Rational Quantitative Trading in Efficient Markets*

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Abstract

We present a model of financial markets where quantitative trading emerges endogenously as an automated price-contingent strategy under human discretion. Price-contingent trading has been argued to be at odds with (semi-strong) market efficiency. In contrast, we show that price-contingent trading is the profitable equilibrium strategy of a large rational agent whose trading strategy, price-contingent or fundamentals-based, is private information. Even when uninformed about fundamentals he will trade non-zero quantities whose direction—trend-following or contrarian—depends on the magnitude of the order flow in a non-monotonic manner. One additional implication of our model is that future order flow is predictable even if returns are not.

JEL classification: G12, G14, D82

Keywords: quantitative trading, market impact, REE, price-contingent trading

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1 Introduction

It is recognized by financial market participants and academic literature that a noticeable part of the trading volume in financial markets is driven by price-contingent strategies, i.e., strategies that are based on past asset prices, rather than fundamental analysis. Importantly, these strategies seem to form a major part of the strategies implemented by funds specialized in quantitative trading. Price-contingent trading can be trend-following (e.g., buying when the price goes up), contrarian (e.g., selling when the price goes up), or can take more complex forms where the direction of price-contingent trading varies across time horizons, magnitudes of past price chances, instruments and trade structures. (see Section 3 in Narang 2013 for characterization of data driven quantitative strategies).

Empirical evidence shows that retail investors and specialized quantitative funds do not equally profit from price-contingent trading. In fact, the evidence shows that pursuing trend-following or contrarian strategies generates losses for individual and retail investors. These findings have motivated a large part of the literature to argue that trend-following or contrarian trading must stem from behavioral biases, imperfect or bounded rationality, non-standard preferences, or institutional frictions (e.g., Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), Hong and Stein (1999); see Shleifer (2000) and Barberis and Thaler (2003) for surveys). At the same time, this literature does not explain the unprecedented growth of quantitative trading by sophisticated large financial institutions (see e.g., Osler (2003) and Hendershott, Jones, and Menkveld (2011))—unlike retail investors, these large institutions seem to systematically profit by price-contingent trading.²

Our paper presents a stylized micro-founded trading model to analyze quantitative trading by large financial institutions. While adopting a setting that maintains standard assumptions of (semi-strong) market efficiency and preferences, we allow for informational friction and prove that price-contingent trading is a rational equilibrium strategy for large traders, who systematically profit from trading in response to past price changes. Furthermore, we show that in general, the direction of optimal price-contingent trading depends on the magnitude of past order flows (or equivalently, the magnitude of past price changes) in a non-monotonic manner, i.e., it tends to be trend-following at small and contrarian at large past order flows. To the best of our knowledge, such non-monotonicity is a novel theoretical finding in the context of

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¹Contrarian trading leads to portfolio losses, as shown for individuals in Odean (1998) and Barber and Odean (2000). Trend-following trading leads to portfolio losses, as shown for mutual funds in Grinblatt, Titman, and Wermers (1995), Carhart (1997), and Coval and Stafford (2007) and for pension funds in Lakonishok, Shleifer, and Vishny (1992). See also Nofsinger and Sias (1999) and Griffin, Harris, and Topaloglu (2003).

²For example, Commodity Trading Advisors (CTAs) profit from trend-following strategies in futures markets (e.g., see Clenow (2013), and Baltas and Kosowski (2014)); and various institutions profit from contrarian strategies in equities (e.g., see Lehmann (1990), and Jegadeesh (1990)).
related literature (discussed in Section 2). It also allows us to highlight empirical implications
that are unique to our model, and which we discuss in Section 6.

Before addressing the drivers of trend-following and contrarian trading strategies, we should
highlight two key assumptions that are important for explaining why quantitative trading is a
rational equilibrium strategy in our setting. First, we assume that a quantitative trader has
market impact, i.e., his trades have the capacity to move prices. Second, there is a positive
probability that the quantitative trader may have fundamental information, while it is not
known with certainty by market makers who set the price, whether or not the trader has such
information. The first assumption must hold by definition when the traders are “large”. The
second assumption is realistic because quantitative trading nearly always involves discretion
and human supervision, which implies that a quantitative trader can and will override the
“trading algorithm” should he become aware of some fundamental information. In fact, Naragn
(2013), who explains and reflects on quantitative trading from a practitioner’s point of view,
repeatedly highlights such discretion to be of key importance to understand quantitative trading
by professional investors.

We present a stylized setting with one risky asset and two trading dates, and model quan-
titative trading by assuming that there is a large risk-neutral trader, called P, who may be
directly informed about the fundamental value (type I) or not informed (type U). Trader P
knows his own type and takes into account the market impact of his order on the expected
market price. As standard in this literature, we impose that prices are set such that the market
is semi-strong efficient by introducing another risk-neutral agent, the Market (often called the
“market maker” in the rational expectations literature)\(^3\) who observes the aggregate order flow
and sets prices such that he breaks even in expectations. Crucially, the Market does not observe
P’s type. Instead, he knows the prior probability of P being type I or U, and updates his beliefs
about the asset value and P’s type based on the order flows.

As a simple example, consider the case when P is the only strategic large trader in the
market, and the rest of the aggregate order flow comes from noise traders whose trading volume
is independent from other random variables, serially uncorrelated and drawn from the normal
distribution.\(^4\) It is intuitive in this setting that the equilibrium price is an increasing function
of date 1 order flow, simply because if P is informed (type I), P’s orders are driven by the

\(^3\)We call this agent “the Market”, as the same outcome would be obtained if prices are set in a competitive
market populated by a large number of small uniformed traders.

