rates $\lambda_t$ in Eq. 7 are not directly observable by an econometrician. A vast literature (Bansal et al., 2014, Campbell and Vuolteenaho, 2004, Campbell et al., 2015) employs a VAR setup with macroeconomic and financial variables to predict $\lambda_t$ and back out the corresponding shocks. A major limitation of such an approach is that it restricts the information set to a small number of variables. Since investors are presumed to condition on all available information, the forecasts from a predictive regression will not equal market expectations. Additionally, even a small bias in the levels of risk premia forecasts often translates into significant misspecification of shocks inferred from a VAR. Indeed, Bansal et al. (2014) and Campbell et al. (2015) estimate similar VARs with slightly different state variables and obtain conflicting predictions. Chen and Zhao (2009a) show that VAR estimates are highly sensitive to small changes in specification.

We employ a novel methodology designed to circumvent the issue. We use future realized returns as an unbiased estimator of the current risk premia required by investors. The following theorem illustrates this idea.

**Theorem 3.** Using future realized returns in place of $\lambda_t$ in Eq. 7 delivers a consistent estimate of the covariance, i.e.,

$$\text{cov}\left(rx^i_t, rx^M_{t+1}\right) = \text{cov}\left(rx^i_t, \lambda_t\right)$$

in population, where $\lambda_t = \mathbb{E}_t[rx^M_{t+1}]$.

**Proof.** For any information set $\mathcal{F}_t$ at time $t$, $rx^M_{t+1} = \mathbb{E}\left[rx^M_{t+1} | \mathcal{F}_t\right] + \epsilon^M_{t+1} \equiv \lambda_t + \epsilon^M_{t+1}$. $\epsilon^M_{t+1}$ is the projection error and thus by definition is orthogonal to any information at time $t$, $\mathbb{E}\left[\epsilon^M_{t+1} | \mathcal{F}_t\right] = 0$. Unconditional covariances are therefore equal in population,

$$\text{cov}\left(rx^i_t, rx^M_{t+1}\right) = \text{cov}\left(rx^i_t, \mathbb{E}\left[rx^M_{t+1} | \mathcal{F}_t\right]\right) + \text{cov}\left(rx^i_t, \epsilon^M_{t+1}\right) = \text{cov}\left(rx^i_t, \lambda_t\right)$$

(9)
Theorem 3 allows us to simply substitute $rx_{t+1}^M$ in place of $\lambda_t$ in the estimate of the covariance in Eq. 7 and shows that this approach delivers a consistent estimate of $\text{cov} (rx_t^i, \lambda_t)$ in population. The in-sample estimator $\hat{\text{cov}} (rx_t^i, rx_{t+1}^M)$ of this covariance is however noisy because $\text{var} (\epsilon^M) \gg \text{var} (\epsilon^\lambda)$. We find that, provided that $\lambda_t$ is persistent, we can increase the power of the estimator by proxying for $\lambda_t$ with the sum of future realized excess returns, $\hat{\lambda}_{t:T} = \sum_{j=0}^T rx_{t+j+1}^M$. We establish via simulation that estimation is robust to varying $T$ significantly, with the optimum around 12 months. In Appendix E.2 we verify this result empirically. The following identity holds in population:

$$\text{cov} (rx_t^i, \hat{\lambda}_{t:T+T}) = \left( \frac{1 - \phi^{T+1}}{1 - \phi} \right) \times \text{cov} (rx_t^i, \lambda_t)$$ (10)

and hence the price of risk corresponding to the covariance $\text{cov} (rx_t^i, \hat{\lambda}_{t:T+T})$ is scaled by a positive constant $\left( \frac{1 - \phi^{T+1}}{1 - \phi} \right)^{-1}$ relative to the price of risk implied by Eq. 7.

Naturally, $\hat{\lambda}_{t:T+T} = \sum_{j=0}^T rx_{t+j+1}^M$ is a rather imprecise estimator of the level of risk premia $\lambda_t$; however, it proves to be informative for the purpose of estimating unconditional covariances $\text{cov} (rx_t^i, \lambda_t)$. Therefore, we are able to estimate Eq. 7 without directly estimating the level of market risk premium $\lambda_t$. We establish statistical and economic significance of our estimator in Section 3.

### 2.5 Empirical Specification

Combining the results in Section 2.3 and 2.4, we arrive at our final empirical specification:

$$\mathbb{E} (R_{i,t}^e) = \text{cov} (rx_t^i, rx_t^M) \times \tilde{\delta}_M + \text{cov} (rx_t^i, \hat{\lambda}_{t:T+T}) \times \tilde{\delta}_\lambda$$ (11)

where $R_{i,t}^e \equiv [\exp (rx_t^i) - 1]$ is the level of excess returns\(^\text{12}\) on an asset $i$ and $\hat{\lambda}_{t:T+T} \equiv rx_{t+1:t+T+1}^M$ are future market excess returns cumulated over $T + 1$ periods.

All quantities in Eq. 11 are observable and thus the relation can be easily estimated by OLS. Furthermore, Theorem 2 and Eq. 10 shows that a negative estimate of the unconditional

\(^{12}\text{We assumed the approximation } \mathbb{E} (rx_t^i) + \frac{V_i^2}{2} \approx \exp [\mathbb{E} (rx_t^i) + \frac{V_i^2}{2}] - 1 \text{ holds.}\)
price of the discount-rate risk $\delta_\lambda$ implies a negative value of the expected conditional price of risk, $E \left[ \delta_\lambda^t \right] < 0$, from Eq. 5.

### 3 Empirical Link Between Cross-Sectional and Aggregate Expected Returns

We estimate and test the expected return relation of Eq. 11 using three sets of test assets. The first is the canonical 25 portfolios formed by a two-way sort of firms on market capitalization (ME) and book-to-market ratio (BE/ME), available at Ken French’s website\(^{13}\). Lewellen et al. (2010) highlight a key issue in estimating and testing asset pricing models. When the test assets have a strong factor structure that captures much of the time-series variation as well as the cross-sectional variation in expected returns, a spurious model with many factors may still produce a remarkably good cross-sectional fit as long as the spurious factors are correlated with the “true” factors. This result is not due to sampling variation; it holds in population. A solution they propose is to add assets which increase the “dimensionality” of the test asset space.

In addition to the canonical 25 portfolios, we construct a second alternative set of test assets. We include fifteen portfolios consisting of five value-weighted quintile portfolios each from independent sorts on size, book-to-market ratio, and momentum (prior 2-12)\(^{14}\). The momentum factor, UMD (Carhart, 1997), is nearly uncorrelated with the size factor, SMB, and is negatively correlated with the book-to-market factor, HML (Fama and French, 1996). Further, sorting firms based on prior performance produces a reliable spread in average returns subsumed by neither the size effect nor the book-to-market effect (Fama and French, 2008). Therefore, including momentum sorted portfolios as test assets makes it decidedly more difficult for a model to fit the cross-section of expected returns. Our preferred estimation uses these fifteen portfolios; for robustness and for comparison with the literature, we perform all estimation using the Fama-French 25 portfolios as well.

Third, we estimate the model using 15 anomaly long-short portfolios from Novy-Marx and Velikov (2014)\(^{15}\) (hereafter NMV). These anomalies capture many prominent features.

---

\(^{13}\)http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

\(^{14}\)Also available at Ken French’s website.

\(^{15}\)See Novy-Marx and Velikov (2014) for details on data construction. The data are available at
of the cross-section of returns\textsuperscript{16}. The data are only available at monthly frequency. Finally, we always include the value-weight market and risk-free returns as test assets, both to show how well the model fits these assets and because they theoretically have the best measured factor loadings.

Following the spirit of Merton (1973), we use portfolio returns measured at daily frequency when possible\textsuperscript{17}. All returns are measured over the period 01-Aug-1963 to 31-Dec-2013. Using daily returns, rather than monthly, reduces the approximation error due to linearization of the exponential function that we rely on in deriving Eq. 11. As noted in Campbell and Vuolteenaho (2004), “July 1963 is when COMPUSTAT data become reliable and most of the evidence on the book-to-market anomaly is obtained from the post-1963 period”. Furthermore, in the pre-1963 sample, the “CAPM explains the cross-section of stock returns reasonably well” (Campbell and Vuolteenaho, 2004). We therefore focus on post-1963 in our analysis.

As a proxy for the excess return on the wealth portfolio, $r^M_t$, we use the log excess return on the value-weight portfolio of all common equity traded on the NYSE, AMEX, and NASDAQ. Of course the standard critique applies that there exist many assets, both traded (foreign securities) and non-traded (real-estate, human capital) that are not included in this portfolio (Roll, 1977). As discussed above, we construct $\hat{\lambda}_t = \sum_{i=1}^H r^M_{t+1}$. For our preferred specification, we set $H = 126$ trading days, or one-half year. Our results are quantitatively robust across various choices of $H$, using daily or monthly frequency of returns (see Appendix E).

Table 1 shows the estimated covariances of asset returns with the factors. Panel A shows $\text{cov} \left( r^t_i, r^M_t \right)$ with Newey-West standard errors in parentheses\textsuperscript{18}. Quintile 1 represents large firms, growth firms, and recent losers in relation to the dimensions, size, book-to-market, and momentum, respectively. Analogously, Quintile 5 represents small firms, value firms, and recent winners. The column to the right of Quintile 5 represents the Q5-Q1 spread portfolio. The FF column gives the estimates for the canonical Fama-French-Carhart (FFC) factors, SMB, HML, and UMD. The covariances match the well known pattern in market betas – their inability to explain the cross-section of size, book-to-market, and momentum

\textsuperscript{15}http://rnm.simon.rochester.edu/data_lib/index.html
\textsuperscript{16}We exclude a few high-frequency anomalies and those that did not survive in the past two decades.
\textsuperscript{17}We also replicated the analysis at monthly frequency and obtain very similar results (see Appendix E).
\textsuperscript{18}Moving block bootstrap gives similar standard errors.
Table 1: Covariances

This table shows covariances and annualized mean returns estimated over 01-Aug-1963 to 31-Dec-2013. Panel A lists the covariances of portfolio returns with the market return, $C_{i,M} = \text{cov} \left( r_{x_i}^t, r_M^t \right)$. Panel B depicts the covariances of portfolio returns with the risk premium factor, $C_{i,\lambda} = \text{cov} \left( r_{x_i}^t, \hat{\lambda}_t \right)$. Panel C shows the annualized expected excess returns on each portfolio, $\mathbb{E} [R_e^i]$. The column "FF" represents the Fama-French oks: SMB, HML, and UMD. The last column represents the Q5-Q1 spread portfolio. t-statistics (in parentheses) are adjusted for serial correlation using Newey-West procedure with 252 lags (1 year). All covariances are scaled by the variance of the daily market excess returns.

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>FF</th>
<th>Q5-Q1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: $C_{i,M}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ME</td>
<td>0.95</td>
<td>0.91</td>
<td>0.90</td>
<td>0.89</td>
<td>0.76</td>
<td>-0.07</td>
<td>-0.19</td>
</tr>
<tr>
<td>BE/ME</td>
<td>0.99</td>
<td>0.89</td>
<td>0.87</td>
<td>0.82</td>
<td>0.86</td>
<td>-0.14</td>
<td>-0.13</td>
</tr>
<tr>
<td>Prior 2-12</td>
<td>1.20</td>
<td>1.00</td>
<td>0.93</td>
<td>0.92</td>
<td>1.07</td>
<td>-0.09</td>
<td>-0.14</td>
</tr>
<tr>
<td>Panel B: $C_{i,\lambda}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ME</td>
<td>0.19</td>
<td>0.15</td>
<td>0.09</td>
<td>0.03</td>
<td>-0.02</td>
<td>-0.12</td>
<td>-0.20</td>
</tr>
<tr>
<td>(2.33)</td>
<td>(1.44)</td>
<td>(0.81)</td>
<td>(0.25)</td>
<td>(-0.14)</td>
<td>(-1.85)</td>
<td>(-2.14)</td>
<td></td>
</tr>
<tr>
<td>BE/ME</td>
<td>0.21</td>
<td>0.15</td>
<td>0.11</td>
<td>0.11</td>
<td>-0.06</td>
<td>-0.18</td>
<td>-0.27</td>
</tr>
<tr>
<td>(2.37)</td>
<td>(1.62)</td>
<td>(1.00)</td>
<td>(1.11)</td>
<td>(-0.60)</td>
<td>(-2.70)</td>
<td>(-2.80)</td>
<td></td>
</tr>
<tr>
<td>Prior 2-12</td>
<td>0.36</td>
<td>0.24</td>
<td>0.14</td>
<td>0.09</td>
<td>0.06</td>
<td>-0.21</td>
<td>-0.30</td>
</tr>
<tr>
<td>(2.50)</td>
<td>(2.53)</td>
<td>(1.71)</td>
<td>(0.92)</td>
<td>(0.49)</td>
<td>(-2.01)</td>
<td>(-2.43)</td>
<td></td>
</tr>
<tr>
<td>Panel C: $\mathbb{E} [R_e^i]$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ME</td>
<td>6.35</td>
<td>8.09</td>
<td>8.47</td>
<td>8.87</td>
<td>8.06</td>
<td>2.24</td>
<td>2.42</td>
</tr>
<tr>
<td>BE/ME</td>
<td>6.05</td>
<td>6.92</td>
<td>7.22</td>
<td>9.28</td>
<td>11.24</td>
<td>4.56</td>
<td>5.85</td>
</tr>
<tr>
<td>Prior 2-12</td>
<td>1.43</td>
<td>5.81</td>
<td>5.64</td>
<td>7.46</td>
<td>10.33</td>
<td>8.08</td>
<td>10.95</td>
</tr>
</tbody>
</table>

sorted portfolios.

