ABSTRACT

Investors in fixed income markets are willing to pay a large premium to be hedged against shocks in expected volatility and the size of this premium can be studied through variance swaps. Using thirty years of options and high-frequency futures data we document the following novel stylized facts: First, exposure to bond market volatility is strongly priced with an annualized Sharpe ratio of $-1.8$, 20% higher than what is observed in the equity market. Second, while there is strong co-movement between equity and bond market variance risk, there are distinct periods when the bond variance risk premium is different from the equity variance risk premium. Third, the conditional correlation between stock and bond market variance risk premia switches sign often and ranges between $-45\%$ and $+90\%$. Finally, variance risk premia on Treasuries predict positive expected bond returns but negative equity returns, and this finding is robust to the inclusion of the equity variance risk premium. We conclude by showing facts pose a serious challenge to consumption-based asset pricing models.

Keywords: Variance Risk Premia, Implied Volatility, Realised Volatility, Covariation, Long Run Risk, Stocks, Bonds.

First version: September 2015.
This version: March 2016.

Philippe Mueller, Andrea Vedolin, and Petar Sabtchevsky are at the London School of Economics. Paul Whelan is at Copenhagen Business School.
I. Introduction

Increased bond market volatility both in the United States and Europe has triggered reactions from investors and central banks around the globe. For example, measures of option-implied bond market volatility have risen by 60% between mid-2014 and 2015 in anticipation of the Federal Reserve’s monetary tightening fuelling fears of liquidity squeezes in the bond market. Similarly, in Europe, Chairman Draghi announced in summer 2015 that “we should get used to periods of higher volatility” in an era of low interest rates. Heightened volatility is usually associated with bad economic times and a plethora of research has documented the negative impact of volatility shocks onto the real economy. One way to hedge volatility exposure is through a variance swap, a contract which measures how much investors are willing to pay to hedge against future changes in variance.

While the modelling and pricing of equity variance risk has received considerable attention in the asset pricing literature over the past decade, the same does not apply to the fixed income markets counterpart of the equity variance risk premium, the Treasury variance risk premium. In particular, much focus has been placed on consumption-based models which feature Epstein and Zin (1989) preferences—in conjunction with a preference for early resolution of uncertainty, investors in these models have a strong desire to hedge time-varying volatility.

In this paper, we establish novel stylized facts about the Treasury variance risk premium and its co-movement with the equity variance risk premium. We summarize them as follows: First, the premium that investors are willing to pay to hedge against changes in expected bond variance is economically significant, but smaller than the equity variance risk premium. Second, while the equity and bond variance risk premium co-move during normal times, there are distinct periods of distress when the two differ. Third, their correlation switches sign more often than the conditional correlation of stock returns and bond yields and it ranges between $-45\%$ and $+90\%$. Finally, we show that while calibrations of standard asset pricing models can successfully explain the size of the equity variance risk premium, they fail to produce realistic moments for the fixed income market and their co-movement.

The first contribution of this paper is to quantify ex-ante variance risk premia for 5-
year, 10-year, and 30-year Treasury futures, as well as the equity variance risk premium for the S&P500 index. The ex-ante variance risk premium is defined as the difference between the expected physical and risk-neutral variance. While the latter can be calculated from the cross section of option prices in a model-free fashion, the calculation of the objective expectation requires some mild auxiliary modeling assumptions. A priori, it is not clear, what the best proxy for this objective expectation should be. For example, Andersen, Bollerslev, and Diebold (2007) show that simple autoregressive type models estimated directly from realized returns often perform better than parametric approaches designed to forecast the integrated volatility. In calculating our benchmark bond variance risk premium, we thus use the HAR-TCJ model for realized variance proposed by Corsi, Pirino, and Renò (2010). We augment the model by including lagged implied variance as additional regressors.\(^1\)

Using data which covers the period from 1990 to 2014, we obtain the following results. First, exposure to bond market volatility is strongly priced, with a Sharpe ratio of \(-1.8\) per annum, 20% higher than the one found in the equity market. This implies that investors in the bond market are willing to pay an economically significant premium to be protected against sudden changes in expected volatility, even higher than in equity markets. This might be surprising at first sight, as bond market volatility is usually perceived to be much smaller than volatility in the equity market. Second, bond market volatility risk premia are particularly large during periods of distress which are unique to the bond market and exhibit less extreme jumps than the ones observed in the equity market. To gauge in more detail the difference between the variance risk premia in bond and equity markets, we study their conditional correlation.

The conditional correlation between stocks and bonds is one of the key inputs into any asset allocation decision. While a vast empirical literature has documented that the correlation between the S&P500 and long-term Treasuries has changed signs multiple times in the past three decades (see Figure 1 upper panel), it is commonly believed that volatilities across different asset classes move in tandem. Interestingly, we find the

\(^1\)Recently, Bollerslev, Sizova, and Tauchen (2012) use a simple heterogeneous autoregressive RV model to construct the stock market variance risk premium while Busch, Christensen, and Nielsen (2011) use the augmented HAR-RV model with lagged IV to improve forecasts of realized volatility. Bekaert and Hoerova (2014) evaluate a series of different models to obtain the “best” estimate of the ex ante equity risk premium.
conditional correlation between bond and equity variance risk premiums to switch sign more often than the conditional correlation of stock returns and bond yields, with values ranging between $-45\%$ to $+90\%$ (see Figure 1, lower panel). Economically, this implies that while bonds serve as good hedges during certain periods for stocks, shocks to bond and equity variance risk seem to be priced differently.

\[ \text{[ Insert Figure 1 ]} \]

We then study the predictive power of the bond and equity variance risk premium for annual excess returns on bonds and equity. First, we find that a linear combination of the bond variance risk premia positively and significantly predicts annual bond excess returns with the strongest predictability for shorter maturity bonds. The statistical significance is also economically relevant, as for any one standard deviation change in the bond variance risk premium, there is a 1.3 standard deviation change in bond risk premia. Interestingly, on the one hand, we find that when we add the equity variance risk premium, that statistical significance for the shorter maturity bonds remains virtually the same. On the other hand, the equity variance risk premium has no predictive power for the shorter maturity bonds, while slope coefficients are significant for longer maturity bonds. Second, in line with earlier research, we find that the equity variance risk premium positively predicts equity returns, the more so at intermediate horizons at around six months. Similarly, we find the bond variance risk premium to be a formidable predictor of the equity variance risk premium, with the strongest predictive power at intermediate horizons.

Finally, we ask what type of equilibrium model can successfully replicate these stylized facts. Consumption-based models which feature Epstein and Zin (1989) preferences seem particularly suited, as in these models high expected volatility is bad news for investors’ lifetime utility. Hence, these agents should be willing to pay a high price in order to hedge themselves against shocks in volatility. Therefore, we derive a very flexible extension of the standard long-run risk model in continuous time and study its theoretical properties for bond and equity variance risk premia. Using calibration parameters similar to Drechsler and Yaron (2011) and Bollerslev, Sizova, and Tauchen (2012), we show that a canonical version of the long-run risk model without jumps, stochastic correlation, and rare disasters is not able to match the empirical regularities that we uncover.
Our paper proceeds as follows. Section II provides details on the estimation of the variance risk premium. In sections III and IV we present and discuss present stylised empirical factors. In section V we present a flexible equilibrium model together with a calibration that seeks to understand the joint properties of variance risk premia across stocks and bonds. Section VII concludes the paper. Technical details and proofs are deferred to the appendix.

**Related Literature**

Our paper relates to two different strands of the literature. The first studies variance risk premia in reduced form. Carr and Wu (2009) approximate the value of the variance swap on individual stocks using portfolios of options. Martin (2013) studies a simple variance swap which can be robustly replicated even in the presence of jumps. Bondarenko (2014) empirically documents negative and large variance risk premia for the S&P500. While these papers only focus on variance risk in the equity market, another strand of literature looks at the compensation for volatility risk in the fixed income market. For example, Trolle (2009) reports that shorting variance swaps in the Treasury futures market, generates Sharpe ratios that are about two to three times larger than the Sharpe ratios of the underlying Treasury futures. Choi, Mueller, and Vedolin (2015) empirically document economically large and negative variance risk premia and argue that there are significant returns to variance trading in Treasury markets that are comparable to those earned in the equity variance market. They also show that the downward sloping term structure of Treasury implied variances is a powerful predictor of future economic activity. Joslin (2014) documents a switching sign in the compensation for volatility risk. In particular, he documents a two-way feedback effect, where in addition to the effect volatility has on the premium for bearing interest rate risk, there is an additional effect where the shape of the yield curve helps determine the sign of the premium. Trolle and Schwartz (2014) study variance across different swap maturities and option tenors in the swaptions market. They find that variance and skewness risk premia are negative and highly time-varying.

The theoretical section of our work is closely related to the literature on long-run risk models, pioneered by Bansal and Yaron (2004) and extended by Bansal, Khatchatrian, and Yaron (2005), Bansal, Kiku, and Yaron (2007a), Bansal, Kiku, and Yaron (2007b), Bansal,
Dittmar, and Kiku (2009), Hansen, Heaton, and Li (2008) and Zhou and Zhu (2009), among others. In a notable extension of the model, Drechsler and Yaron (2011) include infrequent but potentially large spikes in the level of uncertainty (volatility) and infrequent jumps in the small, persistent component of consumption and dividend growth. They argue that such a specification goes a long way towards quantitatively capturing moments of the variance premium and predictability in the data. The particular specification of the long-run risk model we use in this paper is most closely relates to the long-run risk model with short-memory dynamics in Bollerslev, Sizova, and Tauchen (2012). In their paper, Bollerslev, Sizova, and Tauchen (2012) use intra-day high-frequency data on the S&P500 and VIX and successfully match empirical moments observed in the data.

Our paper is also related to Dew-Becker, Giglio, Le, and Rodriguez (2014) who study the variance swap term structure and find that only shocks to realized variance are priced which implies that variance risk is only priced at the short-end of the term structure but that the compensation at the long-end is essentially zero. Similarly, Andries, Eisenbach, Schmalz, and Wang (2015) study a long-run risk economy with Epstein and Zin (1989) preference augmented by horizon-dependent risk aversion to explain the slope of the variance risk premium term structure.