\(^4\)We introduce noise traders as a standard assumption to guarantee that order flows are not fully revealing.
Furthermore, predictability of noise trading (e.g., through serial correlation) would immediately create a reason
for any rational agent to profit by trading against the noise traders’ demand. We abstract from such consid-
erations, and consider noise traders as a large number of retail investors each trading for idiosyncratic reasons
outside the model. As a result, the normal distribution of noise trades follows from the central limit theorem.
In Section 5.4 we discuss the robustness of our findings to relaxing the assumption of normal noise trading and
consider the wide class of log-concave distributions.

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fundamental information. However, if not informed (type U), P still knows his own type and that any deviation of prices from the prior mean is a reflection of noise trading, knowledge of which he can and does benefit from by pursuing a contrarian strategy (at date 2) in a rational expectations equilibrium.

This simple example gives an intuition why P’s market impact and uncertainty about P’s type are both needed for quantitative trading to be an equilibrium strategy. If P’s past trades would not have a market impact, his knowledge of his own past trades would not generate superior information. This helps to explain why it is relatively more difficult for retail investors to benefit from quantitative trading compared to larger institutions. Also, if the trader P’s type was public information, it would also be impossible for P to earn abnormal returns, because the Market and P would have exactly the same information (See Easley and O’Hara 1991 for a similar argument for the case where P’s type is known).

In our main model, we consider a richer setting where there are other informed traders in the market. To account for this possibility, we introduce another large trader, called K, who is also large and risk-neutral, but unlike P, is always informed about the fundamental value. Distribution of the fundamental value of the asset is also important because it determines what informed traders can do in the equilibrium. We show that to obtain all our effects about the direction of P’s trading, there must be, at minimum, three possibilities – fundamental information reflecting positive or negative news, as well as confirming that the fundamental is (close to) to the prior mean (i.e., no or moderate “news”). Our basic model therefore uses a symmetric three-point distribution that covers distributions of very different shapes (from U-shaped to hump-shaped). \(^5\)

We show that at large past order flows, the optimal price-contingent strategy is contrarian; the reason is similar to the simple example above. It always holds that two large informed traders would jointly trade more than one. Hence, in the state where P is uninformed, the Market, who observes a large positive aggregate order flow, tends to optimally set prices too high compared to uninformed P’s best guess. Consequently, by knowing his own type (U), and observing a large past order flow, P knows better than the Market that the past price changes observed at date 1 most likely reflects noise trading rather than fundamental information.

The intuition for trend-following trading at small order flows is just the reverse. When observing a small aggregate order flow, the Market rationally believes that the order flow is most likely generated by noise trading rather than informed trading. As a result, rational equilibrium prices are too insensitive to order flow compared to uninformed P’s best estimate, and the optimal quantitative trading strategy is trend-following. It is important to highlight that such

\(^5\)As we allow any probability of “no news”, our setting also covers the special case of a Bernoulli prior, which is less realistic for most assets as it eliminates the possibility of no major event happening.
trend following incentive would be absent if we eliminated the possibility that informed traders may have chosen not to trade (or to trade little), because they have learned that fundamental is (close to) the prior.\footnote{For this exact reason, we show that a Bernoulli prior implies that only the contrarian trading incentives described above survive in the equilibrium.} We study the robustness of our results to a variety of extensions, including studying the robustness of our results to other distributions of noise trading and prior, and show that the main drivers or contrarian and trend-following incentives apply for a wide set of other (including continuous) prior distributions.

For clarity of exposition, our main setting also assumes that informed traders only trade at date 1 - after all, quantitative strategies are likely to react faster to public signals than purely fundamental-based strategies. Nevertheless, we show that our results are robust to allowing all rational (and forward-looking) traders to trade in all periods - we obtain exactly the same qualitative predictions. It is also straightforward to extend our results to allow for more informed traders and/or more trading rounds, and our results do not rely on assumptions about K’s knowledge of P’s type.

At the most basic level, our theory rationalizes why quantitative trading is profitable on average, over and above standard remunerations for risk, as it is better able to chase information than the rest of the market. It also delivers an additional theoretical result that the order flow is predictable from past information even if returns are not. Order flow predictability is consistent with the evidence of Biais, Hillion, and Spatt (1995), Ellul, Holden, Jain, and Jennings (2007), and Lillo and Farmer (2004). Furthermore, as our main novel predictions are about the non-linear trading strategies, our model also suggests that this non-linearity carries over to the direction of predictability in the order flow, which provides further testable implications (see Section 6).

Our theory also highlights the importance of equilibrium forces that typical practitioners’ accounts (such as Narang 2013) do not consider. While practitioners emphasize the realistic contemporaneous costs of market impact\footnote{E.g., see Kissell (2014). Anecdotal evidence even suggests that hedge funds may decline opportunities to raise funds under management because a larger trading volume would imply too much market impact.}, which we incorporate, a perhaps less intuitive finding of our model is that the same costly market impact is also an important source of subsequent superior private information.

It should also be emphasized that our model is most insightful for quantitative trading strategies at intermediate, relatively short time horizons, i.e., daily, weekly, and/or monthly frequencies. While the forces we highlight may be present also at high frequency level, we recognize that at very high frequencies, a potentially larger part of quantitative trading takes the form of market making. At the same time, at very long horizons the Market would eventually learn P’s type and asset prices would converge to reflect only fundamental information.
The paper proceeds as follows. Section 2 discusses some of the related literature. Section 3 outlines the setup. Section 4 presents the main results. Section 5 considers extensions. Section 6 discusses the empirical implications, and Section 7 concludes.