Panel B reports $\text{cov} \left( r_{x_i}^t, \hat{\lambda}_t \right)$ for the same portfolios. In all three dimensions (size, book-to-market, and momentum), $\text{cov} \left( r_{x_i}^t, \hat{\lambda}_t \right)$ decreases from left to right. That is to say, when the “risk premium”, $\lambda_t$, rises, small stocks are expected to fall more than large stocks, value stocks are expected to fall more than growth stocks, and recent winners are expected to fall more than recent losers. Though using realized market returns in place of expected returns produces consistent covariance estimates, they are less precisely estimated due to
Figure 1: Univariate fit. The top left plot shows sample values of $\mathbb{E}[R^e_t]$ vs $\text{cov} \left(r_x^t, \hat{\lambda}_t\right)$ for the 15 quintile portfolios: 5 size (me), 5 book-to-market (bm) and 5 momentum (m) sorted portfolios. The plot in the top right panel depicts same results for the 25 Fama-French portfolios. The first number in portfolio labels refers to ME quintile (1=large; 5=small); the second number corresponds to BE/ME quintile (1=growth; 5=value). The bottom plot is for Novy-Marx and Velikov (2014) portfolios using monthly data from 01-Aug-1963 to 31-Dec-2013. PC1 and PC2 are the first two principal components of NMV returns.

The noise present in realized returns. Still, the covariances of the spread portfolios with $\hat{\lambda}_t$ are statistically significantly different from zero and the covariances follow a reliable pattern, suggesting that our results are not spurious. Panel C shows sample average returns which increase monotonically from left to right across quintiles, consistent with the well known size,
value, and momentum phenomena. Panels B and C suggest a strong relationship between \( \text{cov} \left( r_{x_t}, \hat{\lambda}_t \right) \) and \( \mathbb{E} [R_{t}^c] \), which can be clearly seen in Figure 1.

Figure 1 (a) plots sample values of \( \mathbb{E} [R_{t}^c] \) vs \( \text{cov} \left( r_{x_t}, \hat{\lambda}_t \right) \) for the 15 quintile portfolios. Figure 1 (b) is the same plot for the 25 Fama-French portfolios and Figure 1 (c) shows the NMV anomalies. We use \( H = 126 \) for daily data (quintile and FF portfolios) and \( H = 6 \) for monthly data (NMV anomalies). The graphs confirm that \( \text{cov} \left( r_{x_t}, \hat{\lambda}_t \right) \) and \( \mathbb{E} [R_{t}^c] \) line up well in the cross-section of assets, suggesting the \( \lambda_t \) risk factor captures the size, value, momentum effects, and NMV anomalies.

3.1 Estimation Results

We estimate the risk price vector \( \delta = [\delta_M \, \delta_\lambda]^\prime \) using GMM with a prespecified block-diagonal weighting matrix (Cochrane, 2001, Chapter 11.5). It is equivalent to the standard two-stage estimation procedure. Covariances \( C_{i,\lambda} = \text{cov} \left( r_{x_t}^i, \hat{\lambda}_t \right) \) and \( C_{i,M} = \text{cov} \left( r_{x_t}^i, r_{x_t}^M \right) \) are estimated in the first stage by just-identified GMM, which yields the standard formulas for sample covariance. In the second stage, we estimate risk prices (SDF coefficients) via an OLS regression of sample mean returns on the covariances estimated from the first stage. In addition to our two-factor model, we estimate the Sharpe-Lintner CAPM and well as the Fama-French-Carhart (FF augmented with the UMD “momentum” factor of Carhart, 1997). For ease of comparison, all models are written and estimated in terms of covariances instead of regression \( \beta \)s. Below is a summary of the pricing equations, where \( \delta \)s are interpreted as risk prices (coefficients in the SDF):

\[
\begin{align*}
\text{2-Factor model: } \mathbb{E} [R_t^c] &= C_{i,M} \delta_M + C_{i,\lambda} \delta_\lambda \\
\text{2-Factor model, Unrestricted: } \mathbb{E} [R_t^c] &= \alpha + C_{i,M} \delta_M + C_{i,\lambda} \delta_\lambda \\
\text{CAPM: } \mathbb{E} [R_t^c] &= C_{i,M} \delta_M \\
\text{CAPM, Unrestricted: } \mathbb{E} [R_t^c] &= \alpha + C_{i,M} \delta_M \\
\text{4-Factor FFC, Restricted: } \mathbb{E} [R_t^c] &= C_{i,M} \delta_M + C_{i,smb} \delta_{smb} + C_{i,hml} \delta_{hml} + C_{i,umd} \delta_{umd} \\
\text{4-Factor FFC, Unrestricted: } \mathbb{E} [R_t^c] &= \alpha + C_{i,M} \delta_M + C_{i,smb} \delta_{smb} + C_{i,hml} \delta_{hml} + C_{i,umd} \delta_{umd}
\end{align*}
\]

where \( C_{i,X} = \text{cov} \left( r_{x_t}^i, X_t \right) \). Estimated risk prices are given in Table 2 along with sample \( R^2 \) and mean absolute pricing errors. Quantitatively, our two-factor model fits the cross-section
This table shows risk prices estimates for our two-factor model, CAPM, and Fama-French-Carhart model. The test assets are value-weighted quintile portfolios sorted on ME, BE/ME, and Prior2-12. $\alpha$ is annualized and "-" indicates that the value is restricted to zero. MAPE is average absolute pricing error, annualized. Moving block bootstrap t-statistics are in parentheses. The sample is from 01-Aug-1963 to 31-Dec-2013.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\delta_M$</th>
<th>$\delta_\lambda$</th>
<th>$\delta_{mb}$</th>
<th>$\delta_{hml}$</th>
<th>$\delta_{umd}$</th>
<th>$R^2$</th>
<th>MAPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-Factor model</td>
<td>4.54</td>
<td>-11</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>80.9</td>
<td>0.959</td>
</tr>
<tr>
<td></td>
<td>(4.46)</td>
<td>(-5.26)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.812</td>
<td>4.17</td>
<td>-10.7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>81.5</td>
<td>0.975</td>
</tr>
<tr>
<td></td>
<td>(1.44)</td>
<td>(4.27)</td>
<td>(-5.18)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CAPM</td>
<td>3.01</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>10.8</td>
<td>1.95</td>
</tr>
<tr>
<td></td>
<td>(3.16)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.12</td>
<td>1.7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>19.2</td>
<td>1.95</td>
</tr>
<tr>
<td></td>
<td>(4.55)</td>
<td>(1.85)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4-Factor FFC</td>
<td>5.54</td>
<td>-</td>
<td>3.83</td>
<td>12.9</td>
<td>8.21</td>
<td>-</td>
<td>87.2</td>
<td>0.874</td>
</tr>
<tr>
<td></td>
<td>(4.65)</td>
<td></td>
<td>(1.12)</td>
<td>(3.83)</td>
<td>(5.03)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.706</td>
<td>5.19</td>
<td>-</td>
<td>3.71</td>
<td>12.6</td>
<td>8.07</td>
<td>87.6</td>
<td>0.896</td>
</tr>
<tr>
<td></td>
<td>(1.84)</td>
<td>(4.51)</td>
<td>(1.07)</td>
<td>(3.78)</td>
<td>(4.95)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

of average returns nearly as well as the 4-factor FFC model. The estimated intercept is nearly zero, both statistically and economically. Though $\text{cov} \left( r_{it}, \hat{\lambda}_t \right)$ is not very well estimated for any individual test asset, the cross-sectional spread in covariances is strong enough to yield precise estimation of $\delta_\lambda$. $H_0 : \delta_\lambda = 0$ is rejected for all conventional significance levels. Covariance with the risk premium factor is able to capture a large portion of the cross-sectional variation in average returns due to the size, book-to-market, and momentum effects. Standard errors are calculated using a moving block bootstrap (Horowitz, 2001) and are consistent across various choices of block size. The cross-sectional fit of our model and 4-factor Fama-French-Carhart model is shown in Figure 2. The graphs plot model implied mean excess returns on the horizontal and sample average returns on the vertical axis. The $45^\circ$ line represents a model with perfect in-sample fit ($100\% R^2$).

---

19We bootstrap the entire GMM system, so uncertainty in the first-stage covariance estimates is fully incorporated in the standard errors of risk prices.
Figure 2: Performance of our 2-factor and 4-Factor Fama-French-Carhart models using quintile portfolios. The plot shows sample average, $E[R^e_i]$, vs model expected, $\hat{E}[R^e_i]$, excess returns. The 2-Factor model with restricted intercept on the left, and 4-factor FFC model with restricted intercept on the right. m1-m5 correspond to momentum quintile portfolios (losers to winners). bm1-bm5 correspond to book-to-market quintiles (growth to value). me1-me5 correspond to size quintiles (large to small). The sample is from 01-Aug-1963 to 31-Dec-2013.

### 3.1.1 A Horserace

A formal test that pricing errors are jointly zero obviously rejects our model. However, a more interesting test is a “horse race” between our equilibrium model and the FFC 4-factor model, which is designed specifically to fit these test assets. As shown in Cochrane (2001, Ch.13), this is equivalent to testing the null

$$H_0 : \alpha_{smb} = \alpha_{hml} = \alpha_{umd} = 0$$

in our 2-factor model. A Wald test$^{20}$ of this joint restriction fails to reject the null with $p = 12\%$.

### 3.2 The Sign of “Risk Premium” Price of Risk ($\delta_\lambda$)

Campbell and Vuolteenaho (2004) perform a similar pricing exercise but with different empirical methodology and theoretic motivation. They find a positive price of expected returns risk, though the estimate isn’t statistically significant. Our estimates reject a positive price of risk with $p \approx 0$. The alternative approaches reach very different conclusions.

$^{20}$cov($\alpha$) is estimated using the bootstrap procedure described above.
We find that growth stocks and large firm stocks outperform value stocks and small firm stocks, respectively, in response to an increase in market expected returns, opposite to the pattern found in Campbell and Vuolteenaho (2004), Campbell et al. (2015). The different pattern in loadings produces a different conclusion about the compensation an investor requires for bearing the risk of time-varying expected returns. We conclude that an increase in the expected market return corresponds to a drop in the investor’s utility and hence an increase in her marginal utility of wealth. This implies she is willing to pay in order to eliminate this risk from her portfolio, as predicted by our model. In contrast, Campbell and Vuolteenaho (2004), Campbell et al. (2015) find that an investor is willing to pay to increase her exposure to this risk. These papers theoretically motivate their finding using a portfolio problem for a long-term investor who treats discount rate shocks as exogenous. Campbell et al. (2015) ignore their model’s theoretical restriction that “the equity premium is proportional to market volatility” since their estimation suggests that, instead, they are nearly uncorrelated. It isn’t surprising that an essentially exogenous increase in expected returns (with no change in risk) is beneficial to an investor. We endogenize time-varying discount rates and find that a representative agent suffers when expected returns rise. The differences in our empirical findings are due to methodology, not sample period. Campbell et al. (2015) update the data to 2010 and find similar factor loadings to Campbell and Vuolteenaho (2004).

3.3 GLS

The two-stage OLS procedure for estimating stochastic discount factors suffers from many problems related to samples size and factor structure in the covariance of test asset returns. Lewellen et al. (2010) highlight these concerns and offer some suggestions:

1. Increase the dimensionality of the test assets relative to the dimension of the SDF.

2. Impose theoretic restrictions: “zero-beta rates should be close to the risk-free rate, the risk premium on a factor portfolio should be close to its average excess return”. This is essentially using GLS instead of OLS in the second stage, with the factors included as test assets.