II. Estimation of Ex-Ante Variance

In this section, we describe the methods used to estimate the expected objective and risk neutral variances, $E^P_t \left( \int_t^T \sigma_u^2 du \right)$ and $E^Q_t \left( \int_t^T \sigma_u^2 du \right)$, and the variance risk premium, defined as the difference between the two, respectively. Using real-time expectations of risk neutral and objective variances over a forecasting horizon of one-month, we construct variance risk premium measures for bonds and equities. For Treasuries, we compute a daily TVRP measure beginning in 1990 using options and futures on 5-year, 10-year, and 30-year Treasury notes. For equities, we use options and futures on the S&P500 index. We directly use the VIX index as our proxy for the expected risk neutral variance.
A. Data

To calculate implied and realized variance measures for Treasury bonds, we use futures and options data from the Chicago Mercantile Exchange (CME). We use high-frequency intra-day price data for 5-year, 10-year, and 30-year Treasury notes futures and S&P500 index futures, and end-of-day prices of options written on the underlying futures. The data runs from 1990 to 2014.

Treasury futures are traded electronically as well as by open outcry. While the quality of electronic trading data is higher, the data only becomes available in August 2000. To maximize our time span, we use data from electronic as well as pit trading sessions. We only consider trades that occur during regular trading hours (07:20–14:00 CT) when the products are traded side-by-side in both markets.\(^2\) The contract months for the Treasury futures are the first five consecutive contracts in the March, June, September, and December quarterly cycle. This means that at any given point in time, up to five contracts on the same underlying are traded. To get one time series, we roll the futures on the 28\(^{\text{th}}\) of the month preceding the contract month.

For options, the contract months are the first three consecutive months (two serial expirations and one quarterly expiration) plus the four months in the March, June, September, and December quarterly cycle. Serials exercise into the first nearby quarterly futures contract, quarterlies exercise into futures contracts of the same delivery period. We roll our options data consistent with the procedure applied to the futures.\(^3\)

B. Physical Variance

To estimate \(\mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \sigma_u^2 du \right)\), the expected variance between \(t\) and \(T\) under the physical measure, we first consider the daily realized variance which is defined as:\(^4\)

\[
\text{RV}_{t,D} = 21 \times \sum_{i=1}^{M} r_{t,i}^2, \tag{1}
\]

\(^2\) Liquidity in the after-hours electronic market is significantly smaller than during regular trading hours.

\(^3\) Detailed information about the contract specifications of Treasury futures and options can be found on the CME website, www.cmegroup.com.

\(^4\) On average, we have 21 trading days per month.
where \( r_{t,i} = \log P(t - 1 + i/M) - \log P(t - 1 + (i - 1)/M) \) is the intra-daily log return in the \( i^{th} \) sub-interval of day \( t \) and \( P(t - 1 + i/M) \) is the asset price at time \( t - 1 + i/M \). For each day, we take \( r_{t,i} \) between 7:25 and 14:00 CT, i.e. during pit trading hours on CME.\(^5\) In line with Andersen, Bollerslev, and Diebold (2007), we use five-minute intervals to calculate the one-day realized variance \( \hat{RV}_{t,daily} \). The normalized weekly and \( h \)-month realized variances are computed from the daily measure as follows:

\[
RV_{t,W} = \frac{1}{5} \times \sum_{j=0}^{4} RV_{t-j,D}, \quad \text{and} \quad RV_{t,M} = \frac{1}{21 \times h} \times \sum_{j=0}^{h \times 21 - 1} RV_{t-j,D}.
\]

To better capture the long memory behavior of volatility, Corsi (2009) proposes the heterogeneous autoregressive model for realized variance using the daily, weekly and monthly realized variance estimates. Andersen, Bollerslev, and Diebold (2007) extend the standard HAR-RV model to show that the predictability for realized variance over different time intervals almost always comes from the continuous component of the total price variation, rather than the discontinuous jump component. Corsi, Pirino, and Renò (2010) introduce the concept of threshold bi-power variation and show that it is well suited for estimating models of volatility dynamics where continuous and jump components are used as explanatory variables. They document that jumps can have a highly significant impact on the estimation of future variance. Their HAR-TCJ model for forecasting daily realized variance is expressed as:

\[
RV_{t,t+1} = \alpha + \beta_D \hat{T}C_{t,D} + \beta_W \hat{T}C_{t,W} + \beta_M \hat{T}C_{t,1M} + \beta_J \hat{T}J_{t,D} + \varepsilon_{t+1},
\]

where the threshold bi-power variation measure \( TBPV_t \) is used to estimate the jump component \( \hat{T}J_{t,D} = I_{C-T_z > \Psi_\alpha} \times (RV_{t,D} - TBPV_t)^+ \), the continuous components are given by\(^6\)

\[
\hat{T}C_{t,D} = RV_{t,D} - \hat{T}J_{t,D},
\]

\[
\hat{T}C_{t,W} = RV_{t,W} - \hat{T}J_{t,W},
\]

\(^5\)For the S&P 500, we sample the returns between 9:30 and 16:00 ET, i.e. during the NYSE opening hours.

\(^6\)The expression for the threshold bipower variation, \( TBPV_t \), is given in Corsi, Pirino, and Renò (2010). We use the confidence level \( \alpha = 99.9\% \).
\[
\widehat{TC}_{t,1M} = RV_{t,1M} - \widehat{TJ}_{t,1M},
\]

and \(\widehat{TJ}_{t,W}\) and \(\widehat{TJ}_{t,1M}\) are normalised averages of \(\widehat{TJ}_{t,D}\).

This simple method avoids some difficulties in long memory time series modeling and the parameters can be consistently estimated by OLS. However, a GMM correction is needed to make appropriate statistical inference. Moreover, such a HAR-TCJ type model can be easily modified, for example, by adding extra covariates that contain predictive power. In our forecasting regressions, we also include implied variance as an additional predictor variable. Since our aim is to compute \(h\)-month variance forecasts we run variance projections of \(h\)-period realized variance on a cascade of lagged variance factors and implied variance from options:

\[
\ln RV_{t,t+21\cdot h} = \alpha + \beta_{C,D} \ln \widehat{TC}_{t-1,D} + \beta_{J,D} \ln (1 + \widehat{TJ}_{t-1,D}) \\
+ \beta_{C,W} \ln \widehat{TC}_{t-5,W} + \beta_{C,M} \ln \widehat{TC}_{t-22,M} \\
+ \sum_{h=1}^{6} \beta_{IV,h} IV_t(h) 
\]

where \(h = 1\)-month. Realized volatility is then given by the following expression:

\[
RV_{Vol_{t,t+21\cdot h}} = \exp \left( E_t \left[ \ln RV_{t,t+21\cdot h} \right] + 1/2 \text{Var} \left[ \ln RV_{t,t+21\cdot h} \right] \right). 
\]

Finally, we obtain real-time estimates for ex-ante variance by implementing this regression using an expanding window with parameters initialised to one trading year (252 days) of data and re-estimating on a daily basis.

Table I reports in-sample point estimates and performance measures for the forecasts model given by equation (2) for the the S&P500 index and 5y, 10y, and 30y Treasury futures, respectively. Considering the point estimates, we find that predictability for realized variance is (almost) always coming from the continuous component of quadratic variation. These findings are consistent with the results of Corsi (2009) and Andersen, Bollerslev,
and Diebold (2007) who also report information about future variance is contained in the
continuous component of quadratic variation.

The final five rows of Table I report a set of performance measures for the specification
given by equation (2) benchmarked against a random walk model for realized variance:

\[
\ln RV_{t, t+21 \times h} = \alpha_{RW} + \beta_{RW} \ln RV_{t-21, t \times h} + \varepsilon_{t, t+21 \times h}.
\]

Following Corsi (2009), Andersen, Bollerslev, and Diebold (2007), Aït-Sahalia and
Mancini (2008) and others, we report the Mincer-Zarnowitz \( R^2 \) for each alternative.\(^7\)
Considering the Mincer-Zarnowitz \( R^2 \) as a performance measure we find the HAR speci-
fication consistently outperforms a random walk model. The final row of Table I reports
the Diebold and Mariano (1995) statistic which is a model free test of forecast accu-
curacy applicable to multi-period forecasts and errors which are non-Gaussian and serially
correlated. Again, we compare the errors from the HAR specification versus a random
walk. After adjusting for the overlapping forecast horizon, we find the HAR model almost
always outperforms the random walk model at conventional statistical levels, consistent
with the alternative forecast metrics. In summary, out-of-sample, the HAR model consist-
tently outperforms a random walk model for variance producing a simple to implement,
yet accurate forecast for future realized variance. This allows us to measure one element
of ex-ante variance risk premiums in real time and without suffering from forward looking
bias.

C. Implied Variance

We directly use the VIX index from CBOE as the VIX is widely used as a proxy for
implied volatility in the equity market. Implied volatility from Treasury futures options
is calculated akin to the VIX by using the cross-section of one-month options written on
Treasury futures. The specifics are discussed in Choi, Mueller, and Vedolin (2015).

\(^7\)The Mincer-Zarnowitz \( R^2 \) is obtained from a projection of ex post realized variance between \( t \) and
\( t + h \) on a constant and the model’s forecasts based on date \( t \) information:

\[
\ln RV_{t, t+21 \times h} = a + bE_t^{\text{model}}[\ln RV_{t, t+21 \times h}] + \text{error}
\]
III. Descriptive Analysis

A. Basic Properties

The variance risk premium is defined as the difference between the expected physical and risk-neutral variance, i.e.,

\[VRP^{(\tau)} = E^P_t \left( \int_t^{t+\tau} \sigma_u^2 du \right) - E^Q_t \left( \int_t^{t+\tau} \sigma_u^2 du \right)\]  \hspace{1cm} (5)

Table II reports summary statistics for the equity and Treasury volatility risk premiums. Note that following market convention, we report statistics for the volatility risk premiums, annualized in percentage points, and not for the variance risk premiums.

[ Insert Table II, Figure 2, and Figure 3 ]

Figures 2 and 3 plot the time-series of the ex-ante physical, blue line, and risk neutral, green line, volatilities of equities and bonds, respectively. Consistent with extant literature, the magnitude of the spikes in equity volatility is much bigger. For example, during the LTCM crisis of 1998 the annualized realized volatility spiked considerably, whereas the spike in fixed income markets was much more subdued and around half the size of the one observed in equity markets. The same holds true at the time of the Lehman default, 15 September 2008 when implied volatility on the equity index spiked at almost 80%, while fixed-income implied volatility was at around 16%.