2 Literature

The broad literature on asset pricing and learning in micro-founded financial markets is surveyed in Brunnermeier (2001) and Vives (2008), among others. Our work relates to the part of the literature that studies trading in markets with asymmetric information. Our results on the profitability of rational price-contingent trading require that informed traders be large, i.e., that their trades have market impact.\(^8\) Our model shows that rational traders with market impact and superior information about their own type can learn from prices better than average market participants. Another strand of the literature studies whether past prices contain useful information for a rational trader (e.g., Grossman and Stiglitz (1980), Brown and Jennings (1989)). However, in these papers there are no profits from uninformed trading in excess of the risk premium. It is worth noting that this is because uninformed traders do not have market impact in these models and the number/share of such traders is known to all market participants.

Our paper also relates to the literature on stock price manipulation, that is, the idea that rational traders may have an incentive to trade against their private information. Provided manipulation is followed by some (exogenously assumed) price-contingent trading, short run losses can be more than offset by long term gains (see Kyle and Viswanathan (2008) for a review). Somewhat closer to our work, Chakraborty and Yilmaz (2004a, 2004b) study the incentives of an informed trader when there is uncertainty about whether such trader is informed, or is a noise trader instead. If this trader turns out to be informed, he may choose to disregard his information and trade randomly, in order to build a reputation as a noise trader. In their model, uninformed traders are assumed to always act as noise traders and are never strategic and rational. Therefore, Chakraborty and Yilmaz do not analyze the trading incentives of rational agents when they are uninformed, which is our main focus.

Goldstein and Guembel (2008) show that if stock prices affect real activity then a form of trade-based manipulation such as short-sales by uninformed speculators can be profitable insofar as it causes firms to cancel positive NPV projects, and justifies ex post the "gamble" for a lower firm value. In their setting, both uninformed trading and successful stock price manipulation stem from the feedback effect between stock prices and real activity. By contrast,

\(^8\)We develop our model in a setting that generalizes the Kyle (1985) framework, but similar implications could be obtained in a Glosten and Milgrom (1985) framework in which trades arrive probabilistically and market makers observe individual trades (see also Back and Baruch (2004)).
in our paper there is price-contingent trading but no manipulation. Therefore, our results demonstrate that price-contingent trading does not make uninformed investors the inevitable prey of (potentially informed) speculators.9

Our paper is also related to the literature that studies the consequences of introducing uncertainty about the types of traders in models with asymmetric or dispersed information. One notable strand of this literature focuses on rational herding (see Avery and Zemski (1998), Park and Sabourian (2011), and Chamley (2004) for a review). Unlike our setting in which traders never disregard their private information, these models characterize conditions under which, when information precision is uncertain, rational traders ignore their noisy private signal and follow the actions of other traders instead.10

A more recent strand of this literature studies static and dynamic competitive rational expectations models with uncertainty about traders’ types (Gao, Song, and Wang (2013), Banerjee and Green (2014)), or a Kyle model with one strategic agent and uncertainty about her information (Li (2013)). Unlike us, none of these papers study the possibility of uninformed traders learning more than market-makers.11

Finally, we should note that our model is most appropriate to understand quantitative strategies that trade daily or weekly, so that there is both some probability that trading reflects information, and some benefit from a relatively fast execution in response to changes in market prices. It is less appropriate for the millisecond environment in which high frequency traders may benefit from momentary imbalances between supply and demand. With this in mind, our paper is also related to a few recent papers that focus on the speed advantage of quantitative traders. Clark-Joseph (2013) studies a partial equilibrium model in which prices are exogenous, and finds empirical support for the idea that high-frequency traders learn from their own trades better than the rest of the market, very closely related to the ideas developed in our model. Biais, Foucault, and Moinas (2013) consider the decision of a financial institution to invest in a high-speed trading technology and derive conditions under which such investment is excessive from a social welfare standpoint; and Pagnotta and Philippon (2012) consider trading exchanges competing on speed to attract future trading activity. Unlike us, these papers do not focus on price-contingent trading.

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9 Allen and Gale (1992) also study a setting with manipulation but without price-contingent trading.

10 One strand of this herding literature has focused on understanding the stock market crash of October 1987 (Grossman (1988), Gennotte and Leland (1990), Jacklin, Kleidon, and Pflederer (1992) and Romer (1993)). A common theme of these papers is that market participants are assumed to have strongly underestimated the extent of portfolio insurance—i.e., positive-feedback trading—which in turn is assumed to be exogenous. Our focus instead is on deriving endogenously price-contingent trading, and characterize conditions under which it is trend-following as opposed to contrarian.

11 Uncertainty about the number of informed traders also shows up in static noisy REE models where agents face short-sale or borrowing constraints, and uninformed traders are uncertain whether the constraint is binding or not (e.g., Yuan (2005)).
3 Setup

In our baseline model we consider a stylized setting with one risky asset that is traded at dates 1 and 2. The fundamental value, \( \theta \), is realized at date 3, and can take the following three values:

\[
\theta = \begin{cases} 
-\bar{\theta} & \text{wpr. } \frac{1-\gamma}{2} \\
0 & \text{wpr. } \gamma \\
\bar{\theta} & \text{wpr. } \frac{1-\gamma}{2}
\end{cases},
\]  

(1)

where \( 0 \leq \gamma < 1 \) and \( \bar{\theta} > 0 \). Such three-point prior provides a simple representation of distributions of many different shapes (from U-shaped to hump-shaped), while the parameter \( \gamma \) is a measure of how much mass is in the centre of the distribution compared to the tails.\(^{12}\)

In Section 5.3 we extend our analysis to consider other priors.