21Theoretically, this depends on the coefficients of relative risk aversion and elasticity of intertemporal substitution. The result obtains for generally accepted parameter values.
The table gives OLS and GLS risk price estimates from \( E[R_e] = C_{i,M} \delta_M + C_{i,\lambda} \delta_\lambda \). OLS estimates are from the standard two-step FM procedure. GLS restricts the model to exactly fit the market and the mimicking portfolio’s in-sample average returns and restricts the zero-beta rate, ignoring pricing errors on other assets. \( E[R_M^e] \) and \( E[R_\lambda^e] \) are the model implied annualized expected excess returns on the market and mimicking portfolios, respectively. For GLS these are, by construction, equal to annualized sample averages.

<table>
<thead>
<tr>
<th></th>
<th>( \delta_M )</th>
<th>( \delta_\lambda )</th>
<th>( E[R_M^e] )</th>
<th>( E[R_\lambda^e] )</th>
<th>( R^2 )</th>
<th>( MAPE )</th>
</tr>
</thead>
<tbody>
<tr>
<td>OLS</td>
<td>4.99</td>
<td>-12.9</td>
<td>6.3</td>
<td>-3.51</td>
<td>83.8</td>
<td>0.939</td>
</tr>
<tr>
<td></td>
<td>(4.53)</td>
<td>(-4.85)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GLS</td>
<td>4.97</td>
<td>-12.9</td>
<td>6.2</td>
<td>-3.54</td>
<td>83.7</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>(4.76)</td>
<td>(-5.18)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Report GLS \( R^2 \) since (a) it “is completely determined by the factor’s proximity to the minimum-variance boundary ... but the OLS \( R^2 \) can, in principle, be anything” and (b) “in practice, obtaining a high GLS \( R^2 \) represents a more stringent hurdle than obtaining a high OLS \( R^2 \).”

4. Report confidence intervals for the cross-sectional \( R^2 \).

We already addressed issue (1) by having only one factor to “explain” three dimensions of average returns. Table 2 shows that estimates with and without restrictions on the zero-beta rate are nearly identical. We implement GLS by forcing the model to price the market and the mimicking (maximally correlated) portfolio for our risk premium factor. For this section, we treat the mimic portfolio as the factor and address OLS vs GLS. Table 3 shows the estimated SDF using both methods. \( E[R_M^e] \) and \( E[R_\lambda^e] \) are the model implied annualized expected excess returns on the market and mimicking portfolios, respectively. For GLS these are, by construction, equal to sample averages. The results are nearly identical.

\[ \text{Because our test assets are highly correlated, in small sample the mimicking portfolio will have unrealistic extreme long/short positions. To mitigate concerns of overfitting, we instead construct } \lambda_t = \text{proj} (\lambda_t | \{PC\}_t) \text{ where } \{PC\}_t \text{ is the first four principal components of returns. When only a subset of test assets are used to construct the mimicking portfolio, there is no guarantee that estimated risk prices, etc will remain unchanged. Still, } \delta \text{ and } R^2 \text{ values are similar. We ignore sampling uncertainty in } C_{i,smb}, C_{i,hml}, C_{i,umd} \text{ when reporting test statistics using the factor mimicking portfolio. This likely does not bias our results greatly since the Newey-West t-statistics on } C_{i,smb}, C_{i,hml}, C_{i,umd} \text{ are between 2 and 3.} \]
Table 4: Risk Price Estimates using NMV Anomaly Portfolios (monthly)

This table shows risk price estimates for our two-factor, CAPM, and FFC models. Test assets are Novy-Marx and Velikov (2014) anomaly portfolios. $\alpha$ is annualized and "-" indicates that the intercept is restricted to zero. MAPE is average absolute pricing error, annualized. Moving block bootstrap t-statistics are in parentheses. The sample is from August 1963 to December 2013.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\delta_M$</th>
<th>$\delta_\lambda$</th>
<th>$\delta_{smb}$</th>
<th>$\delta_{hml}$</th>
<th>$\delta_{umd}$</th>
<th>$R^2$</th>
<th>MAPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-Factor model</td>
<td>-</td>
<td>3.86</td>
<td>-14.7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>64.3</td>
<td>2.36</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(8.69)</td>
<td>(-13.53)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.39</td>
<td>2.58</td>
<td>-11.5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>72.8</td>
<td>2.17</td>
</tr>
<tr>
<td></td>
<td>(4.95)</td>
<td>(5.27)</td>
<td>(-12.48)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CAPM</td>
<td>-</td>
<td>2.43</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-214</td>
<td>7.66</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5.32)</td>
<td>(5.8)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>7.65</td>
<td>-0.668</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5.8</td>
<td>3.92</td>
</tr>
<tr>
<td></td>
<td>(12.85)</td>
<td>(-1.20)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4-Factor FFC</td>
<td>-</td>
<td>4.42</td>
<td>-</td>
<td>2.57</td>
<td>9.62</td>
<td>6.39</td>
<td>37.8</td>
<td>3.09</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(7.48)</td>
<td>(2.05)</td>
<td>(9.67)</td>
<td>(10.71)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.49</td>
<td>1.84</td>
<td>-</td>
<td>0.767</td>
<td>4.53</td>
<td>4.59</td>
<td>73.4</td>
<td>1.99</td>
</tr>
<tr>
<td></td>
<td>(7.82)</td>
<td>(2.92)</td>
<td>(0.60)</td>
<td>(3.93)</td>
<td>(7.93)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

across methods. Bootstrap simulation rejects the null of GLS $R^2 = 0$ with $p < 1\%$. The [1%, 99%] $R^2$ interval under the null is $[-121\%, 83\%]$.

3.4 Anomaly Portfolios

As a further test of the model, we estimate it using the cross-section of anomaly portfolios defined in Novy-Marx and Velikov (2014). These represent a broad set of empirical regularities with seemingly very different fundamental drivers. However, as shown in Kozak et al. (2015), a pricing model using the first two principal components of returns produces high $R^2$ in fitting the cross-section of average returns (though much less in the time-series). This gives us hope that one (or a few) basic economic mechanism is responsible for the variety of anomalies.

\[23\] One-sided test of $H_0: R^2 \leq 0$.

\[24\] These data are only available at monthly frequency.
Figure 3: Performance of our 2-factor and 4-Factor Fama-French models using anomaly portfolios (monthly data). The plot shows sample average, $E[R_e^i]$, vs model expected, $\hat{E}[R_e^i]$, excess returns. The 2-Factor model with restricted intercept on the left, and 4-factor FFC with restricted intercept on the right. Assets are monthly NMV anomaly long-short portfolios from 01-Aug-1963 to 31-Dec-2013. PC1 and PC2 denote two largest principal components of NMV returns. The sample is from August 1963 to December 2013.

Table 4 shows parameters of the pricing models in Equations (12)-(17), estimated using monthly NMV returns\(^{25}\). We also include two principal components (labeled PC1 and PC2) as test assets. The estimated risk prices are similar to those in Table 2. Now, however, our 2-factor model significantly outperforms the 4-factor FFC model in fitting the cross-section of average returns (with restricted intercept). Notably, the t-statistics on risk prices are much higher than in Table 2. This is due to the weaker factor structure in the NMV portfolios as compared to the test assets used before (quintile portfolios sorted on ME, BE/ME, and Prior2-12). Weaker cross-sectional correlation of returns results in more “effective” test assets, improving statistical power. Figure 3 shows graphically the fit of our 2-factor model and 4-factor FFC models (with restricted zero-beta rate). The 2-factor model is visibly

\(^{25}\)Because the NMV portfolios are long-short, they tend to have CAPM $\beta$s near zero. Sample $\beta$ estimates are very noisy and yield unreliable estimates of $\delta_M$. To address this issue, we orthogonalize each anomaly return against the market portfolio before estimating asset pricing models. This procedure is equivalent to giving infinite weight to the market portfolio in the second stage estimation (like GLS). Without this restriction, the model $R^2$ slightly improves, at the cost of very poor fit for $E[R_M^i]$. 

24
superior to the 4-factor model in fitting the average NMV returns.

4 Bond Risks and Risk Premia

4.1 Bond Pricing Model

![Graph]

Figure 4: **Pricing bonds in the 2-factor model.** We plot the fitted and sample mean values of expected returns in the model Eq. 11. Test assets include the stock portfolios we used before as well as 7 bonds with maturities from 1 to 7 years, labeled B1-B7, respectively.

Whereas there are numerous papers which explore risk premia either for equities or for fixed income securities, few study these assets in a unified framework\(^\text{26}\). We extend our analysis to include default-free government bonds and interest rate risk. **Figure 4** illustrates the failure of the unconditional model in Eq. 11 in pricing bond excess returns of maturities from 1 to 7 years. Pricing errors are large and the slope is completely wrong. Clearly, the market and risk premium returns factors are not sufficient to explain the risk compensation required for holding these bonds.

\(^{26}\)Recent work in this area includes Koijen et al. (2015).
In Appendix C we extend the model from Section 2 by adding an additional source of risk: we allow the rate of time preference to be stochastic. Along with time-varying risk aversion, this shock generates valuation risk (Albuquerque et al., 2012). Therefore, in addition to the permanent shocks to consumption, the economy features transitory shocks that directly affect preferences for assets. An increase in risk aversion decreases demand for risky assets. An increase in “impatience” decreases demand for assets with delayed cash flows. This framework generates a plausible model of expected asset returns without a strong counterfactual link between asset prices and “real” quantities\(^{27}\). It resolves another shortcoming of long-run-risk and habit-based models – namely, their inability to generate a positive real term premium and an upward-sloping real yield curve\(^{28}\). In our model, increases in the short rate are “bad” shocks caused by higher investor “impatience”. These “impatience” shocks move bond and stock prices in the same direction.

In Appendix C we show that one can replace the “impatience” shocks with changes in short-term interest rates, \(\Delta r_f t+1\), and derive the following pricing equation:

\[
\mathbb{E}_t \left(r x^i_t, t+1\right) + \frac{1}{2} \text{var}_t \left(r x^i_t, t+1\right) = \delta^M_t \text{cov}_t \left(r x^i_t, r^M t+1, t+1\right) + \delta^\lambda_t \text{cov}_t \left(r x^i_t, \lambda t+1\right) \\
+ \delta^\Delta r_f t \text{cov}_t \left(r x^i_t, \Delta r_f t+1\right) \quad (18)
\]

The pricing model given by Eq. 18 has an additional source of risk compared to the model in Eq. 5 – assets earn extra risk premium due to their covariance with shocks to the risk-free rate — the “level” (bond) factor. The price of this risk is negative, i.e. assets that covary more negatively with shocks to the risk-free rate (long-term bonds, for example) earn higher risk premium. Therefore, in our model long-term bonds command positive and increasing-in-maturity risk premium due to their exposure to the “level” risk (driven by time preference shocks in our model) and negative and decreasing-in-maturity term premium due to their exposure to the “discount-rate” risk (driven by shocks to risk aversion). Our empirical findings match this prediction. Using the same approach as in Section 2.5, we

\(^{27}\)A long-run risks model with stochastic volatility of consumption growth predicts, counterfactually, that the equity premium and risk-free rate are spanned by expected consumption growth and volatility.

\(^{28}\)Under reasonable calibrations. Wachter (2006), for instance, generates upward sloping yield curve by calibrating the model to imply a counter-cyclical pattern of the short rate.
condition down the model in Eq. 18 to obtain the following empirical specification:

\[ \mathbb{E}(R_{i,t}^e) = \text{cov}(rx_i^t, rx_M^t) \times \delta_M + \text{cov}(rx_i^t, \hat{\lambda}_{t:T+1}) \times \delta_{\hat{\lambda}} + \text{cov}(rx_i^t, \Delta r_f^t) \times \delta_{\Delta r_f} \] (19)

where \( R_{i,t}^e \equiv [\exp (rx_i^t) - 1] \) is the level of excess returns\(^{29} \) on asset \( i \), \( \hat{\lambda}_{t:T+1} \equiv rx_M^{t+1:T+1} = \sum_{j=0}^{T} rx_M^{t+j+1} \) are future log market excess returns cumulated over \( T+1 \) periods, and \( \Delta r_f^t \) is the change in the short rate.

### 4.2 Data

We use zero-coupon treasury yields from Gürkaynak et al. (2006) (GSW), which provides a daily constant maturity yield curve from 1961 onward. Though the data are smoothed by the use of a Svensson polynomial (extension of Nelson-Siegel), the yields are usually very close to the unsmoothed yields derived using the methodology of Fama and Bliss (1987) and “for many purposes the slight smoothing in GSW data may make no difference” (Cochrane and Piazzesi, 2008). The advantage of GSW yields is the daily observation frequency, which we have argued in Section 3 is important to our empirical strategy. Prior to 1971, the GSW yields only include maturities up to seven years. Post-1971 they includes maturities to 30 years, though there is some question of the reliability of the very long maturity yields. To match the timing of our stock data, we use maturities up to seven years, starting in 1963.