It is instructive to plot the difference between realized and implied volatility, given in Figure 4. The upper panel reports the equity volatility risk premium. The lower panel reports the five-year Treasury volatility risk premium. Similarly, Figure 5 plots the 10-year and 30-year Treasury volatility risk premiums. Consistent with existing empirical results on volatility risk premiums, they are negative most of the time, i.e., the risk neutral volatility is higher than the realized volatility. We can interpret the volatility risk premium as the net payoff of a party with a net long position in a variance swap-like contract, where the realized leg determines the pay-out at maturity and the risk-neutral volatility is the price of that contract, paid at inception. As we can see from the figure, the net pay-off of the contract is negative most of the time and the agent earns a negative
volatility risk premium. Consequently, Figures 4 and 5 imply that traders active in equity
and Treasury markets are willing to pay volatility risk premiums in order to hedge away
volatility risk, i.e., volatility risk is priced.

[ Insert Figure 4, 5 ]

The volatility risk premiums plotted in Figures 4 and 5 impose restrictions on the true
structural model governing the economy. We can immediately reject a wide class of general
equilibrium models featuring endowment economies populated by a representative agent
with power utility as volatility risk in these economies is not priced. In order for volatility
risk to be priced, we need a representative agent with preference for early resolution of
uncertainty, e.g., Epstein-Zin type of recursive preferences with intertemporal elasticity
of substitution, IES, above one. We will discuss this in a later section.

B. Persistence and Correlation

As discussed above, the equity and variance risk premiums are negative most of the time.
As it is evident from the upper panel in Figure 6, they are also highly persistent. The
180 day autocorrelation of the equity variance risk premium is slightly above 0.52. The
180 day autocorrelation of the 30-year Treasury variance risk premium is lower compared
to its equity markets counterpart, but still above 0.25.

[ Insert Figure 6 ]

It is instructive to analyze the correlation between the equity and Treasury variance
risk premiums. Casual examination of Figures 4 and 5 would suggest that the equity
and Treasury variance risk premiums are positively correlated. The lower panel in Figure
1 hints that such a conclusion would be premature. Even through the two series are
mildly correlated most of the time, they diverge in times of severe market dislocations,
for example the LTCM default in August 1998. We observe a huge spike in the EVPR,
but only a muted response in fixed-income markets. The bottom panel of Figure 7, covers
the period of the mortgage refinancing boom that reached a high-water mark in 2003.
We can see that the resulting stress is contained in the fixed-income markets as there is
a huge spike in the TVRP, but not in the EVRP.
Yet another market development of interest is the bond market calamity that unfolded in the summer of 2013, the Taper Tantrum. It was largely precipitated by a string of comments on the part of Ben Bernanke, Chairman of the Federal Reserve at the time. In his testimony before Congress in May and June 2013 Bernanke hinged that the Fed would likely start tapering the pace of its bond purchases later in the year, conditional on continuing robust economic data. The ensuing market reaction was abrupt. Long-term U.S. bond yields and the dollar spiked substantially leading to an abrupt rise in implied volatility. Figure 8 plots the EVRP and the TVRP(30Y) during the period surrounding the Taper Tantrum. As it is evident from the figure in the aftermath of the Taper Tantrum, between June and December 2013, the 30-year Treasury variance risk premium increased three-fold as wrong-footed market participants were desperate to hedge their volatility exposure in Treasury markets. It is important to note, however, that the Taper Tantrum was largely contained in the realm of the fixed income markets. While the TVRP increased three-fold, the EVRP did not change much over the same period.

The lower panel in Figure 1 warrants an even better understanding of the correlation between the EVRP and the TVRP. The figure plots the 180-day rolling window conditional correlation between EVRP and TVRP. The sign of the conditional correlation frequently changes sign.

The flipping sign of the conditional correlation hints that there is a large time-variation in relative hedging demand for second moments in equity and bond markets. As a result, the lower panels of Figures 1 and 6 provide important information about the functioning of equity and fixed-income markets. The conditional correlation on the upper panel of Figure 1 adds an extra dimension to our understanding of equity-bond market co-movement as inferred from stock-bond correlations. It is important to note that as second moments are measured with greater precision compared to first moments the conditional correlation between EVRP and TVRP has the potential to become the conditional statistic of choice for the analysis of the interrelation between equity and fixed-income markets.
IV. Predictive Regressions

In this section, we investigate to what extent our measures of \textit{ex-ante} variance risk premiums contain predictive power for bond and stock excess returns. In particular, we study the \textit{in-sample} predictive power of bond and equity variance risk premiums for fixed income and equity excess returns. It is instructive to note that while the predictability regressions we run are in-sample, our proxies for the variance risk premiums are constructed without any forward looking bias. More importantly, the predictors we use are observable in real time. To construct a single forecasting factor from the information in 5,10, and 30-year Treasury variance risk premia we adopt the approach of Cochrane and Piazzesi (2005) in forming linear combinations of factors in a 2-stage predictability regression. Specifically, the predictability regression

\[ brx_{t+1}^{(n)} = \alpha_n + \beta_1 TVRP_t^{(5)} + \beta_2 TVRP_t^{(10)} + \beta_3 TVRP_t^{(30)} + \epsilon_{t+1} \]

can be factorised as

\[ brx_{t,t+1}^{(n)} = b_n \left( \gamma_0 + \gamma_1 TVRP_t^{(5)} + \gamma_2 TVRP_t^{(10)} + \gamma_3 TVRP_t^{(30)} \right) + \epsilon_{t+1}^{(n)}. \]

Then, considering the cross-sectional average predictability regression

\[ \frac{1}{n} \sum_n brx_{t,t+1}^{(n)} = \bar{\alpha} + \gamma' TVRP_t^{[2,5,10]} + \bar{\epsilon}_{t+1} \]

allows us to obtain a single Treasury variance risk premium return forecasting factor \( TVRF_t = \bar{\alpha} + \gamma' TVRP_t^{[2510]} \). While this approach condenses the information in each time series it also has the advantage of filtering out noise in our estimates.

We opt for a parsimonious regressions specification of the form

\[ brx_{t,t+12}^{(i)} = \beta_{h}^{(i)} TVRF_t + \gamma_{h}^{(i)} EVRP_t + \epsilon_{t,t+12}^{(i)}, \]

where \( brx_{t+12}^{(i)} \) denotes the \( h \)-period excess return on a zero coupon with maturity.
\[ i = \{5, 10, 20, 30\}\text{-years}, \] 

\( TVRF \) is the Treasury variance risk premium factor described above and \( EVRP \) is the equity variance risk premium. All regression results we report are standardized, i.e., we normalize all regressors and regressands to have a mean of zero and a standard deviation of one. As a result, constants appear nowhere in our specifications. The normalization not only allows us to compare coefficients across different specifications, but it also helps our interpretation of economic significance. We report \( t \)-statistics that are calculated using the Hansen and Hodrick (1983) GMM correction with 12 Newey-West lags.

[ Insert Table III ]

Following convention, we focus on one year excess bond returns. Panel A of Table III reports regression results from the regression of one-year excess returns on the Treasury VRP. In Panel B we add the equity variance risk premium (EVRP) to the regression. The Treasury VRP has strongly statistically significant predictive power for bond excess returns on long maturity bonds. Adjusted \( R^2 \) range between 3% for 10 year bonds and up to 11% for maturities up to 25 years. While Treasury VRP are significant, the coefficient for the equity VRP is close to zero and insignificant. Note that we focus on rather long maturity bonds in the bond excess return regression in line with the underlying maturities of the futures used to construct the Treasury variance risk premiums.

We also run equity predictability regressions. Table IV reports the results for the predictability regression, where we have equity excess returns on the left-hand-side and the holding periods range between one month and two years. Similarly to the regression results for bond excess returns the Treasury variance risk premiums have strong forecasting power for equity excess returns up to a horizon of one year with \( R^2 \) up to 10%. In line with the extant literature (see, e.g., Bekaert and Hoerova (2014) and Bollerslev, Tauchen, and Zhou (2009)) we find that the equity variance risk premium also has non-negligible forecasting power for excess returns between six months and one year. However, our results are somewhat weaker than those reported previously. This can be explained by the fact that in estimating the variance risk premiums we ensure that our forecast under the physical measure is not subject to a forward looking bias. At the same time, the forecasting method we use is more complicated than (and superior to) a simple random walk. As a result, our
time series of equity variance risk premiums differs from both Bekaert and Hoerova (2014) and Bollerslev, Tauchen, and Zhou (2009). Running a bivariate regression with both Treasury and equity variance risk premiums we find that the predictive power in Treasury VRP is not subsumed by information contained in the equity variance risk premium. Both are highly statistically significant and economically meaningful for horizons up to twelve to eighteen months.

[ Insert Table IV ]

In summary, we find that Treasury VRP contain relevant information for both bond and stock excess returns.

V. A Consumption Based Framework

It is well known that representative agent economies with time-separable preferences cannot generate priced volatility risk consistent data. This observation led researchers to consider a number of alternatives and much attention has focused on consumption-based models featuring Epstein and Zin (1989) preferences and long-run risks, as in Bansal and Yaron (2004). The combination of preference for early resolution and time-varying economic uncertainty has been shown to generate a strong demand to hedge time-varying volatility risk; thus, driving a wedge between expected variance under the physical $\mathbb{P}$ and risk-neutral measure $\mathbb{Q}$.

In the context of equity markets, the empirical predictions of the long-run risk model have been well studied. For example, Bollerslev, Sizova, and Tauchen (2012) among others, derive the equilibrium equity variance risk premium implied by an extension of the long-run risk model, cast in continuous time. In the following, using their calibration parameters, we test the model’s predictions for first and second moments for bonds and the joint behavior of risk premiums in equity and fixed income markets.

---

8 A non-exhaustive list of classic contributions to the long run risk literature include Bansal, Khatchatryan, and Yaron (2005), Bansal, Kiku, and Yaron (2007a), Bansal, Kiku, and Yaron (2007b), Bansal, Dittmar, and Kiku (2009), Hansen, Heaton, and Li (2008), Zhou and Zhu (2009), and Bollerslev, Sizova, and Tauchen (2012)
A. Fundamentals

We define a standard Wiener process $W^T = \begin{bmatrix} W^c & W^x & W^v & W^d \end{bmatrix}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We denote by $\mathbb{F}$ the augmented filtration, $\{\mathcal{F}(t)\}_{t \in [0,T]}$, generated by the Wiener process and assume that $\mathcal{F} = \mathcal{F}(T)$. A process in this section is by definition a stochastic process; that is progressively measurable with respect to $\{\mathcal{F}(t)\}$.