As our goal is to understand the incentives of large traders who know that their trades have a market impact, we adopt a setting in the spirit of Kyle (1985). We maintain the assumption that large strategic risk-neutral traders and non-strategic noise traders submit market orders before knowing the execution price, and that the equilibrium prices are set by a hypothetical agent, the Market, who observes the total order flow and implements the market efficiency condition. Namely, he sets period \( t \) price,

\[
p_t = \mathbb{E} [\theta | \Omega_t^M],
\]  

(2)

where \( \Omega_t^M \) is the information set available to the Market in \( t \in \{1, 2\} \), which includes all publicly available information such as the current and past order flows. Denoting the date \( t \) total order flow with \( y_t \), it holds that \( \Omega_1^M = \{y_1\} \) and \( \Omega_2^M = \{y_1, y_2\} \).

To analyze quantitative trading by a large strategic investors, we introduce a risk-neutral trader, \( P \). This trader sets up his quantitative trading strategy before date 1 without knowing the fundamental value \( \theta \). However, we have argued that quantitatitive trading is not void of the possibility of the trader altering his strategy should he learn fundamental information. This implies that when trading takes place, \( P \) may either trade based of his knowledge of the fundamental or based on his original strategy. To formalize this idea, we assume that \( P \)'s type is \( R \), where

\[
R = \begin{cases} 
I & \text{if } P \text{ is "informed" (i.e., knows } \theta) \\
U & \text{if } P \text{ is "uninformed" (i.e., does not know } \theta) 
\end{cases},
\]

\(^{12}\)Naturally, the special case with \( \gamma = 0 \) corresponds to a Bernoulli prior, which is of potential separate interest as settings with two possible outcomes are rather common in the literature. Importantly, this special case would not capture many realistic enviroments as it eliminates the possibility that there has not been any noticeable event affecting the fundamental value of the asset. Allowing for this possibility turns out to be important for some of our main findings.
where that probability that $P$ is informed is $\Pr(I) = \eta$, where $0 < \eta < 1$. While the Market knows $\eta$, it does not know $P$'s type ($R$). We further assume that $R$ is independent of the fundamental and noise trading. This is crucial as $P$ knowing his own type is a source of his, de facto, private information should he remain uninformed about $\theta$.\(^{13}\)

While, we give an illustrative example where $P$ is the only trader able to learn the fundamental value, we consider that setting to be a special case that explains why market impact and uncertainty about $P$'s type are sufficient for generating rational quantitative trading (see Section 5.2), our baseline setting incorporates the idea that there are other sophisticated funds specialized on trading based on fundamental information, and these funds have a relative advance when pursuing fundamentals based strategy compared to $P$. To capture this idea, we assume that in addition to $P$, there is a large neutral trader, we call $K$, who is always informed about the fundamental, rational, strategic and equivalent to the insider in Kyle (1985).

To stack the cards against $P$, we assume that $K$ knows $P$'s type. This assumption highlights that what is crucial for our argument is that the Market does not know $P$'s type, irrespective to other sophisticated traders knowing it or not. In our main setting we also assume that fundamentals based traders are "slower" than quantitative traders (whose trades simply react to observable outcomes), i.e, $K$, and informed $P$, can only trade at date 1, while uninformed $P$ can trade any time. The "speed" assumption, albeit realistic, appears to be unimportant for our qualitative findings (see Section 5.1)

For now, we assume that traders informed about the fundamental can only trade at date 1 as this allows to abstract from the additional effects that arise from the well-known informed traders’ incentives to split their orders. This assumption delivers our main results and intuition in the most transparent manner. We relax this assumption in Section 5.1 and allow both $K$ and $P$ to trade also at date 2 when informed. All our results are qualitatively unchanged.

Obviously $P$ knows his own type. We also assume that $K$ knows $P$'s type with certainty, but this is not crucial for our results.\(^{15}\)

Strategic traders $K$ and $P$ know that their market orders have a non-negligible impact on prices. Denote the market order by trader $J \in \{K, P\}$ in state $R \in \{I, U\}$ at date $t \in \{1, 2\}$.

\(^{13}\)We also assume that $\eta$ is not arbitrarily close to one (see Appendix B). This realistically avoids a situation in which $P$ is uniformed, but the Market is convinced that he is informed and is very reluctant to update his beliefs about $P$'s type.

\(^{14}\)An alternative interpretation that would be captured by exactly the same specification is the one where outside investors (i.e, the Market) do not know for sure whether a large fund is pursuing price-contingent or fundamental based strategy regarding a particular asset at a given point in time. Strategies pursued by sophisticated Hedge Funds, are rarely concentrated to only one approach, and these institutions are known to be highly secretive about their portfolio.

\(^{15}\)This assumption makes it ex ante harder for $P$ to develop an information advantage as $P$ never has more information than $K$. It also helps to highlight that for rational price-contingent trading to emerge as optimal, it is important that typical market participants (the Market) do not know large trader’s types with certainty, even if (some) other sophisticated traders do.
as $h_t^{RJ}$. If both traders are informed, $R = I$, then trader $J$ solves

$$\max_{h_t^{IJ}} \pi_t^{IJ} = \mathbb{E} [h_t^{IJ} (\theta - p_1) | \theta, I],$$

(3)

where $J \in \{K, P\}$. If only $K$ is informed about the fundamental, $R = U$, then $K$ solves

$$\max_{h_t^{UK}} \pi_t^{UK} = \mathbb{E} [h_t^{UK} (\theta - p_1) | \theta, U],$$