To construct zero-coupon bond returns from the GSW yields, we use the daily parameter estimates available online\(^{30} \). This allows us, for example to recover the yield on a bond with 364 days to maturity. This yield is necessary for calculating the daily return on a one-year bond. For excess returns, we subtract the return on a one month T-bill, the same procedure we use for excess stock returns. We use the 3-month zero-coupon yield\(^{31} \) as our proxy for the short rate, \( r_f^t \).

---

\(^{29}\)We use the approximation \( \mathbb{E}(rx_i^t) + \frac{V_i}{2} \approx \exp [\mathbb{E}(rx_i^t) + \frac{V_i}{2}] - 1. \)


\(^{31}\)GSW note that yields on bonds with maturities shorter than three months are strongly affected by liquidity issues.
Figure 5: Univariate Bond Pricing. The plot shows sample average, $E[R_i^e]$, vs model expected, $\hat{E}[R_i^e]$, excess returns from a single factor bond model.

4.3 Estimating the price of “level risk”

In the spirit of Cochrane and Piazzesi (2008), we estimate the single factor bond model $E[R_i^e] = C_{i,B}\delta_B$, where $C_{i,B} = \text{cov}(r x_i^t, \Delta r_f^t)$ and $\Delta r_f^t$ is the change in 3-month T-bill rate. $\delta_B$ is estimated to be approximately $-54$. The cross-sectional $R^2$ is 96% with 0.07% annualized mean absolute pricing error. Figure 5 shows graphically the good fit of the level model for bonds.

In the context of Eq. 19, we argue that $\delta_B$, the price of “level risk”, is commonly underestimated in bond-only models; it is a classic case of omitted variables bias. Eq. 11 and the results of Section 3.1 suggest at least two such missing variables, $C_{i,\lambda} = \text{cov}(r x_i^t, \sum_{i=1}^H r x_{i+i}^M)$ and $C_{i,M} = \text{cov}(r x_i^t, r x_i^M)$. Table 5 shows $C_{i,B}$, $C_{i,\lambda}$ and $C_{i,M}$ across maturities. First note that $C_{i,M} \approx 0$ for all maturities (bonds have almost zero market $\beta$s). More importantly, $\forall i$, $C_{i,\lambda} \approx -15 \times C_{i,B}$. Cross-sectionally, $\text{corr}(C_{i,B}, C_{i,\lambda}) \approx 1.0$. Since we know from Section 3.1 that $\delta_\lambda \neq 0$, the univariate level model suffers greatly from omitted variables bias. Using the estimate of $\delta_\lambda = -11$, a back-of-the-envelope calculation suggests the true $\delta_B = -54 - 15 \times 11 = -219$. In other words, the required compensation for bearing level risk
Table 5: Bond Covariances

This table shows covariances of government bonds with the bond “level” factor, \( C_{i,B} \equiv \text{cov} \left( r_{xi}, \Delta r_f^t \right) \); the stock excess returns market factor, \( C_{i,M} \equiv \text{cov} \left( r_{xi}, r_{xM}^t \right) \); and the discount rate factor, \( C_{i,\lambda} \equiv \text{cov} \left( r_{xi}, \sum_{i=1}^{H} r_{xM}^{t+i} \right) \), estimated over 01-Aug-1963 to 31-Dec-2013. t-statistics in parentheses are adjusted for serial correlation using the Newey-West procedure with 252 lags (1 year). All covariances are scaled by the variance of the market daily excess returns.

<table>
<thead>
<tr>
<th></th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
<th>6Y</th>
<th>7Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{i,B} )</td>
<td>-0.004</td>
<td>-0.006</td>
<td>-0.009</td>
<td>-0.011</td>
<td>-0.012</td>
<td>-0.014</td>
<td>-0.015</td>
</tr>
<tr>
<td>( C_{i,M} )</td>
<td>0.002</td>
<td>0.002</td>
<td>0.003</td>
<td>0.004</td>
<td>0.006</td>
<td>0.008</td>
<td>0.010</td>
</tr>
<tr>
<td>( C_{i,\lambda} )</td>
<td>0.051</td>
<td>0.094</td>
<td>0.129</td>
<td>0.161</td>
<td>0.190</td>
<td>0.217</td>
<td>0.242</td>
</tr>
<tr>
<td></td>
<td>(3.46)</td>
<td>(3.49)</td>
<td>(3.58)</td>
<td>(3.69)</td>
<td>(3.81)</td>
<td>(3.90)</td>
<td>(3.96)</td>
</tr>
</tbody>
</table>

is much higher than is estimated from a univariate model of bond expected returns. Treasury bonds, in addition to loading positively on level risk, also provide investors a hedge against increases in the risk premium on stocks. Thus, bonds earn lower average excess returns than in a hypothetical economy where the expected excess market return is constant.

This intuition is formalized by estimating the 3-factor bond model given by Eq. 19. Table 6 gives estimated risk prices (\( \delta_s \)) from the following models:

\[
\begin{align*}
\text{2-Factor model: } & \quad \mathbb{E} \left[ R_i^t \right] = C_{i,M} \delta_M + C_{i,\lambda} \delta_\lambda \\
\text{Univariate Level Risk: } & \quad \mathbb{E} \left[ R_i^t \right] = C_{i,B} \delta_B \\
\text{3-Factor bond model: } & \quad \mathbb{E} \left[ R_i^t \right] = C_{i,M} \delta_M + C_{i,\lambda} \delta_\lambda + C_{i,B} \delta_B
\end{align*}
\]

where \( C_{i,B} \equiv \text{cov} \left( r_{xi}, \Delta r_f^t \right) \) and \( \delta_B \) is the price of the “level” risk. All models are estimated with the intercept restricted to zero. The 2-factor model is estimated using only the stock portfolios from Section 3 (15 quintile portfolios; both stocks and bonds are used as test assets) and hence the risk price estimates are the same as in Section 3.1. The univariate Level Risk model is estimated using only bond excess returns; bonds are also the only test assets. The 3-factor bond model is estimated using all assets, stock portfolios (15 quintile portfolios) as well as bonds. Estimated values for \( \delta_M \) and \( \delta_\lambda \) are essentially unchanged in the 3-factor bond model (relative to the 2-factor estimates). The \( R^2 \) of the 2-factor model is so low because bonds are included as test assets (see Figure 4). Importantly, \( \delta_B \) in the 3-factor
Table 6: Risk Price Estimates

This table shows premia estimates for our 2-factor model (estimated using 15 stock quintile portfolios only; both bonds and stocks included as test assets), the Level Risk model (estimated using bond returns; only bonds used as test assets), and the 3-factor bond model (estimated using both quintile stock portfolios and bonds to price both). Model intercepts are restricted to zero. MAPE is average absolute pricing error, annualized. Moving block bootstrap t-statistics are in parentheses. The sample is 01-Aug-1963 to 31-Dec-2013.

<table>
<thead>
<tr>
<th>Model</th>
<th>δ_M</th>
<th>δ_λ</th>
<th>δ_B</th>
<th>R^2</th>
<th>MAPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-Factor model (est. stocks only; pricing bonds &amp; stocks)</td>
<td>4.54</td>
<td>-11</td>
<td>-</td>
<td>-8.4</td>
<td>2.5</td>
</tr>
<tr>
<td>(4.45) (-5.29)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Level Risk (bonds only)</td>
<td>-</td>
<td>-</td>
<td>-53.7</td>
<td>95.9</td>
<td>0.0705</td>
</tr>
<tr>
<td>(-1.66)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3-Factor bond model (bonds and stocks)</td>
<td>3.29</td>
<td>-9.28</td>
<td>-211</td>
<td>93.9</td>
<td>0.601</td>
</tr>
<tr>
<td>(3.37) (-5.26) (-5.19)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7: Bond Expected Returns

This table shows bond annualized percent returns by maturity in sample (second column), as implied by the univariate Level Risk model (third column), and as implied by our 3-Factor bond model (last column).

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Sample Mean</th>
<th>Level Risk</th>
<th>3-Factor bond model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-year bond</td>
<td>0.69</td>
<td>0.5</td>
<td>0.63</td>
</tr>
<tr>
<td>2-year bond</td>
<td>0.99</td>
<td>0.88</td>
<td>1</td>
</tr>
<tr>
<td>3-year bond</td>
<td>1.2</td>
<td>1.2</td>
<td>1.4</td>
</tr>
<tr>
<td>4-year bond</td>
<td>1.5</td>
<td>1.5</td>
<td>1.7</td>
</tr>
<tr>
<td>5-year bond</td>
<td>1.6</td>
<td>1.7</td>
<td>1.8</td>
</tr>
<tr>
<td>6-year bond</td>
<td>1.8</td>
<td>1.9</td>
<td>1.9</td>
</tr>
<tr>
<td>7-year bond</td>
<td>2</td>
<td>2</td>
<td>1.8</td>
</tr>
</tbody>
</table>

The bond model is \(-211 \leq -54\). This is nearly equal to the back of the envelope prediction given above. Table 7 gives annualized percent returns by maturity in sample, as implied by the univariate Level Risk model, and as implied by our 3-factor bond model. It shows that our 3-factor model performs on par with the level risk model in pricing bonds.

Figure 6 shows average returns vs our 3-factor bond model expected returns for bonds and stock portfolios, with model implied mean excess returns on the horizontal and sample average returns on the vertical axes. The \(45^\circ\) line represents a model with perfect in-sample
Figure 6: Joint Bond and Stock Pricing. The figure shows average returns vs our 3-factor bond model expected returns for bonds and 15 quintile stock portfolios. The line represents a model with perfect in-sample fit (100% $R^2$).

Stocks fit almost as well as in Figure 2 (using our 2-factor model) and bonds fit quite well. It is worth emphasizing that this result is not merely mechanical. Given two factor models, each fitting either cross-section of stocks or bonds, a combined model with all factors need not fit the joint cross-section of bonds and stocks (see Koijen et al. 2015).

Figure 7 decomposes the expected excess return on the various bonds. The premium due to market risk, $C_{i,M}$, is excluded since it is negligible for bonds. Bonds earn a large premium for loading on the “level risk”, whereas they command a large negative premium for loading on the “risk premium” factor. This is consistent with a “flight-to-quality” (Caballero and Krishnamurthy, 2008) interpretation where investors’ appetite for risk falls and they attempt to rebalance their portfolios towards safer securities (like U.S. government debt and “good companies”). Since it is impossible for everyone to rebalance in this way at the same time, prices adjust instead of quantities. The prices of “risky” assets fall relative to the prices of “safer” assets.

Koijen et al. (2015) have a seemingly similar decomposition, albeit with a very different
interpretation. Our 3-factor bond model as well as their model both feature a level factor and a market factor. Instead of our expected stock return factor, they use an expected bond return factor (CP from Cochrane and Piazzesi, 2005). Whereas bond returns load positively on our factor, \( \lambda \), they load negatively on CP. Koijen et al. (2015) find a positive price of CP risk whereas we find a negative price of \( \lambda \) risk. The product of loading \( \times \) risk price yields a negative number in both cases, and hence the pictures look quite similar, but with different interpretation. We find that bonds hedge against increases in expected stock returns but Koijen et al. (2015) find that bonds respond negatively to increases in expected bond returns. Finally, our estimated model produces a term structure of expected returns which is consistent with the data (Table 7). In contrast, the estimates in Koijen et al. (2015) result in a flat term structure.
5 Predicting the Future Market using Cross-Section

In our empirical methodology, we use future realized excess returns as a proxy for today’s market expectation of future excess returns. We further show that this proxy is key in explaining the cross section of stock returns. This observation can be viewed from the reverse perspective. If time-varying expected returns manifest in the cross-section, the cross-section of stock returns can provide information about expected future returns. Indeed, “priced factors ... are innovations in state variables that predict future returns.” (Brennan et al., 2004). It is therefore natural to ask whether cross-sectional variables can predict future returns and to what extent. Few recent papers have looked at this question. Kelly and Pruitt (2015) use the cross-section of dividend-price ratios and show that they indeed predict future returns significantly better than the aggregate dividend-price ratio alone.

Our aim is not to construct the optimal predictor; we only want to show that predictability is indeed present and use it as a robustness check of our methodology. If future returns help to explain the cross-section, the cross-section of returns themselves mechanically must predict future returns. We want to ensure the covariances reported in Table 1 and Figure 1 are economically significant. As such, we use the returns on \( \text{SMB} \), \( \text{HML} \), and \( \text{UMD} \) portfolios to forecast future market returns:

\[
\sum_{i=1}^{H} r_{M,i+1} = a + [DP_t \: MRKT_{t-63:t} \: SMB_{t-63:t} \: HML_{t-63:t} \: UMD_{t-63:t}] b + \varepsilon_{M,t+1} \quad (23)
\]

Each of the \( \text{MRKT}, \text{SMB}, \text{HML}, \text{UMD} \) factors is computed using the past 63 trading days (3 months). Results are robust to varying the lag length.