We consider a finite-horizon endowment economy in continuous time. Under the physical measure, the consumption and dividend processes follow

$$dC_t / C_t = (\mu_c + x_t)dt + \sqrt{V_t}dW^c_t,$$

$$dx_t = -k_x x_t dt + \varphi_x \sqrt{V_t}dW^x_t,$$

$$dV_t = k_v (\bar{V} - V_t)dt + \varphi_v \sqrt{V_t}dW^v_t,$$

$$\frac{dD_t}{D_t} = (\mu_D + \phi x_t)dt + \xi_c \sqrt{V_t}dW^c_t + \xi_x \sqrt{V_t}dW^x_t + \varphi_d dW^d_t,$$

where $dC/C$ is the growth rate of consumption and $x$ is the persistently varying component of the expected consumption growth rate. The conditional variance of consumption growth, $V$, follows a mean-reverting stochastic process that represents time-varying economic uncertainty in consumption growth. The model exhibits persistent variations in both the growth rate and volatility of aggregate consumption by construction. Increases in consumption growth drive up stock prices through the cash-flow channel, whereas increases in consumption volatility drive up the equity premium and decrease stock prices.

Prices of equilibrium quantities in the long-run risk model vary in response to highly persistent fluctuations in the mean and volatility of aggregate consumption growth. In early calibrations of the model, e.g., Bansal and Yaron (2004), persistent fluctuations in the mean of consumption growth play a dominant role. However, as noted in Beeler and Campbell (2009), the original Bansal and Yaron (2004) calibration counter-factually implies that long-run consumption and dividend growth should be highly persistent and predictable from stock prices. A later calibration of the long-run risk model by Bansal, Kiku, and Yaron (2007a) ameliorates this issue by significantly increasing the persistence of volatility fluctuations. It increases the importance of discount-rate news and reduces the importance of cash-flow news for stock prices.
B. Stochastic Differential Utility

Consider a representative agent with the continuous time equivalent of Epstein and Zin (1989) and Weil (1989) recursive preferences. The normalized continuous time limit of the Kreps and Porteus (1978) aggregator of the stochastic differential utility (SDU) of the representative agent is given by

\[ f(c, v) = \beta(h^{-1}(v))^{1-\gamma}u(c/h^{-1}(v)), \]

\[ u(x) = \frac{x^{1-\psi} - 1}{1 - \frac{1}{\psi}}, \psi > 0, \]

\[ h(x) = \frac{x^{1-\gamma}}{1 - \gamma}, \gamma \geq 0, \gamma \neq 1. \]

Here, \( \beta \) refers to the instantaneous discount factor. The preferences of the representative agent in the long-run risk model are characterised by high elasticity of inter-temporal substitution (EIS), \( \psi > 1 \), and high risk aversion, \( \gamma > 1 \). While high EIS leads to positive correlation between expected future consumption growth and stock prices, high risk aversion is instrumental in achieving high risk premiums. In models with Epstein-Zin utility, and their continuous time equivalents, risk premiums are driven not only by the covariances of asset returns with current consumption, but also by the covariances of asset returns with expected future consumption growth.

C. Stochastic Discount Factor and the Equilibrium Interest Rate

The stochastic discount factor (SDF) is given by

\[
\frac{d\Lambda_t}{\Lambda_t} = -r^f_t dt - \Theta_c(t) dW^c_t - \Theta_x(t) dW^x_t - \Theta_v(t) dW^v_t
\]

\[
r^f(t) = \beta + \frac{1}{\psi} (\mu_c + x_t) - \frac{\gamma}{2} \left( 1 + \frac{1}{\psi} \right) V_t + \frac{1}{2}(\theta - 1)(A^2_v \varphi^2_v - A^2_x \varphi^2_x)V_t
\]

\[
\Theta_c(t) = \gamma \sqrt{V_t}
\]

\[
\Theta_x(t) = -(\theta - 1) \varphi_x A_x \sqrt{V_t}
\]

\[
\Theta_v(t) = -(\theta - 1) \varphi_v A_v \sqrt{V_t} \quad \text{where } \theta = \frac{1-\gamma}{1-\frac{1}{\psi}}
\]

All else equal, a more persistent \( V(\cdot) \) process increases the value of \( A_v \) and, in turn,
the sensitivity of the SDF to $W^v$ shocks. Preference for early resolution means $ψ > 1$ and risk aversion is larger then EIS. This implies $θ < 0$ but since $A_v$ can be shown to be (always) negative, $Θ_v(t)$ is also always negative.

D. Equilibrium Variance Risk premiums

The technical appendix derives semi-closed form solutions for default free bonds and the leverages Lucas Tree (the equity market). These solutions allow simple characterisation of equilibrium variance risk premiums. The integrated variance on a zero coupon Treasury with maturity $T$ is given by

$$IV(0, \bar{t}, T) = \int_0^{\bar{t}} d\langle \ln P, \ln P \rangle_t$$

Note that $P = P(t, T)$ is the equilibrium bond price and $\bar{t} \in [0, T]$. By definition, the equilibrium fixed income variance risk premium is equal to the difference between the expected values of the integrated variance under the physical and the equivalent martingale measures,

$$VRP(0, \bar{t}, T) = \mathbb{E}^P IV(0, \bar{t}, T) - \mathbb{E}^Q IV(0, \bar{t}, T)$$

$$= \int_0^{\bar{t}} (\mathbb{E}^P d\langle \ln P, \ln P \rangle_t - \mathbb{E}^Q d\langle \ln P, \ln P \rangle_t),$$

(6)

The variance risk premium is 0 at time 0, but it accumulates over time. The conditional Treasury variance risk premium is given by

$$TVRP(0, \bar{t}, T) = \int_0^{\bar{t}} (\mathbb{E}^P d\langle \ln P, \ln P \rangle_t - \mathbb{E}^Q d\langle \ln P, \ln P \rangle_t)$$

$$= \int_0^{\bar{t}} (\varphi_a^2 B^2(\tau) + \varphi_a^2 C^2(\tau)) (\mathbb{E}^P V_t - \mathbb{E}^Q V_t) dt$$

An immediate implication is that the Treasury variance risk premium is bounded to be negative when agents have a preference for early resolution of uncertainty. This shows that, as in equity markets, the long-run risk model generates a demand to hedge volatility risk in Treasury markets. This provides an important prediction that we address in the
following. In a similar vain, the equity variance risk can be shown to be

\[ \text{EVRP}(0, \tilde{t}, T) = \int_0^{\tilde{t}} ((A_x^d \varphi_x + \xi_x)^2 + (A_v^d \varphi_v)^2 + \xi_v^2 (E^P V_t - E^Q V_t)) dt \] (7)

With this result at hand we move to understanding the joint dynamics of Treasury bond and stock market variance risk premiums.

E. Covariation Between TVRP and EVRP

The covariation between the TVRP and the EVRP is given by

\[
\text{Cov(TVRP}(T), \text{EVRP}) = \left( \varphi_x^2 B^2(T) + \varphi_v^2 C^2(T) \right) \left[ (A_x^d \varphi_x + \xi_x)^2 + (A_v^d \varphi_v)^2 + \xi_v^2 (E^P V_t - E^Q V_t) \right] \\
= g(\text{structural Params}) \times (E^P V_t - E^Q V_t)
\]

from which we see the covariation between EVRP and TVRP(T) is bounded to be negative when agents have a preference for early resolution of uncertainty. In other words, the long-run risk model predicts that hedging demands for time-varying volatility risks move in opposite directions. Moreover, it is easy to show this result holds both conditionally and unconditionally.

VI. Quantitative Exercise

In this section, we calibrate the model and quantitatively investigate its equilibrium properties. We use parameter values agreed upon in the literature and try to jointly match basic equity and fixed income market moments, such as the unconditional expectation of the equilibrium risk-free rate, the unconditional volatility of the risk-free rate, the equity risk premium, the unconditional volatility of equity, EVRP, TVRP, and their quadratic co-variation among each other.

[ Insert Table V ]

Table V reports the parameter values we use in the quantitative exercise. Most of the parameters are in line with Bansal and Yaron (2004). We make the simplifying assumption
that $\phi_d = \xi_e = \xi_x$ and set them in such a way so that the volatility of dividends is three times the volatility of consumption. We have five free parameters: $\varphi_x, \phi, \varphi_v, k_1,$ and $k_{1d}$. Parameter $\varphi_x$ enters the diffusion coefficient of the persistently varying component of the expected consumption growth rate, $x(\cdot)$. Parameter $\phi$ controls the sensitivity of dividends to the persistently varying component of the expected consumption growth rate. $\varphi_v$ enters into the diffusion coefficient of the variance of consumption and $k_1$ and $k_{1d}$ are linearization parameters in the equilibrium equations for the price-dividend ratios of the unleveraged and leveraged Lucas trees, respectively.

[ Insert Table VI ]

Not very surprisingly, the model simultaneously resolves the equity premium and risk-free rate puzzles by generating a sizable equity risk premium of 7.36% and at the same time keeping the equilibrium risk-free interest rate at bay. The risk premium is large because the representative agent in the long-run risk model has a preference for early resolution of uncertainty. She dislikes future economic uncertainty (volatility) and demands a high equity risk premium. At the same time, increases in volatility drive down short term interest rates. This is because the representative agent tries to save more and consume less, but since consumption is fixed to clear the market rates have to decrease.

The present calibration of the model also matches the unconditional volatility of equity, but fails to generate a reasonable bond risk premiums. The model-implied bond risk premium on one-year bonds is $-2.78\%$, against 1.47% in the data. An initial value for $x$ well-above its unconditional expectation delivers an upward-sloping term structure of interest rates in our calibration. If, however, the initial value of $x$ were equal or below its unconditional expectation the term structure of interest rates would be downward-sloping, consistent with Beeler and Campbell (2009).

Consistent with the empirical literature, the equity and fixed income variance risk premiums are negative. It is instructive to notice that the co-variation between the equity and fixed income variance risk premiums is also negative in the model. The present calibration of the model fails, however, to match the magnitude of the equity and variance risk premiums reported in the literature.
A careful analysis of the model exposes its deficiencies. First, variance in the long-run risk model is exogenous. The model tells us how much variance do we need in order to generate sufficient amount of asset pricing volatility, but remains agnostic about the sources of this variability. Second, the treasury variance risk premium implied by widely accepted calibrations of the Long-run risk model, such as Bansal and Yaron (2004), Zhou and Zhu (2009), and Drechsler and Yaron (2011), is infinitesimal. According to Drechsler and Yaron (2011), the model generates an equity variance risk premium close to what we observe in the data, but our own calculations show that the model miserably fails to generate a seizure treasury variance risk premium, let alone the covariation between EVRP and TVRP close to what we observe in the market.