(4)

and $P$ solves\textsuperscript{16}

$$\max_{h_t^{UP}} \pi_t^{UP} = \mathbb{E} [h_t^{UP} (\theta - p_1) + h_t^{UP} (\theta - p_2) | U]\quad (5)$$

$$\max_{h_t^{UP}} \pi_t^{UP} = \mathbb{E} [h_t^{UP} (\theta - p_2) | y_1, U].$$

The total order flow is

$$y_1 = h_t^{RK} + h_t^{RP} + s_1 \text{ for } R = \{I, U\}$$

(6)

$$y_2 = \begin{cases} 
  s_2 & \text{if } R = I \\
  h_t^{UP} + s_2 & \text{if } R = U,
\end{cases}$$

where $s_t$ is date $t \in \{1, 2\}$ demand by noise traders.\textsuperscript{17} We assume that noise traders demand is drawn from a normal distribution with mean zero and variance $\sigma_s^2$, serially uncorrelated and independent of fundamental and state. We denote the probability density function with $\varphi_s(s_t)$ for $t \in \{1, 2\}$. While being a standard assumption in this literature, there is also a natural economic argument for this choice of distribution. In fact, we can interpret noise trading as the total demand by a large number of small traders who trade for idiosyncratic reasons unrelated to the fundamental (such as liquidity shocks, private values etc.). In such case, the normality of the distribution of noise trading follows directly from the central limit theorem.

Technically, a useful property of the normal distribution is that it is strictly log-concave, allowing us to use some general properties of log-concave functions.\textsuperscript{18} Log-concavity of noise trading also guarantees some desirable properties of the model, and we discuss generality further.

\textsuperscript{16}We condition $P$‘s expectation on the order flow (instead of the price or both), because date 1 order flow is always at least as informative as date 1 price. If price is monotonic in the order flow, then the two have the same information content.

\textsuperscript{17}As usual, the presence of noise traders is needed to avoid the Grossman and Stiglitz’s (1980) paradox about the impossibility of a fully revealing price in equilibrium.

\textsuperscript{18}A function $f(x)$ (where $x$ is a $n$-component vector) is log-concave if $\ln (f(x))$ is concave. In the univariate and differentiable case, the following are equivalent: 1) $\partial^2 \ln (f(x)) / \partial x \partial x < 0$, 2) $f'(x) / f(x)$ is decreasing in $x$, 3) $f''(x) f(x) - (f'(x))^2 < 0$. 

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Further, while in settings with known types the total order flow provides noisy information about the fundamental, $\theta$, things are more complex in ours setting. In our model the total order flow reveals noisy information about both the fundamental and $P$’s type. In our further derivations we benefit from the law of total expectations, as the Market efficiency condition (2) can be expanded as

\begin{align*}
p_1 &= \mathbb{E}[\theta|y_1] = Q_1 \mathbb{E}[\theta|y_1, I] + (1 - Q_1) \mathbb{E}[\theta|y_1, U] \\
p_2 &= \mathbb{E}[\theta|y_1, y_2] = Q_2 \mathbb{E}[\theta|y_1, y_2, I] + (1 - Q_2) \mathbb{E}[\theta|y_1, y_2, U],
\end{align*}

where $Q_1 \equiv \text{Pr}(I|y_1)$ and $Q_2 \equiv \text{Pr}(I|y_1, y_2)$ are the probabilities of $P$ being informed conditional on the observed total order flows. We also use notation $p_1(y_1), p_2(y_2), Q_1(y_1)$ and $Q_2(y_2)$ to express these prices and probabilities as functions of contemporaneous order flows.

To summarize the setup, the timing of events is as follows:

- **date 0** - Nature draws $R \in \{I, U\}$ and $\theta$. $K$ and $P$ learn $R$. If $R = I$, then both $K$ and $P$ learn $\theta$. If $R = U$, only $K$ learns $\theta$.

- **date 1** - $K$, $P$, and noise traders submit market orders before knowing the price. The Market observes total order flow and sets the price $p_1$ based on the market efficiency condition (2).

- **date 2** - Noise traders submit market orders. If $R = U$, then $P$ also submits a market order before knowing the price. The Market observes total order flow and sets the price $p_2$ based on the market efficiency condition (2).

- **date 3** - uncertainty resolves and $P$ and $K$ consume profits given the realization of $\theta$.

As standard in the literature we focus on equilibria in pure strategies by $K$ and $P$.

### 4 Results

#### 4.1 Importance of uncertainty about types

We start by highlighting why uncertainty about $P$’s type is crucial by considering a benchmark case where $P$ is known to be uninformed with certainty, i.e., $\text{Pr}(I) = \eta = 0$.

\footnote{Many other well known distributions are log-concave and symmetric. Notable examples include the beta (with parameters $\alpha = \beta > 1$) and truncated normal. See Bagnoli and Bergstrom (2005) for an overview and further examples of log-concave densities.}
Proposition 1 If $\eta = 0$: $E[\theta|y_1, U] = E[\theta|y_1] = p_1 = p_2$; $P$ can never earn positive expected profits from trading.

Proof. See Appendix A.