The top panel of Table 8 reports the coefficient estimates, \( t \)-statistics of estimated coefficients in Eq. 23, and \( R^2 \) at various horizons, \( k \), (3, 6, 9, and 12 months) with only the market and dividend-price ratio included as predictors. There is little evidence of return predictability at horizons up to one year, as evidenced by the insignificant \( t \)-statistics and low \( R^2 \). The bottom panel shows results when including \( \text{SMB}, \text{HML}, \) and \( \text{UMD} \) as additional predictors. We find that all of the coefficients for each variable at 3-9 months horizon are significant and negative and \( R^2 \) increases greatly. We conclude that covariances of FFC factors with expected future market returns are economically significant. Related evidence of predictability is documented by Liew and Vassalou (2000). They show that \( \text{SMB} \) and
Table 8: Time-series predictability of market excess returns using Fama-French factors

The table shows time-series predictability of the stock market risk premium using returns on SMB, HML, and UMD. We estimate the following regression:

$$\sum_{t+1}^{t+k} r_{xM}^t = a + [DP_t \text{ MRKT}_{t-63:t} \text{ SMB}_{t-63:t} \text{ HML}_{t-63:t} \text{ UMD}_{t-63:t}] b + \varepsilon_{M,t+1}.$$

Daily sample from 01-Aug-1963 to 31-Dec-2013. Newey-West t-statistics in parentheses.

<table>
<thead>
<tr>
<th></th>
<th>DP</th>
<th>MRKT</th>
<th>SMB</th>
<th>HML</th>
<th>UMD</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 months</td>
<td>0.018</td>
<td>0.049</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>(1.27)</td>
<td>(0.80)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 months</td>
<td>0.036</td>
<td>0.0074</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.014</td>
</tr>
<tr>
<td></td>
<td>(1.24)</td>
<td>(0.10)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 months</td>
<td>0.052</td>
<td>-0.012</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.019</td>
</tr>
<tr>
<td></td>
<td>(1.21)</td>
<td>(-0.13)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 months</td>
<td>0.066</td>
<td>-0.047</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.023</td>
</tr>
<tr>
<td></td>
<td>(1.20)</td>
<td>(-0.47)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>DP</th>
<th>MRKT</th>
<th>SMB</th>
<th>HML</th>
<th>UMD</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 months</td>
<td>0.019</td>
<td>0.039</td>
<td>-0.27</td>
<td>-0.2</td>
<td>-0.19</td>
<td>0.069</td>
</tr>
<tr>
<td></td>
<td>(1.51)</td>
<td>(0.47)</td>
<td>(-2.49)</td>
<td>(-2.72)</td>
<td>(-3.30)</td>
<td></td>
</tr>
<tr>
<td>6 months</td>
<td>0.038</td>
<td>-0.024</td>
<td>-0.39</td>
<td>-0.39</td>
<td>-0.27</td>
<td>0.078</td>
</tr>
<tr>
<td></td>
<td>(1.48)</td>
<td>(-0.26)</td>
<td>(-2.42)</td>
<td>(-2.45)</td>
<td>(-2.48)</td>
<td></td>
</tr>
<tr>
<td>9 months</td>
<td>0.053</td>
<td>-0.077</td>
<td>-0.39</td>
<td>-0.47</td>
<td>-0.35</td>
<td>0.079</td>
</tr>
<tr>
<td></td>
<td>(1.41)</td>
<td>(-0.67)</td>
<td>(-2.18)</td>
<td>(-2.16)</td>
<td>(-2.57)</td>
<td></td>
</tr>
<tr>
<td>12 months</td>
<td>0.067</td>
<td>-0.1</td>
<td>-0.41</td>
<td>-0.37</td>
<td>-0.42</td>
<td>0.073</td>
</tr>
<tr>
<td></td>
<td>(1.36)</td>
<td>(-0.75)</td>
<td>(-1.71)</td>
<td>(-1.60)</td>
<td>(-2.84)</td>
<td></td>
</tr>
</tbody>
</table>

HML help forecast future rates of economic growth.

Finally, we repeat the forecasting exercise using the first principal component of the NMV anomaly returns (PC1). Table 9 shows that PC1 alone has similar forecasting ability\(^{32}\) to the three FFC factors combined. The statistical significance for PC1 is substantially higher, likely because SMB, HML, and UMD each contain substantial idiosyncratic “noise”, adding uncertainty to the estimates.

\(^{32}\)As measured by $R^2$
Table 9: Time-series predictability of market excess returns using NMV anomalies

The table shows time-series predictability of the stock market risk premium using the first principal component of NMV long-short portfolios. We estimate the following regression:

\[ \sum_{t+1}^{t+k} r_{xM} = a + [DP_t \ NMV_{t-63:t}] b + \varepsilon_{M,t+1}. \]


<table>
<thead>
<tr>
<th></th>
<th>DP</th>
<th>MRKT</th>
<th>NMVPC1</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 months</td>
<td>0.019</td>
<td>0.047</td>
<td>-</td>
<td>0.011</td>
</tr>
<tr>
<td></td>
<td>(1.33)</td>
<td>(0.67)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 months</td>
<td>0.038</td>
<td>0.015</td>
<td>-</td>
<td>0.016</td>
</tr>
<tr>
<td></td>
<td>(1.33)</td>
<td>(0.19)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 months</td>
<td>0.056</td>
<td>-0.02</td>
<td>-</td>
<td>0.022</td>
</tr>
<tr>
<td></td>
<td>(1.33)</td>
<td>(-0.20)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 months</td>
<td>0.072</td>
<td>-0.035</td>
<td>-</td>
<td>0.027</td>
</tr>
<tr>
<td></td>
<td>(1.33)</td>
<td>(-0.34)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>DP</th>
<th>MRKT</th>
<th>NMVPC1</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 months</td>
<td>0.021</td>
<td>-0.022</td>
<td>-0.28</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>(1.56)</td>
<td>(-0.34)</td>
<td>(-5.44)</td>
<td></td>
</tr>
<tr>
<td>6 months</td>
<td>0.041</td>
<td>-0.079</td>
<td>-0.38</td>
<td>0.069</td>
</tr>
<tr>
<td></td>
<td>(1.52)</td>
<td>(-0.93)</td>
<td>(-4.03)</td>
<td></td>
</tr>
<tr>
<td>9 months</td>
<td>0.059</td>
<td>-0.12</td>
<td>-0.42</td>
<td>0.065</td>
</tr>
<tr>
<td></td>
<td>(1.49)</td>
<td>(-1.13)</td>
<td>(-3.24)</td>
<td></td>
</tr>
<tr>
<td>12 months</td>
<td>0.075</td>
<td>-0.14</td>
<td>-0.43</td>
<td>0.062</td>
</tr>
<tr>
<td></td>
<td>(1.46)</td>
<td>(-1.39)</td>
<td>(-3.13)</td>
<td></td>
</tr>
</tbody>
</table>

6 Conclusions

In this paper we link two well-documented empirical facts: (1) time-series variation in aggregate discount rates and (2) cross-sectional dispersion in average returns. We present a model with time-varying risk aversion which generates endogenous variation in expected excess market returns. Investors’ hedging demand with respect to discount rate shocks results in an equilibrium pricing kernel in which those shocks appear as an additional factor. In the cross-section of assets, differences in return covariance with these shocks produces differ-
ences in expected returns. Crucially, the model predicts that shocks to risk premia increase investors’ marginal utility and in equilibrium have a negative price of risk.

We confront the model with return data on a large set well known asset pricing anomalies, including size, value, and momentum. Instead of using the typical VAR approach, we overcome the unobservability of expected market returns by using future realized returns as a proxy. This allows us to estimate factor loadings without actually observing the factor itself. Across empirical specifications, we find consistent support for a negative risk price on our “risk premium” factor. Our theoretically motivated pricing model performs nearly as well as standard empirical factor models in fitting the cross-sectional dispersion in average returns. This is surprising since we augment the standard CAPM with a single additional economically motivated factor. These empirical results and theoretical motivation contrast sharply with previous related work. Our main conclusion is that shocks to aggregate expected returns are “bad” for the representative agent, who requires higher expected returns for holding exposure to these shocks. Previous studies conclude that such shocks increase investor utility (decreasing marginal utility) and hence, higher exposure results in lower equilibrium expected returns.

Finally, we extend the model and empirical specification to jointly price stocks and bonds. We find that bonds hedge against discount rate shocks, earning negative expected returns (and downward sloping term premium). This is offset by exposure to risk-free rate shocks. Overall, the model generates a small and upward sloping term premium, as found in the data.

Taken as a whole, the empirical evidence is consistent with our model in which increases in investor risk aversion, or “flight-to-quality”, generate time-varying aggregate discount rates. In the model and data, cross-sectional heterogeneity in sensitivity to these shocks generates the observed spread in average returns. Like much of the literature, we examine the equilibrium conditions of an asset pricing model. It remains uncertain as to precisely what generates the different covariances of asset returns with discount rate shocks.
 References


Appendix

A The Model: Solution

Consider the specification in Section 2.1. Since preferences are homogeneous of degree one in wealth, we define the value function as

\[ J (W_t, \alpha_t) = \phi (\alpha_t) W_t \equiv \phi_t W_t \]  

(24)

Taking the log of Eq. 1 and evaluating first-order conditions for consumption yields:

\[ C_t = (1 - \delta) W_t \]  

(25)

A unit elasticity of substitution therefore implies that consumption is proportional to wealth, i.e. that agents possess a form of (rational) myopia in consumption and savings decisions. The optimal portfolio choice is fully dynamic though unless risk aversion is also unity (Giovannini and Weil, 1989).

Substitute Eq. 24 into Eq. 1:

\[ \phi_t = (1 - \delta)^{1 - \delta} \delta \left( E_t \left[ \phi_t^{\alpha_t} \left( R_{t+1}^M \right)^{\alpha_t} \right] \right) \]  

(26)

or

\[ E_t \left[ B_t \phi_t^{\alpha_t} \left( R_{t+1}^M \right)^{\alpha_t} \right] = 1 \]  

(27)

where \( B_t = \left( \frac{1 - \delta}{\phi_t} \right)^{\alpha_t} \).

Take logs and guess that

\[ \ln \phi_t = a_0 + a_1 \gamma_t \]  

(28)

and substitute in:

\[ a_0 + a_1 \gamma_t = (1 - \delta) \ln (1 - \delta) + \delta (a_0 + a_1 [\mu_\gamma + \phi_\gamma \gamma_t]) \\ + \frac{1}{2} \delta (1 - \gamma_t) \left( a_1^2 \sigma_\gamma^2 + \sigma_M^2 \right) \]  

(29)

where we used the fact that \( E_t r_{t+1}^M = \mu_c - \ln \delta \) (follows from Eq. 2). It follows that

\[ a_1 = - \frac{1}{2} \delta \frac{a_1^2 \sigma_\gamma^2 + \sigma_M^2}{1 - \delta \phi_\gamma} < 0. \]  

(30)

Portfolio problem of an investor is given by

\[ \max_{\theta_t} E_t \left[ \phi_t^{\alpha_{t+1}} (\theta_t' R_{t+1})^{\alpha_{t+1}} \right] \]  

(31)
subject to $\theta'1_n = 1$. First-order conditions:

$$E_t \left[ \phi_t^{\alpha_t} \left( R^{M}_{t+1} \right)^{\alpha_t-1} \left( R^i_{t+1} - R^i_{t+1} \right) \right] = 0 \tag{32}$$

where $R^i_{t+1}$ denotes the return on any asset within a portfolio (for instance, risk-free rate). Combining this equation with Eq. 27, we obtain the Euler equation for any asset:

$$E_t \left[ B_t \phi_t^{\alpha_t} \left( R^{M}_{t+1} \right)^{\alpha_t-1} R^i_{t+1} \right] = 1 \tag{33}$$

We assume that log returns are jointly normally distributed\textsuperscript{33}. Taking logs and computing conditional expectations of log-normal, it follows that the risk premium on any asset is given by:

$$E_t \left( r_{x_{t+1}}^i \right) + \frac{1}{2} \text{var}_t \left( r_{x_{t+1}}^i \right) = \gamma_t \text{cov}_t \left( r_{x_{t+1}}^i, r_{x_{t+1}}^M \right) + a_1 (\gamma_t - 1) \text{cov}_t \left( r_{x_{t+1}}^i, \gamma_{t+1} \right) \tag{34}$$

where $E_t \left( r_{x_{t+1}}^i \right) \equiv E_t r^i_{t+1} - r^f_t$ denotes the risk premium on asset $i$. Specializing this to the market return we get the expression for the market risk premium

$$E_t \left( r_{x_{t+1}}^M \right) + \frac{1}{2} \text{var}_t \left( r_{x_{t+1}}^M \right) = \gamma_t \sigma^2_M \tag{35}$$

Using the fact that $E_t r^M_{t+1} = \mu_c - \ln \delta$, we can also solve for the risk-free rate:

$$r^f_t = -\ln \delta + \left( \mu_c + \frac{1}{2} \sigma^2_M \right) - \gamma_t \sigma^2_M. \tag{36}$$

**B Operationalizing the Model**

**B.1 Unconditional Pricing Relation**

Let the conditional model be

$$E_t \left( r_{x_{t+1}}^i \right) + \frac{1}{2} \text{var}_t \left( r_{x_{t+1}}^i \right) = \text{cov}_t \left( r_{x_{t+1}}^i, f_{t+1} \right) \times \delta_{f,t}$$

$$+ \text{cov}_t \left( r_{x_{t+1}}^i, \lambda_{t+1} \right) \times \delta_{\lambda,t} \tag{37}$$

where $f_{t+1} = \left[ f_{t+1}^{(1)}, f_{t+1}^{(2)}, \ldots f_{t+1}^{(k)} \right]$ denotes $k$ factors that are log excess returns, $\lambda_{t+1} = E_{t+1} f_{t+2}$ denotes risk premia on those excess returns starting one period ahead; $\delta_{f,t}$ and $\delta_{\lambda,t}$ are of size $k \times 1$ and denote corresponding factor risk prices.