One might be tempted to ameliorate this problem by augmenting the dynamics of the canonical long-run risk model by adding jumps, stochastic correlation, or rare disasters. These extensions to the standard stochastic discount factor in the long-run risk model would mechanically increase the level of the Treasury variance risk premium at the expense of counter-factually high equity and equity variance risk premium.

**VII. Conclusion**

In this paper, we document four novel stylized facts. First, investors in fixed income markets require large premia for being exposed to volatility shocks. These risk premia are negative and economically significant. More specifically, we find that the price carries an annual Sharpe ratio of $-1.8$, 20% larger than the one observed in the index equity market. Second, variance risk premia in bond and equity markets co-move during normal times, but diverge during periods of distress. In particular, the bond variance risk premia jumps during periods which are specific to bond market news. Third, the conditional correlation between the bond and equity variance risk premium is highly time-varying and switches sign, with values ranging between $-45\%$ and $+90\%$. Forth, both the equity and bond variance risk premia are formidable predictors of annual bond and equity returns. For bond returns, we find that a linear combination of the bond variance risk premia predicts shorter maturity bond returns, while the equity variance risk premium significantly predicts longer maturity bonds. Both the bond and equity variance risk premia highly
significantly predict equity returns at intermediate horizons.

We try to reconcile these empirical facts within a very flexible long-run risk model where agents feature Epstein and Zin (1989) recursive utility. If investors have a preference for early resolution, then high volatility implies bad news for investors’ lifetime utility. Our calibration exercise shows that while these models match well first and second moments of equity market risk premia, they perform badly when explaining bond variance risk premia and the joint behavior of equity and bond variance risk.
References


VIII. Appendix: Figures
Figure 1. CAPM Bond Betas and Conditional Correlation of TVRP and EVRP
The top panel of this figure plots time series of 10-year nominal bond CAPM betas. The numerator of the bond beta is defined as the covariance between the return on 10y Treasury bonds and the return on the stock market index. Bottom Panel: Conditional correlation between the equity and treasury variance risk premiums (5-year, 10-year, and 30-year). We use 180 day rolling window. The data runs from 1990 to 2014.
Panel a) of this figure plots the time series of the ex-ante physical, blue line, and risk neutral, green line, equity volatility. Panel b) plots the time series of the ex-ante physical, blue line, and risk neutral, green line, 5-year treasury volatility. To calculate implied and realized variance measures, we use futures and options data from the Chicago Mercantile Exchange (CME). We use high-frequency intra-day futures price data and end-of-day prices of options written on the underlying futures. The data runs from 1990 to 2014. We construct expected variances and variance risk premiums at the daily frequency and then sample end of month observations.
Figure 3. Ex-ante Physical and Risk Neutral Volatility

Panel a) of this figure plots the time series of the ex-ante physical, blue line, and risk neutral, green line, 10-year treasury volatility. Panel b) plots the time series of the ex-ante physical, blue line, and risk neutral, green line, 30-year treasury volatility. To calculate implied and realized variance measures, we use futures and options data from the Chicago Mercantile Exchange (CME). We use high-frequency intra-day futures price data and end-of-day prices of options written on the underlying futures. The data runs from 1990 to 2014. We construct expected variances and variance risk premiums at the daily frequency and then sample end of month observations.
Figure 4. Equity and Treasury Volatility Risk Premia
This figure plots the EVRP, panel (a), and the 5-year TVRP, panel (b). Red lines are daily data and the thick black line is a one-month moving average. To calculate implied and realized variance measures, we use futures and options data from the Chicago Mercantile Exchange (CME). We use high-frequency intra-day futures price data and end-of-day prices of options written on the underlying futures. The data runs from 1990 to 2014. We construct expected variances and variance risk premia at the daily frequency and then sample end of month observations.
Figure 5. Equity and Treasury Volatility Risk Premia
This figure plots the 10-year TVRP, panel a), and the 30-year TVRP, panel b). To calculate implied and realized variance measures, we use futures and options data from the Chicago Mercantile Exchange (CME). We use high-frequency intra-day futures price data and end-of-day prices of options written on the underlying futures. The data runs from 1990 to 2014. We construct expected variances and variance risk premia at the daily frequency and then sample end of month observations.
Figure 6. Auto-Correlations and Cross-Correlations
The top panel of this figure plots the auto-correlations of EVRP, TVRP(5-year), TVRP(10-year) and TVRP(30-year). Lags are in days. The bottom panel of the figure plots the cross-correlation between EVRP and TVRP. To calculate implied and realized variance measures, we use futures and options data from the Chicago Mercantile Exchange (CME). We use high-frequency intra-day futures price data and end-of-day prices of options written on the underlying futures. The data runs from 1990 to 2014. We construct expected variances and variance risk premia at the daily frequency and then sample end of month observations.
Figure 7. EVRP and TVRP
Top panel: EVRP and TVRP between January 1997 and December 1999. Bottom panel: EVRP and TVRP between January 2003 and December 2004. To calculate implied and realized variance measures, we use futures and options data from the Chicago Mercantile Exchange (CME). We use high-frequency intra-day futures price data and end-of-day prices of options written on the underlying futures. We construct expected variances and variance risk premia at the daily frequency and then sample end of month observations.
This figure plots EVRP and TVRP between January 2013 and December 2013. Bottom panel: EVRP and TVRP between January 2003 and December 2004. To calculate implied and realized variance measures, we use futures and options data from the Chicago Mercantile Exchange (CME). We use high-frequency intra-day futures price data and end-of-day prices of options written on the underlying futures. We construct expected variances and variance risk premia at the daily frequency and then sample end of month observations.
This figure plots the volatility of the EVRP, red line, the volatility of the TVRP (5-year), yellow line, the volatility of the TVRP (10-year), blue line, and the volatility of the TVRP (30-year), green line. The data runs from 1990 to 2014.
IX. Appendix: Tables
Table I. Out-of-Sample-Forecasting

This table reports results from forecasting regressions of the one month realised variance of 5y, 10y, and 30y Treasury bond futures and of the S&P 500 index on a lagged cascade of HAR factors and the 1-month implied variance. Factor loadings are estimated in-sample and t-statistics reported in parenthesis are computed using GMM correction. $R^2_{MZ,mod}$ and $R^2_{MZ,RW}$ report the $R^2$ from a Mincer and Zarnowitz, (1969) regression where expected variance is computed using the HAR or random-walk models. $RMSE_{OS}$ and $RMSE_{RW}$ report out-of-sample and random walk root-mean-square-errors. The final row reports the Diebold and Mariano (1995) statistic for the difference between the squared out-of-sample HAR errors versus the squared errors from a random walk model. Standard errors in $DM(h)$ are adjusted using an h-lag Newey-West correction. Daily frequency between 1990 and 2014.

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P 500</th>
<th>5yr Bond</th>
<th>10yr Bond</th>
<th>30yr Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{RV,D}$</td>
<td>0.05</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(4.87)</td>
<td>(3.46)</td>
<td>(3.78)</td>
<td>(3.78)</td>
</tr>
<tr>
<td>$\beta_{RV,W}$</td>
<td>0.16</td>
<td>0.13</td>
<td>0.13</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>(4.64)</td>
<td>(3.54)</td>
<td>(4.06)</td>
<td>(3.04)</td>
</tr>
<tr>
<td>$\beta_{RV,M}$</td>
<td>0.24</td>
<td>0.34</td>
<td>0.17</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>(5.71)</td>
<td>(5.68)</td>
<td>(3.28)</td>
<td>(8.07)</td>
</tr>
<tr>
<td>$\beta_{TCI,J}$</td>
<td>54.62</td>
<td>-249.82</td>
<td>-141.61</td>
<td>-44.40</td>
</tr>
<tr>
<td></td>
<td>(0.55)</td>
<td>(-0.63)</td>
<td>(-0.50)</td>
<td>(-0.36)</td>
</tr>
<tr>
<td>$\beta_{IV,lag1}$</td>
<td>0.73</td>
<td>0.33</td>
<td>0.43</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td>(14.33)</td>
<td>(7.45)</td>
<td>(10.12)</td>
<td>(10.26)</td>
</tr>
<tr>
<td>$\beta_{IV,lag2}$</td>
<td>-0.04</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>(-1.63)</td>
<td>(4.14)</td>
<td>(4.13)</td>
<td>(4.88)</td>
</tr>
<tr>
<td>$\beta_{IV,lag3}$</td>
<td>-0.04</td>
<td>0.04</td>
<td>0.05</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(-1.49)</td>
<td>(1.97)</td>
<td>(2.57)</td>
<td>(1.48)</td>
</tr>
<tr>
<td>$\beta_{IV,lag4}$</td>
<td>-0.06</td>
<td>0.01</td>
<td>0.02</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>(-2.13)</td>
<td>(0.68)</td>
<td>(1.03)</td>
<td>(-0.58)</td>
</tr>
<tr>
<td>$\beta_{IV,lag5}$</td>
<td>-0.06</td>
<td>-0.02</td>
<td>0.00</td>
<td>-0.03</td>
</tr>
<tr>
<td></td>
<td>(-2.41)</td>
<td>(-0.81)</td>
<td>(-0.08)</td>
<td>(-1.39)</td>
</tr>
<tr>
<td>$\beta_{IV,lag6}$</td>
<td>0.02</td>
<td>-0.01</td>
<td>0.05</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td>(0.48)</td>
<td>(-0.28)</td>
<td>(1.39)</td>
<td>(-0.44)</td>
</tr>
<tr>
<td>$MZ \hat{R}^2_{mod}$</td>
<td>0.75</td>
<td>0.52</td>
<td>0.57</td>
<td>0.53</td>
</tr>
<tr>
<td>$MZ \hat{R}^2_{RW}$</td>
<td>0.68</td>
<td>0.46</td>
<td>0.45</td>
<td>0.52</td>
</tr>
<tr>
<td>$RMSE_{model}$</td>
<td>0.48</td>
<td>0.47</td>
<td>0.37</td>
<td>0.33</td>
</tr>
<tr>
<td>$RMSE_{RW}$</td>
<td>0.55</td>
<td>0.55</td>
<td>0.45</td>
<td>0.36</td>
</tr>
<tr>
<td>$DM$</td>
<td>5.26</td>
<td>3.91</td>
<td>5.72</td>
<td>1.67</td>
</tr>
</tbody>
</table>
Table II. Summary Statistics: Volatility Risk Premia
The table reports summary statistics for the volatility risk premiums on 10-year and 30-year Treasury bond futures and the S&P500 index for a 1-month horizon. Volatility Risk premiums are obtained as the difference between physical and risk neutral volatilities. Estimates are expressed in annualised percentage points. The sample period is 1990–2014.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>S&amp;P 500</th>
<th>5yr Bond</th>
<th>10yr Bond</th>
<th>30yr Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obs</td>
<td>5964.00</td>
<td>5964.00</td>
<td>5964.00</td>
<td>5964.00</td>
</tr>
<tr>
<td>Mean</td>
<td>-4.26</td>
<td>-1.21</td>
<td>-1.56</td>
<td>-1.91</td>
</tr>
<tr>
<td>Std</td>
<td>2.95</td>
<td>0.75</td>
<td>0.87</td>
<td>1.38</td>
</tr>
<tr>
<td>Min</td>
<td>-18.31</td>
<td>-3.97</td>
<td>-7.69</td>
<td>-9.43</td>
</tr>
<tr>
<td>Max</td>
<td>-0.25</td>
<td>1.84</td>
<td>2.17</td>
<td>0.85</td>
</tr>
<tr>
<td>Skew</td>
<td>-0.87</td>
<td>-0.48</td>
<td>-0.72</td>
<td>-1.33</td>
</tr>
<tr>
<td>Kurt</td>
<td>4.28</td>
<td>3.18</td>
<td>4.86</td>
<td>5.78</td>
</tr>
<tr>
<td>AR(1)</td>
<td>0.92</td>
<td>0.80</td>
<td>0.84</td>
<td>0.92</td>
</tr>
</tbody>
</table>
Table III. Variance Risk Premia and Excess Bond Returns