Proposition 1 shows that in the special case in which $\eta = 0$, our model supports the Easley and O’Hara (1991) argument against the possibility of uninformed traders profiting from rational price-contingent trading. Indeed, in such a case uninformed $P$ cannot earn positive profits, because prices already reflect all information that an uninformed $P$ could have. To be more specific, at date 1 uninformed $P$’s best guess of the fundamental is the prior mean and any non-zero quantity traded would move prices and lead to an expected loss from trading at date 1 (this is because the price is an increasing function of the order flow). Therefore, it is optimal for $P$ not to trade, which yields a zero profit. At date 2, uninformed $P$ does learn new information from the order flow, but the information he obtains is exactly the same as the information that the Market has already obtained, $E[\theta|y_1, U] = E[\theta|y_1] = p_1$. Because prices will not change between date 1 and 2, he cannot earn positive profits from trading.\footnote{Note that in our setting the Market knows that date 2 order flow is not informative and thus prices will not change, $p_2 = p_1$. As a result $P$ would earn zero profits from any quantity traded. In this setting $h_{2}^{UP} = 0$, but also any other constant quantity traded by uninformed $P$ at date 2 can be sustained as an equilibrium. However, the latter only holds because of risk neutrality of $P$. Only the equilibrium with $h_{2}^{UP} = 0$ could be sustained with even a very small degree of risk aversion.} It should also be noted that sufficient conditions for Proposition 1 to hold are that the prior is symmetric, and noise trading is log-concave and symmetric. The latter is important as it guarantees that the price is an increasing function of the order flow.\footnote{Log-concavity of noise trading implies that the likelihood function $f(y_1|\theta, U)$ has the monotone likelihood ratio property (MLRP). As known from Milgrom (1981) the MLRP guarantees that the expected value of the fundamental is increasing in the order flow due to first-order stochastic dominance. See Appendix A.}

4.2 Date 2 problem

Assume $\eta > 0$ and notice that all date 1 quantities, $E[\theta|y_1, R]$, $p_1$ and $Q_1 = \text{Pr}(I|y_1)$ depend on $y_1$ and are known to $P$ and the Market before date 2. Date 2 problem is only interesting if there is a difference between $P$’s and the Markets expectations about the fundamental ($E[\theta|y_1, U] \neq p_1$ or equivalently $E[\theta|y_1, U] \neq E[\theta|y_1, I]$) and the Market has not fully learned $P$’s type ($Q_1 > 0$). For now, we conjecture that this is the case. We verify it later when analyzing the date 1 problem.

As there is no informed trading at date 2, it holds that conditional on a given state $R \in \{I, U\}$ and $y_1$, the date 2 order flow only depends on $\theta$ through $y_1$, which is already incorporated in prices and expectations and therefore $E[\theta|y_1, y_2, R] = E[\theta|y_1, R]$. Using (7) we obtain

$$p_2 = p_1 + \frac{(Q_1 - Q_2)}{Q_1} (E[\theta|y_1, U] - p_1).$$

\footnote{\textsuperscript{20}Note that in our setting the Market knows that date 2 order flow is not informative and thus prices will not change, $p_2 = p_1$. As a result $P$ would earn zero profits from any quantity traded. In this setting $h_{2}^{UP} = 0$, but also any other constant quantity traded by uninformed $P$ at date 2 can be sustained as an equilibrium. However, the latter only holds because of risk neutrality of $P$. Only the equilibrium with $h_{2}^{UP} = 0$ could be sustained with even a very small degree of risk aversion.}
Clearly prices change between date 1 and 2 only if $Q_2 \neq Q_1$, which implies that they only change if the Market updates its beliefs about $P$'s type after observing date 2 order flow. Provided that the true state is $R = U$, the Market updates in the "correct" direction if $Q_2 < Q_1$. In such case prices increase (decrease) if $\mathbb{E} [\theta | y_1, U] > (<) p_1$. Using (8), we can restate $P$’s problem (5) as

$$
\max_{h_2^{UP}} \pi_2^{UP} = h_2^{UP} \mathbb{E} [Q_2 | y_1, U] \left( \frac{\mathbb{E} [\theta | y_1, U] - p_1}{Q_1} \right) = h_2^{UP} \left( \int_{-\infty}^{\infty} Q_2 (h_2^{UP} + s_2) \varphi_s(s_2) \right) \frac{(\mathbb{E} [\theta | y_1, U] - p_1)}{Q_1}. \tag{9}
$$

We can make some immediate observations. Suppose that $\mathbb{E} [\theta | y_1, U] > p_1$, i.e., uninformed $P$ expects the fundamental to be higher than date 1 price. On the one hand, $P$ can profit from trading any positive quantity. Ignoring the effect of his trade on $Q_2$ would make him to want to buy an infinitely large quantity of the asset at date 2. On the other hand, the term $\mathbb{E} [Q_2 | y_1, U]$ captures the expected updating of $P$’s type by the Market. Because $Q_2$ depends on date 2 order flow, $P$ knows that his trade will affect the Markets' beliefs about his type. Since these beliefs directly affect $p_2$, one would expect the traditional trade-off between transaction size and information disclosure to be present, but to establish this formally we need to investigate further the properties of $Q_2$.

As we focus on pure strategies and uninformed $P$’s trading strategy, we can see that the beliefs of the Market are characterized by the quantity it expects $P$ to trade. Thus, consider that the market expects $P$ to trade some quantity $\bar{h}_2$, whereby $\bar{h}_2$ can take any value in $\mathbb{R}$. Then, from (6) $y_2 = \bar{h}_2 + s_2$ if $R = U$ and $y_2 = s_2$ if $R = I$, we can derive $Q_2$ by using Bayes’ rule, as

$$
Q_2 = \frac{Q_1 f (y_2 | y_1, I)}{Q_1 f (y_2 | y_1, I) + (1-Q_1) f (y_2 | y_1, U)} = \frac{Q_1}{Q_1 + (1-Q_1) r(y_2)}, \tag{10}
$$

where

$$
r(y_2) = \frac{\varphi_s(y_2 - \bar{h}_2)}{\varphi_s(y_2)} \tag{11}
$$

is the likelihood ratio and we used the fact that conditional on the state $R$ the date 2 order flow is normally distributed with density $\varphi_s(.)$.

**Lemma 1** The following properties hold for $Q_2 = \Pr (I | y_1, y_2)$

1. $Q_2$ is decreasing (increasing) in $y_2$ for any $\bar{h}_2 > (<) 0$.  