\textsuperscript{33}The following results are exact in this case or can be considered an approximation when the assumption is relaxed.
Note that the model in Appendix A is the special case of Eq. 37 in which \( f_{t+1} \equiv r_{t+1}^M \) and \( \lambda_{t+1} \equiv \mathbb{E}_{t+1} r_{t+2}^M \).

**Assumption 1.** All conditional covariances \( c'_{if} \equiv \text{cov}_t (r_{t+1}^i, f_{t+1}) \), \( c'_{i\lambda} \equiv \text{cov}_t (r_{t+1}^i, \lambda_{t+1}) \), and variance \( V_i = \text{var}_t (r_{t+1}^i) \) are constant over time.

This assumption is automatically satisfied in our model for the market portfolio. We therefore only assume that systematic components of excess returns on other assets (which are exogenous to the model) are homoskedastic. Idiosyncratic components (orthogonal to \( \varepsilon^c_t \) and \( \varepsilon^\gamma_t \)) are allowed to be heteroskedastic though.

With this notation in hand, we can rewrite

\[
\mathbb{E}_t (r_{t+1}^i) + \frac{V_i}{2} = c'_{if} \times \delta_{f,t} + c'_{i\lambda} \times \delta_{\lambda,t}
\]

where \( C_i = \begin{bmatrix} c'_{if} & c'_{i\lambda} \end{bmatrix} \) and \( D_t = \begin{bmatrix} \delta_{f,t} \\ \delta_{\lambda,t} \end{bmatrix} \).

**Assumption 2.** Risk premia \( \lambda_{t+1} = \mathbb{E}_{t+1} f_{t+2} \) follow a VAR(1) process:

\[
\lambda_{t+1} = \Lambda_0 + \Lambda \lambda_t + \Omega_{\lambda,t+1}
\]

where \( \Omega_{\lambda} \) are mean-zero errors uncorrelated with \( \lambda_t \).

Note that this assumption is automatically satisfied in our model given Eq. 4 and the expression for the market risk premium in Eq. 35.

**Theorem 4.** Given Assumption 1 and Assumption 2 and the conditional model (37), we obtain the following linear pricing relation:

\[
\mathbb{E} (r_{t+1}^i) + \frac{V_i}{2} = \text{cov}_t (r_{t+1}^i, f_t) \times \hat{\delta}_f
\]

\[
+ \text{cov}_t (r_{t+1}^i, \lambda_t) \times \hat{\delta}_\lambda
\]

\[
\equiv \text{cov}_t (r_{t+1}^i, [f_t \ lambda]) \times \hat{D}
\]

where \( \hat{D} = \Theta^{-1} \mathbb{E} [D_t] \)

\[
\Theta = \mathbb{I}_{2k} + \Phi_{f \lambda} \times \begin{bmatrix} \mathbb{I}_k & \Lambda' \end{bmatrix}
\]

\[
\Phi_{f \lambda} = \begin{bmatrix} \Gamma_f \Sigma_f + \Gamma_f \lambda \Sigma_{f \lambda}' \\ \Gamma_{f \lambda} \Sigma_f + \Gamma_{f \lambda} \Sigma_{f \lambda} \end{bmatrix}
\]

where \( \mathbb{I}_{2k} \) denotes an identity matrix of size \( 2k \times 2k \); \( \Sigma_f = \text{var}_t [f_{t+1}], \Sigma_{f \lambda} = \begin{bmatrix} c_{f_1 \lambda} & c_{f_2 \lambda} & \cdots & c_{f_k \lambda} \end{bmatrix}' \),

and \( c'_{f \lambda} \equiv \text{cov}_t (f_{t+1}, \lambda_{t+1}) \) denote conditional covariances; \( \Gamma_f = \text{cov} [\delta_f, \delta_\lambda], \Gamma_{f \lambda} = \text{var} [\delta_f], \Gamma_{\lambda} = \text{var} [\delta_\lambda] \) denote unconditional covariances.
Proof. Take expectations and apply the law of total covariance to the RHS of Eq. 37 to get

\[
\begin{align*}
\mathbb{E}\left( r_{xt}^i \right) + \frac{V_i}{2} &= \text{cov} \left( r_{xt}^i, f_t \right) \times \mathbb{E} \left[ \delta_{f,t} \right] + \text{cov} \left( r_{xt}^i, \lambda_t \right) \times \mathbb{E} \left[ \delta_{\lambda,t} \right] \\
&- \text{cov} \left( \mathbb{E}_t r_{xt+1}^i, \mathbb{E}_t f_{t+1} \right) \times \mathbb{E} \left[ \delta_{f,t} \right] - \text{cov} \left( \mathbb{E}_t r_{xt+1}^i, \mathbb{E}_t \lambda_{t+1} \right) \times \mathbb{E} \left[ \delta_{\lambda,t} \right]
\end{align*}
\]

(46)

Using Assumption 1, we can write \( \mathbb{E}_t \lambda_{t+1} = \Lambda_0 + \Delta \lambda_t \). Then

\[
\begin{align*}
\mathbb{E}\left( r_{xt}^i \right) + \frac{V_i}{2} &= C_i \times D = \text{cov} \left( r_{xt}^i, \left[ f_t, \lambda_t \right] \right) \times D \\
&- \text{cov} \left( \mathbb{E}_t r_{xt+1}^i, \mathbb{E}_t f_{t+1} \right) \times \left[ \mathbb{I}_k, \Lambda' \right] \times D
\end{align*}
\]

(47)

where \( D = \mathbb{E} \left[ \delta_{f,t,\lambda,t} \right] \).

We now use Eq. 38 to substitute the expressions for two covariates in \( \text{cov} \left( \mathbb{E}_t r_{xt+1}^i, \mathbb{E}_t f_{t+1} \right) \) term

\[
\begin{align*}
\text{cov} \left( c'_f \times \delta_{f,t} + c'_f \times \delta_{\lambda,t} - \frac{V_i}{2} \Sigma_f \times \delta_{f,t} + \Sigma_f \times \delta_{\lambda,t} - \frac{1}{2} V_f \right)
\end{align*}
\]

(48)

\[
\begin{align*}
= c'_f \text{var} \left[ \delta_f \right] \Sigma_f + c'_f \text{var} \left[ \delta_{\lambda} \right] \Sigma'_{f\lambda} + c'_f \text{cov} \left[ \delta_f, \delta_{\lambda} \right] \Sigma_{f\lambda} + c'_f \text{cov} \left[ \delta_f, \delta_{\lambda} \right] \Sigma'_{f\lambda}
\end{align*}
\]

(49)

\[
\begin{align*}
= C_i \times \left[ \Gamma_f \Sigma_f + \Gamma_f \Sigma'_{f\lambda} \right. \\
\left. \Gamma_{f\lambda} \Sigma_{f\lambda} + \Gamma_{f\lambda} \Sigma_f \right]
\end{align*}
\]

(50)

\[
\equiv C_i \times \Phi_{f\lambda}
\]

(51)

where \( V_f = [V_{f1}, V_{f2}, ..., V_{f_k}]' \).

Plugging this back in Eq. 47 and collecting terms on the LHS and RHS gives

\[
\begin{align*}
C_i \times \left[ \mathbb{I}_{2k} + \Phi_{f\lambda} \times \left[ \mathbb{I}_k, \Lambda' \right] \right] \times D = \text{cov} \left( r_{xt}^i, \left[ f_t, \lambda_t \right] \right) \times D
\end{align*}
\]

(52)

Denote

\[
\Theta = \mathbb{I}_{2k} + \Phi_{f\lambda} \times \left[ \mathbb{I}_k, \Lambda' \right]
\]

(53)

and assume that \( \Theta \) is invertible. Since Eq. 52 has to hold for any \( D \), we get

\[
\begin{align*}
C_i = \text{cov} \left( r_{xt}^i, \left[ f_t, \lambda_t \right] \right) \times \Theta^{-1}
\end{align*}
\]

(54)

43
and hence
\[ E\left(rx_t^i\right) + \frac{V_t}{2} = C_i \times D \]
\[ = \text{cov}\left(rx_t^i, [f_t, \lambda_t]\right) \times \{\Theta^{-1}D\} \]
\[ = \text{cov}\left(rx_t^i, [f_t, \lambda_t]\right) \times \tilde{D} \]  
\[ (55) \]
\[ (56) \]
\[ (57) \]

Finally, we prove the following theorem that allows us to link unconditional price of discount-rate to risk to its conditional counterpart.

**Theorem 5.** In the model specification as defined in Section 2.1, negative unconditional price of discount-rate risk implies negative expected conditional price of risk:

\[ \hat{\delta}_\lambda < 0 \implies E\left[\delta_t^\lambda\right] < 0 \]
\[ (58) \]

where \( \delta^\lambda \) is the price of the discount-rate risk in the conditional model in Eq. 5 and \( \hat{\delta}_\lambda \) is the price of the discount-rate risk in the unconditional pricing relation in Eq. 7.\(^{34}\)

**Proof.** Solve the quadratic equation in Eq. 30:

\[ a_1 \sigma^2_\gamma = -\left(\frac{1 - \delta \phi}{\delta}\right) \pm \sqrt{\left(\frac{1 - \delta \phi}{\delta}\right)^2 - \sigma^2_\gamma \sigma^2_M} \]
\[ (59) \]

Both roots are negative. We pick the one closer to zero\(^{35}\) and bound it by

\[ a_1 \sigma^2_\gamma \geq -\frac{1 - \delta \phi}{\delta}. \]
\[ (60) \]

Theorem 4 implies that \( E\left[D_t\right] = \Theta D \), where \( \Theta \) is given by Eq. 44. Substitute \( \Sigma_f \equiv \var_t\left(rx_{t+1}^M\right) = \sigma^2_M \), \( \Sigma_{f\lambda} = 0 \), \( \Gamma_f \equiv \var\left(\delta_M\right) \), and \( \Gamma_{f\lambda} \equiv \text{cov}\left(\delta_M, \delta_\lambda\right) = \frac{\alpha}{\sigma^2_M} \left(\frac{\sigma^2}{1 - \phi^2}\right) \) to obtain:

\[ E\delta_\lambda = \text{cov}\left(\delta_M, \delta_\lambda\right) \text{var}_t\left(rx_{t+1}^M\right) \delta_M + \left[1 + \text{cov}\left(\delta_M, \delta_\lambda\right) \text{var}_t\left(rx_{t+1}^M\right) \phi\right] \hat{\delta}_\lambda. \]
\[ (61) \]

\(^{34}\)We require the following two technical conditions to hold to obtain this result: (i) the determinant of the quadratic equation in Eq. 30 is non-negative: \( \left(\frac{1 - \delta \phi}{\delta}\right)^2 - \sigma^2_\gamma \sigma^2_M \geq 0 \); and (ii) rate of time discounting \( \delta \geq \phi \), or, using the identity in Eq. 25, that consumption-to-wealth ratio \( \frac{C}{W} \leq 1 - \phi \) (this condition must be satisfied for all plausible calibrations).