This table reports return predictability regressions of 1-year excess returns on zero coupon bonds with maturities $n = 5, 10, 20,$ and 30-years on a single linear combination of bond variance risk premium factors and equity variance risk premia defined in the main body of the paper:

$$brx_{t,t+12}^{(n)} = \beta^{(n)} TVRP_t + \gamma^{(n)} EVRP_t + \varepsilon^{(n)}_{t,t+12},$$

Panel A reports univariate predictability regressions on $TVRP$ and Panel B reports multivariate predictability regressions on $TVRP$ and $EVRP$. $t$-statistics are reported in parentheses and computed using the Hansen and Hodrick (1983) GMM correction with 12 Newey-West lags. The sample period is 1990:01–2014:01 for all regressions. Left and right hand variables are standardized (a constant is included but not reported).

<table>
<thead>
<tr>
<th></th>
<th>$TVRP$</th>
<th>t-stat</th>
<th>$EVRP$</th>
<th>t-stat</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$brx_{t,t+12}^{(5)}$</td>
<td>1.28</td>
<td>(2.74)</td>
<td></td>
<td></td>
<td>11.07</td>
</tr>
<tr>
<td>$brx_{t,t+12}^{(10)}$</td>
<td>1.29</td>
<td>(3.30)</td>
<td></td>
<td></td>
<td>11.23</td>
</tr>
<tr>
<td>$brx_{t,t+12}^{(20)}$</td>
<td>0.86</td>
<td>(1.99)</td>
<td></td>
<td></td>
<td>4.96</td>
</tr>
<tr>
<td>$brx_{t,t+12}^{(30)}$</td>
<td>0.58</td>
<td>(1.26)</td>
<td></td>
<td></td>
<td>2.24</td>
</tr>
<tr>
<td><strong>Panel B</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$brx_{t,t+12}^{(5)}$</td>
<td>1.33</td>
<td>(2.43)</td>
<td>0.05</td>
<td>(0.33)</td>
<td>11.30</td>
</tr>
<tr>
<td>$brx_{t,t+12}^{(10)}$</td>
<td>1.40</td>
<td>(3.10)</td>
<td>0.11</td>
<td>(0.83)</td>
<td>12.32</td>
</tr>
<tr>
<td>$brx_{t,t+12}^{(20)}$</td>
<td>1.08</td>
<td>(2.46)</td>
<td>0.22</td>
<td>(1.98)</td>
<td>9.26</td>
</tr>
<tr>
<td>$brx_{t,t+12}^{(30)}$</td>
<td>0.90</td>
<td>(2.08)</td>
<td>0.32</td>
<td>(2.37)</td>
<td>11.49</td>
</tr>
</tbody>
</table>
Table IV. Variance Risk Premia and Excess Stock Returns

This table reports results from regressing $h$-period returns on the value weighted market portfolio from CRSP in excess of the $h$-period risk free rate on the bond and equity variance risk premia defined in the main body of the paper:

$$err_{t,t+h} = \beta_h TVRP_t + \gamma^{(n)} EVRP_t + \varepsilon_{t,t+h},$$

t-statistics are reported in parenthesis and computed using the Hansen and Hodrick (1983) GMM correction with 12 Newey-West lags. The sample period is 1990:01–2014:01 for all regressions.

<table>
<thead>
<tr>
<th>$h$</th>
<th>EVRP</th>
<th>t-stat</th>
<th>TVRP</th>
<th>t-stat</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.10</td>
<td>(-1.07)</td>
<td>0.96</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.10</td>
<td>(-1.38)</td>
<td>2.65</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.10</td>
<td>(-2.55)</td>
<td>4.47</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-0.07</td>
<td>(-1.87)</td>
<td>3.54</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-0.06</td>
<td>(-1.68)</td>
<td>3.21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>-0.05</td>
<td>(-1.38)</td>
<td>2.70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>-0.05</td>
<td>(-1.17)</td>
<td>3.62</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$h$</th>
<th>EVRP</th>
<th>t-stat</th>
<th>TVRP</th>
<th>t-stat</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.52</td>
<td>(-2.08)</td>
<td>1.82</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.44</td>
<td>(-2.01)</td>
<td>3.40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.48</td>
<td>(-2.40)</td>
<td>6.64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-0.48</td>
<td>(-2.24)</td>
<td>8.91</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-0.45</td>
<td>(-2.29)</td>
<td>9.41</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>-0.28</td>
<td>(-1.49)</td>
<td>4.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>-0.16</td>
<td>(-0.88)</td>
<td>2.07</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$h$</th>
<th>EVRP</th>
<th>t-stat</th>
<th>TVRP</th>
<th>t-stat</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.66</td>
<td>(-2.64)</td>
<td>-0.14</td>
<td>(-1.64)</td>
<td>3.02</td>
</tr>
<tr>
<td>3</td>
<td>-0.57</td>
<td>(-2.40)</td>
<td>-0.14</td>
<td>(-2.07)</td>
<td>7.42</td>
</tr>
<tr>
<td>6</td>
<td>-0.62</td>
<td>(-2.67)</td>
<td>-0.13</td>
<td>(-3.62)</td>
<td>13.96</td>
</tr>
<tr>
<td>9</td>
<td>-0.58</td>
<td>(-2.52)</td>
<td>-0.10</td>
<td>(-2.60)</td>
<td>15.20</td>
</tr>
<tr>
<td>12</td>
<td>-0.53</td>
<td>(-2.54)</td>
<td>-0.09</td>
<td>(-2.34)</td>
<td>15.07</td>
</tr>
<tr>
<td>18</td>
<td>-0.34</td>
<td>(-1.88)</td>
<td>-0.06</td>
<td>(-1.98)</td>
<td>8.76</td>
</tr>
<tr>
<td>24</td>
<td>-0.22</td>
<td>(-1.35)</td>
<td>-0.06</td>
<td>(-1.46)</td>
<td>6.30</td>
</tr>
</tbody>
</table>
This table reports parameter values used in the calibration of the theoretical model discussed above. BY stands for Bansal and Yaron (2004). BST stands for Bollerslev, Sizova, and Tauchen (2012). We make the simplifying assumption that $\phi_d = \xi_c = \xi_x$ and set them in such a way so that dividends are three times more volatile than consumption.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$-\ln(0.999) \times 12$</td>
<td>BY</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>7.5</td>
<td>BY</td>
</tr>
<tr>
<td>$\psi$</td>
<td>1.5</td>
<td>BY</td>
</tr>
<tr>
<td>$\mu_c$</td>
<td>$0.0015 \times 12$</td>
<td>BY</td>
</tr>
<tr>
<td>$k_x$</td>
<td>$-\ln(0.979) \times 12$</td>
<td>BY</td>
</tr>
<tr>
<td>$\phi_x$</td>
<td>0.6590</td>
<td>search</td>
</tr>
<tr>
<td>$\mu_d$</td>
<td>$0.0013 \times 12$</td>
<td>BY</td>
</tr>
<tr>
<td>$\phi$</td>
<td>3.0529</td>
<td>BST</td>
</tr>
<tr>
<td>$\phi_d$</td>
<td>0.2449</td>
<td>BST</td>
</tr>
<tr>
<td>$\xi_c$</td>
<td>0.2449</td>
<td>BST</td>
</tr>
<tr>
<td>$\xi_x$</td>
<td>0.2449</td>
<td>BST</td>
</tr>
<tr>
<td>$k_v$</td>
<td>$-\ln(0.979) \times 12$</td>
<td>BY</td>
</tr>
<tr>
<td>$\bar{V}$</td>
<td>$(0.0078 \times \sqrt{12})^2$</td>
<td>BY</td>
</tr>
<tr>
<td>$\varphi_v$</td>
<td>0.0254</td>
<td>BST</td>
</tr>
<tr>
<td>$k_1$</td>
<td>0.1</td>
<td>BST</td>
</tr>
<tr>
<td>$k_2^d$</td>
<td>0.0993</td>
<td>BST</td>
</tr>
</tbody>
</table>
Table VI. Calibrated Populated Moments
This table presents a set of sample moments from the data against corresponding calibrated moments from the model described in section V. ERP is the annualised return on the S&P500 index in excess of the 1-month real risk free rate. \( Rm \) is the annualised return on the S&P500. \( RF^r \) and \( y^{1,r} \) are the real risk free rate and the 1-year real yield, respectively. \( E_Q[\sigma^2(S&P)] \) and \( E_Q[\sigma^2(10yr)] \) are the 1-month implied variances of the S&P500 and 10-year Treasury bond, respectively. \( EVRP \) and \( TVRP(10) \) are the equity and 10-year Treasury variance risk premiums. Moments in Panel A are annualised percentage units. Moments in Panel B are quoted in monthly percentage units.