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22Naturally, in equilibrium it must hold that the Market’s expectations and $P$’s optimal trade are internally consistent, i.e., $h_2^{UP} = \bar{h}_2$. 

13
2. If \( h_2 > 0 \) then \( Q_2 > (\cdot) Q_1 \) for any \( y_2 < (\cdot) \frac{h_2}{2} \). If \( h_2 < 0 \) then \( Q_2 > (\cdot) Q_1 \) for any \( y_2 > (\cdot) \frac{h_2}{2} \).

3. \( Q_2 (0) = Q_1 \cdot \left( Q_1 + (1 - Q_1) \frac{\varphi_s(h_2)}{\varphi_s(0)} \right)^{-1} = Q_1 \cdot \left( Q_1 + (1 - Q_1) \exp \left( -\frac{(h_2)^2}{2\sigma^2} \right) \right)^{-1} \),

4. If \( h_2 > (\cdot) 0 \) then \( \lim_{y_2 \to -\infty} Q_2(y_2) = 0 (= 1) \) and \( \lim_{y_2 \to -\infty} Q_2(y_2) = 1 (= 0) \).

5. \( Q_2(y_2) \) is a log-concave function.

**Proof.** Part 1: Differentiating and simplifying we obtain \( \partial Q_2 / \partial y_2 = -Q_2^2 (1/Q_1 - 1) r'(y_2) \). Lemma A.1 in Appendix A shows that log-concavity of \( \varphi_s \) implies the monotone likelihood ratio property (MLRP), i.e., \( r'(y_2) > (\cdot) 0 \) for any \( h_2 > (\cdot) 0 \). This is because \( \varphi_s \left( y_2 - h_2 \right) / \varphi_s \left( y_2 \right) > (\cdot) \varphi_s \left( y_2 - h_2 \right) / \varphi_s \left( y_2 \right) \) for any \( y_2 > 0 \) and \( h_2 > (\cdot) 0 \). Parts 2-4 are straightforward from (10), (11) and the expression for the normal density. Part 5: Taking logs and differentiating, we obtain that \( \frac{\partial^2 \ln(Q_2)}{\partial y_2 \partial y_2} = -\frac{(1 - Q_1)^2 \left[ (1/Q_1 - 1)r''(y_2) + r'(y_2)r(y_2) - r'(y_2) \right]^2}{(Q_1 + (1 - Q_1)r(y_2))^2} \). It is sufficient to show that the likelihood ratio (11) is (at least weakly) log-convex. Indeed from using the normal density in (11) we find that \( \ln \left( r(y_2) \right) \) is linear in \( y_2 \) and therefore weakly convex.

Part 1 of Lemma 1 implies that the Market updates its beliefs about \( P \)'s type (the state \( R \)) in a "sensible" manner. For example, if the Market believes that trader \( P \) in state \( R = U \) trades a finite and positive quantity, then observing a higher order flow always leads the Market to assign a lower probability on \( P \) being informed. This also confirms that \( P \) indeed faces a meaningful trade-off - the presence of a profit opportunity gives \( P \) incentives to trade, but trading too aggressively will reduce \( P \)'s expected profit as he expects the Market to assign a higher probability on him being uninformed and to adjust the price accordingly. It is worth emphasizing that such realistic tradeoff is always guaranteed because the likelihood ratio (11) is monotone (for a similar argument, see also Milgrom (1981)).

While Bayesian updating itself guarantees that the Market updates its beliefs in the correct direction on average, we can see from part 2 of Lemma 1 that depending on the realized date 2 order flow, the Market can update the probability that \( P \) is informed, \( Q_2 \), in the "correct" or "incorrect" direction. This is because the total order flow includes a random noise trading component. Namely, if the realized order flow is relatively small (i.e., less than half of the volume that the Market expects uninformed \( P \) to trade) or has an opposite sign to \( P \)'s expected trade, then the Market updates in the "correct" direction if the state is \( R = I \) and in the "incorrect"

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23 \( r(y_2) \) is log-convex if \( \ln \left( r(y_2) \right) \) is convex. Equivalently, it must hold that \( r''(y_2) r(y_2) - (r'(y_2))^2 \geq 0 \). This, together with \( r'(y_2) > 0 \) also implies that \( r''(y_2) > 0 \).

24 The monotone likelihood ratio property holds for the whole family of log-concave distributions, to which the normal belongs (see Lemma A.1 in Appendix A).
direction if the state is $R = U$. It is also immediate from parts 2-4 of Lemma 1 that the Market never learns $P$’s type perfectly for finite order flows. Therefore, despite some learning about $P$’s type, it is clear from (9) that $P$ would always earn positive profits from trading any finite quantity that has the same sign as the difference $\mathbb{E}[\theta | y_1, U] - p_1$.

Part 4 of Lemma 1 confirms that the Market’s learning about $P$’s type is unbounded. This is necessary to guarantee that $P$ has an incentive to trade a finite amount.\footnote{Suppose instead that learning was bounded (i.e., $Q_2$ was such that $0 < |Q_2| < 1$ even at the limit when $y_1 \to \pm\infty$) and consider a candidate equilibrium where $P$ trades a finite amount. It is easy to see that this cannot be an equilibrium as $P$ would benefit from deviating to trade an infinite quantity. See also Section 5.}

While the previous analysis gives some confidence that it may be optimal for uninformed $P$ to trade a finite quantity in equilibrium, it is not yet clear whether $P$’s problem has a unique (interior) solution. Namely, from (9) $P$’s expected profit involves an integral over a non-trivial function $Q_2(\cdot)$ that depends on uninformed $P$’s demand and is always positive for $h_2^{UP} > (\cdot) > 0$ provided that $(\mathbb{E}[\theta | y_1, U] - p_1) > (\cdot) > 0$.