\(^{35}\)This root implies that agents are more sensitive to variation in risk aversion when consumption volatility is high.
It is clear that if the term in square brackets is non-negative, then \( \mathbb{E} \delta \lambda \) is always negative when \( \hat{\delta}_\lambda < 0 \). The sufficient condition:

\[
1 + \text{cov} (\delta_M, \delta_\lambda) \text{var} (r x_{t+1}^M) \phi \quad = \quad 1 + a_1 \sigma_y^2 \frac{\phi}{1 - \phi^2}
\]

\[
\geq 1 - \left( \frac{1 - \delta \phi}{\delta} \right) \frac{\phi}{1 - \phi^2} \geq 0
\]

This inequality holds when

\[
\delta \geq \phi.
\]

The restriction is equivalent to \( \frac{C}{\phi} \leq 1 - \phi \) as implied by Eq. 25. All plausible calibrations must satisfy this restriction. Therefore, \( \mathbb{E} \delta \lambda \) is always negative when \( \hat{\delta}_\lambda < 0 \).

\[\Box\]

**B.2 Realized-Returns Estimator**

In sample, \( \hat{\text{cov}} (r x_{t+1}^i, \epsilon_{t+1}^M) \) is not equal to zero. Moreover, because \( \text{var} (\epsilon_t^M) \gg \text{var} (\epsilon_t^\lambda) \) and \( \lambda_t \) is persistent, we can increase the power of the estimate by proxying for \( \lambda_t \) in (41) with the moving sum of future realized excess returns, \( \hat{\lambda}_{t:t+T} = \sum_{j=0}^{T} r x_{t+j+1}^M \). Note that by combining Eq. 4 and Eq. 6, we can write the process for excess returns as

\[
\lambda_{t+1} = \mu \lambda_t + \phi \lambda_t + \epsilon_{t+1}^\lambda
\]

where \( \phi = \phi_y \). We can then rewrite the moving-sum estimator as

\[
\hat{\lambda}_{t:t+T} = \sum_{j=0}^{T} (\lambda_{t+j} + \epsilon_{t+j+1}^M) = \text{const} + \left( \frac{1 - \phi^{T+1}}{1 - \phi} \right) \lambda_t
\]

\[
+ \left( \sum_{j=0}^{T-1} \phi^j \epsilon_{t+1}^\lambda \right) + \left( \sum_{j=0}^{T-2} \phi^j \epsilon_{t+2}^\lambda \right) + \ldots + \epsilon_{t+T}^\lambda + \sum_{j=0}^{T} \epsilon_{t+j+1}^M
\]

For any information set \( \mathcal{F}_t \) at time \( t \) and any \( j \geq 1 \), \( \mathbb{E} [\epsilon_{t+j}^\lambda | \mathcal{F}_t] = 0 \) and \( \mathbb{E} [\epsilon_{t+j}^M | \mathcal{F}_t] = 0 \) since \( \epsilon_{t+j}^M \) and \( \epsilon_{t+j}^\lambda \) are projection errors. Therefore the following identity holds in population:

\[
\text{cov} (r x_t^i, \hat{\lambda}_{t:t+T}) = \left( \frac{1 - \phi^{T+1}}{1 - \phi} \right) \times \text{cov} (r x_t^i, \lambda_t)
\]

Let,

\[
C_T = \left( \frac{1 - \phi^{T+1}}{1 - \phi} \right)^{-1} \hat{\text{cov}} (r x_t^i, \hat{\lambda}_{t:t+T})
\]

be the estimate of \( \text{cov} (r x_t^i, \lambda_t) \) using horizon \( T \) for the moving-sum. Then “optimal” \( T \) is the horizon which minimizes the RMSE of \( C_T \). We find the optimal \( T \) through simulation. We simulate 10,000
histories of length 50 years (of daily data) using the calibrated parameters given in Appendix B.3 of the paper. For each history, we compute $C_T$ for $T=1..36$ months for an asset with zero market $\beta$ and the same expected return as HML. Figure 8 shows RMSE $[C_T]$ computed across the simulations. Recall that covariance builds monotonically with horizon so the $\frac{1-\phi^T+1}{1-\phi}$ term ensures $C_T$ has the same $plim$ for all $T$. As $T$ is increased, sampling uncertainty initially decreases rapidly, showing the power of cumulating returns. The estimator is robust for horizon of 6 to 24 months, and the minimum is attained at ~13 months. Appendix E.2 gives estimates of unconditional risk prices for the horizons. The precision of the estimates and model fit are quite similar across horizons, with the maximal values occurring at 12 months.

![Figure 8: Simulated estimate uncertainty.](image)

**B.3 Risk Prices**

We perform a simple calibration using daily estimates to assess plausible magnitudes of unconditional risk prices predicted by the model. We pick the following parameter values (annualized):

- time preference parameter (from Bansal and Yaron, 2004) $\delta = 0.976$;
- persistence of risk aversion (from data) $\phi = 0.32$;
- average risk aversion of $\bar{\gamma} = 4.5$, which implies $\mu_{\gamma} = 4.5 (1 - \phi) = 0.02$;
- volatility of risk aversion $\sigma_{\gamma} = 1.5$;
- length of the cumulative sum of market returns $T = 126$ trading days;
- volatility of market returns (from data) $\sigma_M \equiv \sigma_c = 16\%$.
Based on these parameters, expected conditional risk prices are $E\delta_t^M = \bar{\gamma} = 4.5$ and $E\delta_t^\lambda = a_1 \frac{\bar{\gamma}}{\bar{R}_M} (E\gamma_t - 1) = -937$, where $a_1$ is given by Eq. 59. To find unconditional prices of risk, we rely on Theorem 4; namely, we first compute $\Theta$ using Eq. 44 and then plug it into Eq. 43 to get unconditional risk prices. As a result, we obtain $\tilde{\delta}_M = 4.5$ and $\tilde{\delta}_\lambda = -9.9$, nearly matching the estimates in Section 3.1.

C Bond Model

We assume that the rate of time preference follows an AR(1) process,

$$\delta_{t+1} = \mu_\delta + \phi_\delta \delta_t + \sigma_\delta \varepsilon_{t+1}$$  \hspace{1cm} (69)

and generalize the Bellman Eq. 1 to the case when $\delta$ is stochastic:

$$J(W_t, X_t) = \max_{\{C_t, \Theta_t\}} \left\{ C_t^{1-\delta_t} \left( E_t \left[ J(W_{t+1}, X_{t+1})^{\alpha_t} \right] \right)^{\frac{\alpha_t}{\alpha_t}} \right\}$$  \hspace{1cm} (70)

where $X_t = \{\alpha_t, \delta_t\}$ is the state vector. Since preferences are homogeneous of degree one in wealth, we define

$$J(W_t, X_t) = \phi(X_t) W_t \equiv \phi_t W_t$$  \hspace{1cm} (71)

Taking the log of Eq. 70 and evaluating first-order conditions for consumption yields:

$$C_t = (1 - \delta_t) W_t$$  \hspace{1cm} (72)

A unit elasticity of substitution therefore implies that consumption-to-wealth ratio changes only in response to shocks to the time preference parameter. Substitute Eq. 71 and Eq. 72 into Eq. 70:

$$E_t \left[ B_t \phi_{t+1}^{\alpha_t} \left( R_{t+1}^M \right)^{\alpha_t} \right] = 1$$  \hspace{1cm} (73)

where $B_t = \left( \frac{(1-\delta_t)^{1-\delta_t} \delta_t^{\delta_t}}{\phi_t} \right)^{\frac{\alpha_t}{\alpha_t}}$. Take logs:

$$\ln \phi_t = (1 - \delta_t) \ln (1 - \delta_t) + \delta_t \ln \delta_t + \delta_t \left[ E_t \ln \phi_{t+1} + E_t r_{t+1}^M + \frac{1}{2} \alpha_t V \right]$$  \hspace{1cm} (74)

where $V \equiv \text{Var}_t \left( \ln \phi_{t+1} + r_{t+1}^M \right)$. Eq. 2 and Eq. 72 imply that

$$E_t r_{t+1}^M = \mu_c - \ln \delta_t + \ln (1 - \delta_t) - E_t \ln (1 - \delta_{t+1})$$  \hspace{1cm} (75)
Let $\bar{\delta}$ denote the unconditional mean of $\delta_t$. Guess that

$$\ln \phi_t = a_0 + a_1 \gamma_t + a_2 \delta_t$$

and plug the guess into Eq. 74:

$$a_0 + a_1 \gamma_t + a_2 \delta_t = \ln (1 - \delta_t) + \delta_t (a_0 + a_1 [\mu_\gamma + \phi_\gamma \gamma_t] + a_2 [\mu_\delta + \phi_\delta \delta_t]) + \delta_t \mu_c - \delta_t \ln (1 - \delta_{t+1}) + \frac{1}{2} \delta_t (1 - \gamma_t) \mathcal{V}$$

We employ the following first-order approximations:

$$\ln (1 - \delta_t) \approx \kappa_0 + \kappa_1 \delta_t$$

$$\delta_t \gamma_t \approx \tilde{\delta} \gamma + \delta (\gamma_t - \bar{\gamma}) + \bar{\gamma} (\delta_t - \bar{\delta})$$

$$\delta_t^2 \approx \bar{\delta}^2 + 2 \bar{\delta} (\delta_t - \bar{\delta})$$

where $\kappa_1 < 0$. Plug in:

$$a_0 + a_1 \gamma_t + a_2 \delta_t = \kappa_0 + \left( \kappa_1 + a_0 + a_1 \mu_\gamma + a_2 \mu_\delta + \mu_c - \kappa_0 - \kappa_1 \mu_\delta + \frac{1}{2} \mathcal{V} \right) \delta_t + \left( a_1 \phi_\gamma - \frac{1}{2} \mathcal{V} \right) \left[ \bar{\delta} \gamma + \delta (\gamma_t - \bar{\gamma}) + \bar{\gamma} (\delta_t - \bar{\delta}) \right] + \phi_\delta (a_2 - \kappa_1) \left[ \bar{\delta}^2 + 2 \bar{\delta} (\delta_t - \bar{\delta}) \right]$$

where

$$\mathcal{V} = \text{Var}_t \left( \ln \phi_{t+1} + r_{t+1}^M \right) \approx a_1^2 \sigma_\gamma^2 + (a_2^2 + \kappa_1^2) \sigma_\delta^2 + \sigma_M^2$$

By equalizing constants and coefficients multiplying $\gamma_t$ and $\delta_t$, we get

$$a_1 = -\frac{1}{2} \bar{\delta} \mathcal{V} \left( 1 - \bar{\delta} \phi_\gamma \right)^{-1} < 0$$

$$a_2 = \kappa_1 + \left( 1 - \mu_\delta - 2 \bar{\delta} \phi_\delta \right)^{-1} \left[ a_0 + \mu_c - \kappa_0 + a_1 (\mu_\gamma + \bar{\gamma} \phi_\gamma) - \frac{1}{2} (\bar{\gamma} - 1) \mathcal{V} \right]$$

$$a_0 = \kappa_0 - \bar{\delta} \gamma \left( a_1 \phi_\gamma - \frac{1}{2} \mathcal{V} \right) - \bar{\delta}^2 \phi_\delta (a_2 - \kappa_1)$$

We need to solve this system of quadratic equations to get expressions for $a_0$, $a_1$, and $a_2$. Note that value function should be decreasing in risk aversion $\gamma_t$ and increasing in the time preference rate $\delta_t$ ($\delta = 1$ corresponds to no discounting).