<table>
<thead>
<tr>
<th></th>
<th>Sample Moments</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>est</td>
<td>se</td>
</tr>
<tr>
<td>Panel A</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>ERP</strong></td>
<td>7.3262</td>
<td>(3.2440)</td>
</tr>
<tr>
<td>( \sigma(Rm) )</td>
<td>18.2900</td>
<td>(0.7200)</td>
</tr>
<tr>
<td>( RF^r )</td>
<td>1.2695</td>
<td>(0.3404)</td>
</tr>
<tr>
<td>( \sigma(RF^r) )</td>
<td>2.4024</td>
<td>(0.0944)</td>
</tr>
<tr>
<td>( y^{1,r} )</td>
<td>1.4680</td>
<td>(0.3381)</td>
</tr>
<tr>
<td>( \sigma(y^{1,r})^r )</td>
<td>6.2215</td>
<td>(0.2440)</td>
</tr>
<tr>
<td>Panel A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_Q[\sigma^2(S&amp;P)] )</td>
<td>38.8227</td>
<td>(0.3301)</td>
</tr>
<tr>
<td>( E_Q[\sigma^2(10yr)] )</td>
<td>5.3312</td>
<td>(0.0282)</td>
</tr>
<tr>
<td><strong>EVRP</strong></td>
<td>-1.7400</td>
<td>(0.0185)</td>
</tr>
<tr>
<td>( AR(180d)EVRP )</td>
<td>0.2376</td>
<td>(0.0230)</td>
</tr>
<tr>
<td><strong>TVRP</strong></td>
<td>-0.3200</td>
<td>(0.0023)</td>
</tr>
<tr>
<td>( AR(180d)TVRP )</td>
<td>0.1457</td>
<td>(0.0110)</td>
</tr>
<tr>
<td>( \rho(EVRP,TVRP) )</td>
<td>0.3873</td>
<td>(0.0106)</td>
</tr>
</tbody>
</table>
X. Appendix: Technical

A. Stochastic Discount Factor and the Equilibrium Interest Rate

The stochastic discount factor (SDF) in this economy is given by (see e.g., Duffie and Epstein (1992a)):

\[ \Lambda_t = e^{\int_0^t f(s)ds} f_c C_s J_s, \]

where \( f(\cdot) \) is the normalised aggregator of the stochastic differential utility and \( J(\cdot) \) is the value function. The total return on the Lucas tree satisfies

\[ \frac{dR_t}{R_t} = \frac{dS_t + C_t dt}{S_t} = \frac{d(\Psi(X_t) C_t)}{\Psi(X_t) C_t} + \frac{d}{\Psi(X_t)}, \]

where \( \Psi(X_t) \) is the price-dividend ratio. Using the fact that

\[ d \ln \Lambda_t + (1 - \theta) d \ln R_t + \frac{\theta}{\Psi} d \ln C_t = -\rho \theta dt, \]

we derive the equilibrium PDE for the price-dividend ratio, \( \Psi(X_t) \).

\[ D(\Lambda_t R_t) = 0, \]

\[ \frac{1}{dt} E \left( d \ln \Lambda_t R_t + \frac{1}{2} d \langle \ln \Lambda, \ln R \rangle_t \right) = 0, \quad (8) \]

where \( D(\cdot) \) is Dynkin’s operator. We calculate

\[ d \ln \Lambda_t R_t = -\beta \theta dt - (1 - \theta) d \ln R_t - \frac{\theta}{\Psi} d \ln C_t + d \ln R_t \]

\[ = -\beta \theta dt + \theta \left( d \ln \Psi + d \ln C_t + \frac{dt}{\Psi} \right) - \frac{\theta}{\Psi} d \ln C_t \]

\[ = -\beta \theta dt + \theta d \ln \Psi + \theta \left( 1 - \frac{1}{\Psi} \right) d \ln C_t + \theta \frac{dt}{\Psi} \]

\[ = -\beta \theta dt + \theta d \ln \Psi + (1 - \gamma) d \ln C + \theta \frac{dt}{\Psi}, \quad (9) \]

and

\[ d \langle \ln \Lambda, \ln \Lambda \rangle_t = \theta^2 d \langle \ln \Psi, \ln \Psi \rangle_t + (1 - \gamma)^2 d \langle \ln C, \ln C \rangle_t. \quad (10) \]

Next we substitute (9) and (10) in (8) and obtain the equilibrium PDE for the price-dividend ratio,

\[ \theta D \ln \Psi(X_t) + \frac{\theta}{\Psi(X_t)} + (1 - \gamma) \ln C - \theta \beta \]

\[ + \frac{\theta^2}{2} \frac{d \langle \ln \Psi(X), \ln \Psi(X) \rangle_t}{dt} + \frac{1}{2} (1 - \gamma)^2 \frac{d \langle \ln \Psi(X), \ln \Psi(X) \rangle_t}{dt} = 0. \]

A straightforward simplification leads to

\[ \theta D \ln \Psi(X_t) + \frac{\theta}{\Psi(X_t)} + (1 - \gamma) \left( \mu + x_t - \frac{1}{2} V_t \right) - \theta \beta \]

\[ + \frac{\theta^2}{2} \frac{d \langle \ln \Psi(X), \ln \Psi(X) \rangle_t}{dt} + \frac{1}{2} (1 - \gamma)^2 V_t = 0. \]
Following Bollerslev, Sizova, and Tauchen (2012) we conjecture a solution of the form

$$
\ln \Psi(X_t) = A_0 + A_x x_t + A_v V_t.
$$

In order to obtain a closed-form solution we linearise $1/\ln \Psi(X_t)$ and the equilibrium PDE for the price-dividend ratio.

$$
\frac{1}{\Psi(X_t)} = -k_0 - k_1 (A_0 + A_x x_t + A_v V_t),
$$

where $k_0 = e^{-\ln \Psi(1 + \ln \Psi)}$, and $k_1 = e^{-\ln \Psi}$. We then solve the resulting system of three ODEs and obtain

$$
A_x = \frac{1 - \frac{1}{\psi}}{k_1 + k_x}, \quad A_v = \frac{(k_1 + k_v) - \sqrt{(k_1 + k_v)^2 - \theta^2 A_x^2 \varphi^2 \varphi^2}}{\theta \varphi^2}, \quad A_0 = \frac{1}{k_1} \left( k_v \bar{A}_v x - k_0 + \left( 1 - \frac{1}{\psi} \right) \mu_c - \beta \right).
$$

Since $\gamma > 1$ and $\psi > 1$, $\theta < 0$ and $A_v < 0$, i.e., the price-dividend ratio (the market) falls on positive volatility news. The processes for $d \ln R$ and the SDF, $d \Lambda/A$, are as follows:

$$
d \ln R = d \ln \Psi + d \ln C + \frac{dt}{\psi} = -k_x A_x x dt + \varphi_x \sqrt{V_t} A_x dW_t^x + A_v k_v (V - V_t) dt + A_v \varphi_v \sqrt{V_t} dW_t^v + (\mu_c + x_t) dt + \sqrt{V_t} dW_t^c - \frac{1}{2} V_t dt - (k_0 + k_1(A_0 + A_x x_t + A_v V_t)) dt
$$

$$
= -k_x x_t A_x dt + \varphi_x \sqrt{V_t} A_x dW_t^x + A_v k_v (V - V_t) dt + A_v \varphi_v \sqrt{V_t} dW_t^v + (\mu_c + x_t) dt + \sqrt{V_t} dW_t^c - \frac{1}{2} V_t dt - k_0 dt - k_1 A_0 dt - k_1 A_x x_t dt - k_1 A_v V_t dt + \varphi_x \sqrt{V_t} A_x dW_t^x + A_v k_v (V - V_t) dt + A_v \varphi_v \sqrt{V_t} dW_t^v
$$

We substitute $A_x$ and note that

$$
\theta A_v^2 \varphi_v^2 - 2(k_1 + k_v)A_v - \gamma \left( 1 - \frac{1}{\psi} \right) + \theta A_x^2 \varphi_x^2 = 0,
$$

$$
d \ln R_t = \left[ \frac{\mu_c + x_t}{\psi} + \beta - \frac{1}{2} \theta (A_v^2 \varphi_v^2 + A_x^2 \varphi_x^2) V_t + \left( -\frac{1}{2} + \gamma \left( 1 - \frac{1}{\psi} \right) \right) V_t \right] dt + \sqrt{V_t} dW_t^c + \varphi_x \sqrt{V_t} A_x dW_t^x + A_v \varphi_v \sqrt{V_t} dW_t^v
$$

(11)

From (11) we derive the processes of the SDF, the equilibrium interest rate, and the market prices of risk.

$$
\frac{d \Lambda_t}{A_t} = \left[ \frac{\gamma}{2} \left( 1 + \frac{1}{\psi} \right) V_t - \frac{1}{\psi} (\mu_c + x_t) - \beta - \frac{1}{2} (\theta - 1)(A_v^2 \varphi_v^2 + A_x^2 \varphi_x^2) V_t \right] dt - \gamma \sqrt{V_t} dW_t^c + (\theta - 1)(\varphi_x A_x \sqrt{V_t} dW_t^x + \varphi_v A_v \sqrt{V_t} dW_t^v)
$$

(12)

The equilibrium interest rate and market prices of risk follow from (12), the process of the SDF.

$$
r_t = \frac{\mu_c + x_t}{\psi} + \beta + \frac{1}{2} (\theta - 1)(A_v^2 \varphi_v^2 + A_x^2 \varphi_x^2) V_t - \frac{\gamma}{2} \left( 1 + \frac{1}{\psi} \right) V_t,
$$

(13)
\[ MPR_x = -(\theta - 1)\varphi_x \sqrt{V_t} A_x, \]
\[ MPR_v = -(\theta - 1)\varphi_v \sqrt{V_t} A_v, \]

where \( MPR \) stands for market price of risk.

**B. Equilibrium Bond Price**

We conjecture that the bond price is exponentially affine in the state variables,
\[ P(\tau, x, V) = e^{A(\tau) + B(\tau)x + C(\tau)V}. \]

The PDE for the equilibrium bond price is
\[
0 = \frac{1}{2} \frac{\partial^2 P}{\partial x^2} \frac{d(x, x)_t}{dt} + \frac{1}{2} \frac{\partial^2 P}{\partial V^2} \frac{d(V, V)_t}{dt} + \frac{\partial P}{\partial x} (-k_x x_t - MPR_x \varphi_x \sqrt{V_t})
+ \frac{\partial P}{\partial V} (k_v (\bar{V} - V_t) - MPR_v \varphi_v \sqrt{V_t}) + \frac{\partial P}{\partial t} - rP.
\]

After a straightforward simplification of the PDE for the bond price we obtain
\[
\frac{1}{2} B^2(\tau) \varphi_x^2 V_t + \frac{1}{2} C^2(\tau) \varphi_v^2 V_t + B(\tau)(-k_x x_t + (\theta - 1)\varphi_x^2 V_t A_x)
+ C(\tau)(k_v (\bar{V} - V_t) + (\theta - 1)\varphi_v^2 V_t A_v) - (\dot{A}(\tau) + \dot{B}(\tau)x_t + \dot{C}(\tau)V_t) - r = 0,
\]

where
\[ r_t = \mu_c + x_t + \beta + \frac{(\theta - 1)}{2} (A_x^2 \varphi_x^2 + A_v^2 \varphi_v^2) V_t - \frac{\gamma}{2} \left( 1 + \frac{1}{\psi} \right) V_t. \quad (14) \]

The following system of three ODEs follows from the PDE for the equilibrium bond price.
\[
\dot{A}(\tau) = -\frac{1}{\psi} \mu_c - \beta + C(\tau) k_v \bar{V}, \quad (15)
\]
\[
\dot{B}(\tau) = -k_x B(\tau) - \frac{1}{\psi}, \quad (16)
\]
\[
\dot{C}(\tau) = \frac{\gamma}{2} \left( 1 + \frac{1}{\psi} \right) - \frac{(\theta - 1)}{2} (A_x^2 \varphi_x^2 + A_v^2 \varphi_v^2) + (\theta - 1)\varphi_v^2 A_x B(\tau), \quad (17)
\]
\[
+ ((\theta - 1)\varphi_v^2 A_v - k_v) C(\tau) + \frac{1}{2} B^2(\tau) \varphi_x^2 + \frac{1}{2} C^2(\tau) \varphi_v^2.
\]

Whereas (16) admits a closed-form solution, we solve the Riccati equation (17) numerically.
\[
\dot{B}(\tau) = -k_x B(\tau) - \frac{1}{\psi},
\]
\[
dB(t)e^{k_x t} = -\frac{1}{\psi} e^{k_x t},
\]
\[
B(\tau) = -\frac{1}{\psi} k_x \left( 1 - e^{-k_x \tau} \right).
\]

Consequently, the following equation describes the equilibrium bond price process.
\[
\frac{dP(\tau, x, V)}{P(\tau, x, V)} = (\cdot) dt + B(\tau) \varphi_x \sqrt{V_t} dW^x_t + C(\tau) \varphi_v \sqrt{V_t} dW^v_t.
\]
C. **Model-implied TVRP**

The integrand of the TVRP implied by the long-run risk model is

\[
\frac{E^P d(\ln P, \ln P)_t - E^Q d(\ln P, \ln P)_t}{dt} = (\phi^2_2 B^2(\tau) + \phi^2_v C^2(\tau))(E^P V_t - E^Q V_t).
\]

Note that

\[
dV_t = k_v (\bar{V} - V_t) dt + \phi_v \sqrt{\bar{V}} dW^v_t, \quad P-a.s.,
\]

\[
V_t = V_0 e^{-k_v t} + \bar{V} (1 - e^{-k_v t}) + \phi_v \int_0^t \sqrt{\bar{V}} e^{k_v (\tau - t)} dW^v_s
\]

and

\[
dV_t = (\phi^2_v (\theta - 1) A_v - k_v) V_t dt + k_v \bar{V} dt + \phi_v \sqrt{\bar{V}} dW^v_t, \quad Q-a.s.,
\]

\[
V_t = V_0 e^{\chi t} - \frac{k_v \bar{V}}{\chi} (1 - e^{\chi t}) + \phi_v \int_0^t \sqrt{\bar{V}} e^{\chi (t - s)} dW^v_s,
\]

where

\[
\chi = \phi^2_v (\theta - 1) A_v - k_v.
\]

Consequently,

\[
\frac{E^P d(\ln P, \ln P)_t - E^Q d(\ln P, \ln P)_t}{dt} = (\phi^2_2 B^2(\tau) + \phi^2_v C^2(\tau))
\]

\[
\times \left[ V_0 e^{-k_v t} + \bar{V} (1 - e^{-k_v t}) - V_0 e^{\chi t} + \frac{k_v \bar{V}}{\chi} (1 - e^{\chi t}) \right]
\]

and the unconditional equilibrium fixed income variance risk premium implied by the long-run risk model is

\[
TVRP_{MLR}(0, t, T) = \int_0^t (\phi^2_2 B^2(\tau) + \phi^2_v C^2(\tau))
\]

\[
\times \left[ V_0 e^{-k_v t} + \bar{V} (1 - e^{-k_v t}) - V_0 e^{\chi t} + \frac{k_v \bar{V}}{\chi} (1 - e^{\chi t}) \right] dt.
\]

D. **Model-implied EVRP**

As a first step towards the derivation of the equity variance risk premium, we derive the total return on the leveraged Lucas tree, paying a dividend \(D\), and its PDE. We impose a drift restriction of the form

\[
D(\Lambda_t R^d_t) = 0,
\]

i.e.,

\[
\frac{1}{dt} E \left[ d\ln \Lambda_t R^d_t + \frac{1}{2} d(\ln \Lambda R^d_t, \ln \Lambda R^d_t)_t \right] = 0.
\]

and pre-calculate \(d\ln \Lambda_t R^d_t\),

\[
d\ln \Lambda_t R^d_t = d\ln \Lambda_t + d\ln R^d_t
\]

\[
= -\beta \theta dt - (1 - \theta) d\ln R_t - \frac{\theta}{\psi} d\ln C_t + d\ln R^d_t
\]

\[
= (\theta - 1) \left( d\ln \Psi_t + d\ln C_t + \frac{dt}{\Psi_t} \right) - \frac{\theta}{\psi} d\ln C_t - \beta \theta dt
\]

\[
+ \left( d\ln \Psi_t^d + d\ln D_t + \frac{dt}{\Psi_t^d} \right)
\]

46
and conjecture a solution of the form
\[
A^d = A^d_0 + A^d_x x_t + A^d_v v_t,
\]
\[
\frac{1}{\Psi^d(x_t)} = k^d_0 - k^d_1(A^d_0 + A^d_x x_t + A^d_v v_t).
\]
We proceed under the assumption that the dividend process follows
\[
(\theta - 1)\frac{d\ln \Psi_t}{\Psi_t} + \frac{d\ln C_t}{C_t} + \frac{d\ln D_t}{D_t} \gamma \ln C_t + (\theta - 1) \frac{dt}{\Psi_t} - \beta \theta dt
\]
Consequently, the dividend-paying asset satisfies
\[
(\theta - 1)\frac{D}{C} \ln \Psi_t - \gamma D \ln C_t + (\theta - 1) \frac{d\ln \Psi^d(X_t)}{\Psi^d(X_t)} - \beta D \ln \Psi^d(X_t) + D \ln D_t
\]
We next plug the conjecture in the above equation and obtain
\[
\ln \Psi^d(X_t) = A^d_0 + A^d_x x_t + A^d_v v_t,
\]
\[
\frac{1}{\Psi^d(X_t)} = k^d_0 - k^d_1(A^d_0 + A^d_x x_t + A^d_v v_t).
\]
Solving the resulting system of three equations, we obtain the coefficients in the conjectured solution.
\[
A^d_x = \frac{\phi - 1}{k^d_1 + k_x},
\]
\[
A^d_v = \frac{k^d_0 + k_v v_t - \sqrt{\Delta}}{\sqrt{\varphi_v} - (\theta - 1)A_v},
\]
where,
\[
\Delta = \left(\frac{k^d_1 + k_v}{\sqrt{\varphi_v}}\right)^2 - ((\theta - 1)A_x + A^d_x)\varphi_x^2 - 2(\theta - 1)A_v(k^d_1 - k_1)\]
+ 2(\theta - 1)A_x \varphi_x \xi_x - 2A_d^d \varphi_x \xi_x + 2\gamma \xi_c.

Next we derive the return process, \( d\ln R^d \), on the dividend-paying asset. Using

\[
 d\ln R^d_t = d\ln D_t + d\ln \Psi^d(X_t) + \frac{dt}{\Psi^d(X_t)}
\]

we obtain

\[
 d\ln R^d_t = (\cdot)dt + \varphi_d dW^d_t + (A^d_\varphi \xi_x + \xi_x)\sqrt{V^d_t}dW^c_t + A^d_\varphi \varphi_v \sqrt{V^d_t}dW^v_t + \xi_c \sqrt{V^d_t}dW^c_t.
\]

Using this result, we can calculate the unconditional equity variance risk premium,

\[
 EVRP(0,\bar{t},T) = \mathbb{E}^{P} IV(0,\bar{t}) - \mathbb{E}^{Q} IV(0,\bar{t})
\]

\[
 = \int_{0}^{\bar{t}} ((A^d_\varphi \xi_x + \xi_x)^2 + (A^d_\varphi \varphi_v)^2 + \xi_c^2) (\mathbb{E}^{P} V_t - \mathbb{E}^{Q} V_t) dt
\]

\[
 = \int_{0}^{\bar{t}} ((A^d_\varphi \xi_x + \xi_x)^2 + (A^d_\varphi \varphi_v)^2 + \xi_c^2)
\]

\[
 \times \left[ V_0 e^{-k_v \chi t} + \bar{V} (1 - e^{-k_v \chi t}) - V_0 e^{\chi t} + \frac{k_v V}{\chi} (1 - e^{\chi t}) \right] dt.
\]

From the process of the total return on the leveraged Lucas tree we derive the equity risk premium, \( ERP \), for further reference.

\[
 ERP_t = - \frac{1}{\Lambda_t} \frac{d\langle R^d, \Lambda \rangle_t}{dt}
\]

\[
 = - \frac{d\langle \ln R^d, \ln \Lambda \rangle_t}{dt}
\]

\[
 = (\gamma \xi_c - (\theta - 1) [\varphi_x A_x (A^d_\varphi \xi_x + \xi_x) + A_v A^d_\varphi \varphi_v]) V_t.
\]