We now solve the portfolio problem of an investor. Following the derivations in Appendix A,
we obtain the Euler equation for any asset:

$$E_t \left[ B_t \phi_{t+1}^{\alpha_t} \left( R_{t+1}^M \right)^{\alpha_t-1} R_{t+1}^i \right] = 1 \tag{86}$$

Taking logs and computing conditional expectations of log-normal, it follows that the risk premium on any asset is given by:

$$\mathbb{E}_t \left( r_{x_{i,t+1}} \right) + \frac{1}{2} \text{var}_t \left( r_{x_{i,t+1}} \right) = \gamma_t \text{cov}_t \left( r_{x_{i,t+1}}, R_{t+1}^M \right) + a_1 (\gamma_t - 1) \text{cov}_t \left( r_{x_{i,t+1}}, \gamma_{t+1} \right)$$

$$+ a_2 (\gamma_t - 1) \text{cov}_t \left( r_{x_{i,t+1}}, \delta_{t+1} \right) \tag{87}$$

Specializing this to the market return we get the expression for the market risk premium

$$\mathbb{E}_t \left( r_{x_{M,t+1}} \right) + \frac{1}{2} \text{var}_t \left( r_{x_{M,t+1}} \right) = \gamma_t \left( \sigma^2_M + \kappa_1^2 \sigma^2_\delta \right) + a_2 (\gamma_t - 1) \left( -\kappa_1 \right) \sigma^2_\delta \tag{88}$$

Using Eq. 75, we solve for the risk-free rate,

$$r^f_t = \mu_c - \ln \delta_t + (1 - \delta_0) \kappa_1 \delta_t - \gamma_t \left( \sigma^2_M + \kappa_1^2 \sigma^2_\delta \right) + a_2 (\gamma_t - 1) \kappa_1 \sigma^2_\delta \tag{89}$$

Finally, we can log-linearize $\ln \delta_t$ term in Eq. 89, use Eq. 88 and Eq. 89 to express $\delta_t$ and $\gamma_t$ in terms of $r^f_t$ and $\lambda_t$, and substitute them into Eq. 87 to arrive at the following recasted conditional model:

$$\mathbb{E}_t \left( r_{x_{i,t+1}} \right) + \frac{1}{2} \text{var}_t \left( r_{x_{i,t+1}} \right) = \delta^i_{M} \text{cov}_t \left( r_{x_{i,t+1}}, R_{t+1}^M \right) + \delta^\lambda \text{cov}_t \left( r_{x_{i,t+1}}, \lambda_{t+1} \right)$$

$$+ \delta^{r^f} \text{cov}_t \left( r_{x_{i,t+1}}, r^f_{t+1} \right) \tag{90}$$

where risk prices $\delta^{r^f}$ and $\delta^\lambda$ are both negative. To see that $\delta^{r^f}$ is negative, note that $\frac{\partial r^f_t}{\partial \delta_t} < 0$ and that the price of time-preference risk $a_2 (\gamma_t - 1)$ in Eq. 37 is positive. Negativity of $\delta^\lambda$ follows from Eq. 89 and Eq. 37. Bonds therefore command positive and increasing-in-maturity risk premium due to their exposure to risk-free rate shocks (corresponding covariance is negative and falling in maturity) and negative and decreasing-in-maturity risk premium due to their exposure to discount-rate (or risk aversion) shocks (corresponding covariance is positive and increasing in maturity; see Table 5).
D Bootstrap

We construct standard errors for risk prices using the moving block bootstrap procedure as follows. There are $N$ test assets, $k$ factors, and $T$ periodic observations. All moments are sample moments taken as expectations across $T$. The general model is $r_t = C'\lambda + \varepsilon_t$. $C$ is an $N \times k$ matrix of univariate covariances, $\text{cov}(r_t, f_t)$, where $f_t$ are the $k$ factors. Notice the model is homoskedastic. $\lambda$ is the vector of risk prices, and $\varepsilon_t$ is the vector of pricing errors. The null hypothesis is that $\lambda = 0$ and $E[\varepsilon_t] = 0$. The alternative is $\lambda \neq 0$.

Bootstrap procedure:

1. Estimate $\hat{C}$ and $\hat{\lambda}$ via usual two-stage regression

2. Construct $\tilde{r}_t = r_t - E[r_t]$ 

   (a) $\tilde{r}_t$ is satisfies the null hypothesis of risk-neutrality and maintains all other properties of the true DGP which are shared with the null. In particular, $\text{cov}(\tilde{r}_t, f_t) = \text{cov}(r_t, f_t)$

3. Let $L$ be the bootstrap window width. Let $X = \begin{bmatrix} \tilde{r}_1' & f_1' \\ \vdots & \vdots \\ \tilde{r}_T' & f_T' \\ \tilde{r}_1' & f_1' \\ \vdots & \vdots \\ \tilde{r}_L' & f_L' \end{bmatrix}$. To generate bootstrap sample $i$, randomly draw $j$ from $U[1, T]$. Let $s_j = X(i : i + L, :)$ in Matlab’s indexing convention. Append $s_j$ to $X_i$, which is initialized as $[\emptyset]$. Repeat until $X_i$ is of length $T$. Unless $T/L$ is an integer, the process yields a bootstrap sample of incorrect length. Build $X_i$ to be at least length $T$ then trim.

4. Estimate the two-stage regression on sample $X_i$ and save the estimate $\hat{\lambda}_i$

5. Repeat $B$ times (we use 100,000 replications). The estimated $\hat{\lambda}_i$ should be approximately mean zero, and std $\left(\hat{\lambda}_i\right) \approx \text{SE}(\hat{\lambda})$

6. Perform usual asymptotic tests

E Robustness

We present additional results showing the sensitivity of our results to changes in specification (or lack thereof).
Table 10: Risk Price Estimates (Monthly Returns)

This table shows risk prices estimated using monthly returns from August 1963 to December 2013 for the two-factor model, the CAPM, and the augmented Fama-French model. The test assets are value-weighted quintile portfolios sorted on ME, BE/ME, and Prior2-12. \( \alpha \) is annualized and "-" indicates that the intercept is restricted to zero. MAPE is average absolute pricing error, annualized. Moving block bootstrap t-statistics are in parentheses.

<table>
<thead>
<tr>
<th>Model</th>
<th>( \alpha )</th>
<th>( \delta_M )</th>
<th>( \delta_\lambda )</th>
<th>( \delta_{smb} )</th>
<th>( \delta_{hml} )</th>
<th>( \delta_{umd} )</th>
<th>( R^2 )</th>
<th>MAPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-Factor model</td>
<td>-</td>
<td>3.28</td>
<td>-9.39</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>78.3</td>
<td>0.995</td>
</tr>
<tr>
<td></td>
<td>(8.51)</td>
<td>(-6.60)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.39</td>
<td>2.75</td>
<td>-9.41</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>79.9</td>
<td>1.01</td>
</tr>
<tr>
<td></td>
<td>(3.72)</td>
<td>(6.71)</td>
<td>(-6.62)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CAPM</td>
<td>-</td>
<td>2.76</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>30.9</td>
<td>1.63</td>
</tr>
<tr>
<td></td>
<td>(7.07)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.31</td>
<td>2.25</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>32.3</td>
<td>1.72</td>
</tr>
<tr>
<td></td>
<td>(3.46)</td>
<td>(5.62)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4-Factor FFC</td>
<td>-</td>
<td>4</td>
<td>-</td>
<td>1.66</td>
<td>6.67</td>
<td>4.46</td>
<td>94.9</td>
<td>0.469</td>
</tr>
<tr>
<td></td>
<td>(7.47)</td>
<td>(1.32)</td>
<td>(8.53)</td>
<td>(10.22)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0855</td>
<td>3.96</td>
<td>-</td>
<td>1.68</td>
<td>6.65</td>
<td>4.45</td>
<td>94.9</td>
<td>0.475</td>
</tr>
<tr>
<td></td>
<td>(1.10)</td>
<td>(1.34)</td>
<td>(8.44)</td>
<td>(10.15)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

E.1 Monthly Estimation

Table 10 presents risk price estimates using monthly returns on our primary test assets. The estimated parameters and model fit are very similar to the daily results in Table 2. The 4-factor FF model fit has improved to nearly perfect, but the estimated risk prices \((\delta_{smb}, \delta_{hml}, \delta_{umd})\) are half of the corresponding values in Table 2. In an i.i.d serially uncorrelated model, the SDF coefficients should be identical no matter what the frequency of observation\(^{36}\). This result suggests the 4-factor model is overfit, and hence, the estimates are not consistent across frequency.

E.2 Future Market Return Horizon

Table 11 shows estimated \( \delta_\lambda \) and cross-sectional \( R^2 \) using alternative horizons, \( T \), to define \( \bar{\lambda} = \sum_{j=2}^{T} rx_{t+j} \) ranging from six months to two years (using daily return data). All estimates restrict the zero-beta rate. \( \delta_\lambda \) declines almost monotonically with \( T \), which is expected since cov \((rx_{t+1}, \bar{\lambda}_{t+1:t+T})\) increases with \( T \) and hence \( \delta_\lambda \) must decline. Cross-sectional \( R^2 \) are fairly stable across horizon, with a peak at one year. Table 12 shows the results of repeating the exercise using monthly returns.

\(^{36}\)Subject to log-linearization error
Table 11: Alternative Horizons (Daily Returns)

Estimated risk price of discount rate factor and cross-sectional $R^2$ for alternative choices of moving average horizon. Data are daily returns with value-weighted quintile portfolios sorted on ME, BE/ME, and Prior2-12 as test assets. Moving block bootstrap t-statistics are in parentheses.

<table>
<thead>
<tr>
<th></th>
<th>6m</th>
<th>9m</th>
<th>12m</th>
<th>15m</th>
<th>18m</th>
<th>21m</th>
<th>24m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>-11</td>
<td>9.6</td>
<td>-10</td>
<td>-8.4</td>
<td>-9</td>
<td>-5.8</td>
<td>-5.3</td>
</tr>
<tr>
<td></td>
<td>(-5.31)</td>
<td>(-5.03)</td>
<td>(-5.79)</td>
<td>(-5.26)</td>
<td>(-5.68)</td>
<td>(-4.85)</td>
<td>(-4.94)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>81</td>
<td>83</td>
<td>88</td>
<td>79</td>
<td>71</td>
<td>71</td>
<td>75</td>
</tr>
</tbody>
</table>

Table 12: Alternative Horizons (Monthly Returns)

Estimated risk price of discount rate factor and cross-sectional $R^2$ for alternative choices of moving average horizon. Data are monthly returns with value-weighted quintile portfolios sorted on ME, BE/ME, and Prior2-12 as test assets. Moving block bootstrap t-statistics are in parentheses.

<table>
<thead>
<tr>
<th></th>
<th>6m</th>
<th>9m</th>
<th>12m</th>
<th>15m</th>
<th>18m</th>
<th>21m</th>
<th>24m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>-9.4</td>
<td>-7.2</td>
<td>-7.3</td>
<td>-7.2</td>
<td>-7.3</td>
<td>-5.3</td>
<td>-5.4</td>
</tr>
<tr>
<td></td>
<td>(-6.65)</td>
<td>(-6.02)</td>
<td>(-6.74)</td>
<td>(-6.78)</td>
<td>(-7.32)</td>
<td>(-6.86)</td>
<td>(-7.01)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>78</td>
<td>72</td>
<td>75</td>
<td>73</td>
<td>70</td>
<td>73</td>
<td>75</td>
</tr>
</tbody>
</table>

The point estimates and patterns are similar, confirming that our results aren’t driven by the choice of horizon for future market returns.

E.3 Fama-French 25

Our main results are presented using value-weighted quintile portfolios sorted on ME, BE/ME, and Prior2-12. Table 13 gives estimates using daily returns on the Fama-French 25 portfolios sorted on ME and BE/ME. As before, the 4-factor model has better fit than our 2-factor model but at the expense of less stable estimates across horizons and test assets.
Table 13: Risk Price Estimates (FF 25 portfolios sorted on ME and BE/ME)

This table shows premia estimated using monthly returns from 01-Aug-1963 to 31-Dec-2013 for the two-factor model, the CAPM, and the augmented Fama-French model. The test assets are the 25 portfolios sorted on ME and BE/ME. $\alpha$ is annualized and "-" indicates that the intercept is restricted to zero. MAPE is average absolute pricing error, annualized. Moving block bootstrap t-statistics are in parentheses.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha$</th>
<th>$\delta_M$</th>
<th>$\delta_{\lambda}$</th>
<th>$\delta_{smb}$</th>
<th>$\delta_{hml}$</th>
<th>$\delta_{umd}$</th>
<th>$R^2$</th>
<th>MAPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-Factor model</td>
<td></td>
<td>4.65</td>
<td>-9.3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>66</td>
<td>1.39</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(4.11)</td>
<td>(-3.86)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.22</td>
<td>3.6</td>
<td>-8.7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>68.7</td>
<td>1.36</td>
</tr>
<tr>
<td></td>
<td>(2.35)</td>
<td>(3.48)</td>
<td>(-3.84)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CAPM</td>
<td></td>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-7.85</td>
<td>2.47</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.75)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5.46</td>
<td>1.53</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>10.5</td>
<td>2.26</td>
</tr>
<tr>
<td></td>
<td>(3.62)</td>
<td>(1.54)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4-Factor FFC</td>
<td></td>
<td>7.31</td>
<td>-</td>
<td>4.23</td>
<td>17.7</td>
<td>16.4</td>
<td>79.3</td>
<td>1.02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(4.49)</td>
<td></td>
<td>(1.34)</td>
<td>(4.12)</td>
<td>(2.09)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.931</td>
<td>6.78</td>
<td>-</td>
<td>4.1</td>
<td>17.2</td>
<td>15.8</td>
<td>79.7</td>
<td>1.03</td>
</tr>
<tr>
<td></td>
<td>(2.23)</td>
<td>(4.39)</td>
<td></td>
<td>(1.31)</td>
<td>(4.05)</td>
<td>(2.04)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>