**Lemma 2** If $(\mathbb{E}[\theta | y_1, U] - p_1) > (\cdot) > 0$ then uninformed $P$’s expected profit (9) is strictly log-concave in $h_2^{UP} > (\cdot) > 0$.

**Proof.** Assume without loss of generality that $(\mathbb{E}[\theta | y_1, U] - p_1) > 0$ and $h_2^{UP} > 0$. Taking logs of (9), we obtain

$$\ln \left( \pi_2^{UP} \right) = \ln \left( h_2^{UP} \right) + \ln \left( \mathbb{E}[Q_2 | y_1, U] \right) + \ln \left( \mathbb{E}[\theta | y_1, U] - p_1 \right) - \ln (Q_1)$$

and

$$\partial^2 \ln \left( \pi_2^{UP} \right) / \partial h_2^{UP} \partial h_2^{UP} = - \left( h_2^{UP} \right)^{-2} \partial^2 \ln \left( \mathbb{E}[Q_2 | y_1, U] \right) / \partial h_2^{UP} \partial h_2^{UP},$$

which is negative if $\mathbb{E}[Q_2 | y_1, U]$ is log-concave. By change of variables $y_2 = h_2^{UP} + s_2$, we can express

$$\mathbb{E}[Q_2 | y_1, U] = \int_{-\infty}^{\infty} Q_2(y_2) \varphi_s(y_2 - h_2^{UP}) dy_2. \tag{12}$$

Using Theorem 6 of Prékopa (1973), restated as Theorem A.3 in Appendix A, a sufficient condition for (12) to be log-concave is that the function $Q_2(y_2) \varphi_s(y_2 - h_2^{UP})$ is log-concave in $h_2^{UP}$ and $y_2$. We can then derive

$$\partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial h_2^{UP} \partial h_2^{UP} = -\sigma_s^{-2} \partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial h_2^{UP} \partial y_2 = \partial^2 \ln \varphi_s(y_2 - h_2^{UP}) / \partial y_2 \partial h_2^{UP} = \sigma_s^{-2}. \tag{13}$$

As by part 5 of Lemma 1 $\partial^2 \ln (Q_2(y_2)) / \partial y_2 \partial y_2 < 0$, it is immediate that the Hessian\footnote{The Hessian is $\begin{vmatrix} -\sigma_s^{-2} & \sigma_s^{-2} \\ \sigma_s^{-2} & \partial^2 \ln (Q_2(y_2)) / \partial y_2 \partial y_2 - \sigma_s^{-2} \end{vmatrix}$.} is negative definite, and therefore $\mathbb{E}[Q_2 | y_1, U]$ and $\pi_2^{UP}$ are log-concave. The proof for the case $(\mathbb{E}[\theta | y_1, U] - p_1) < 0$ and $h_2^{UP} < 0$ is similar.\blacksquare

Since any univariate log-concave function is also quasi-concave with a unique maximum, we can now state our main result\footnote{An alternative proof of quasi-concavity is to require that the negative of the first derivative of the objective function is single crossing in $h_2^{UP}$. The sufficient conditions for single crossing under uncertainty have also been explored by Athey (2002) and Quah and Strulovici (2012). In both cases we can derive that log-concavity of $Q_2(\cdot)$ and $\varphi_s(\cdot)$ are sufficient for strict single crossing.}:
Theorem 1 Uninformed $P$’s unique equilibrium strategy is to demand a finite amount

\[ h_{UP}^2 = \bar{h}_2 = \begin{cases} 
\sigma_s \kappa & \text{if } \mathbb{E} [\theta | y_1, U] - p_1 > 0 \\
-\sigma_s \kappa & \text{if } \mathbb{E} [\theta | y_1, U] - p_1 < 0
\end{cases}, \tag{13} \]

where $\kappa > 1$ for any $Q_1 \in (0, 1)$ and $\kappa$ depends on $Q_1$ only.

Proof. It is immediate from (9) that $h_{UP}^2 < (>) 0$ when $\mathbb{E} [\theta | y_1, U] - p_1 > (>) 0$ cannot be optimal as it leads to strictly negative profits. By Lemma 2, the uninformed $P$’s problem then has a unique maximum at a non-negative $h_{UP}^2$. Therefore it is sufficient to look at the first order condition only and then impose that in equilibrium beliefs must be consistent with the optimal strategy $h_{UP}^2 = \bar{h}_2$. The first order condition, the expression for $\kappa$ and the proofs of the statements that $\kappa > 0$ and only depends on $Q_1$ are in Appendix B. ■

Because $\kappa$ only depends on $Q_1$, it is most illustrative to present the solution\textsuperscript{28} on a graph (see Figure 1). We find that whenever $\mathbb{E} [\theta | y_1, U] \neq p_1 = \mathbb{E} [\theta | y_1]$, it is generally optimal for uninformed $P$ to trade at date 2. The volume traded by $P$ is proportional to the standard deviation of noise trading and is increasing in $Q_1$. Both effects are intuitive. When the order flow is more noisy (high $\sigma_s$), it is harder for the Market to update its beliefs about the state and it is less costly for uninformed $P$ to trade more aggressively. Because the Market’s posterior belief that the state is $R = I$, $Q_2 = \Pr (I | y_1, y_2)$, is increasing in $Q_1$ (its belief about $P$’s type before date 2 trading), it is clear that a higher $Q_1$ also makes it less costly for an uninformed $P$ to trade.

\textsuperscript{28}It is relatively easy to show that if $Q_1 = 0.5$, then $\kappa = \sqrt{2}$. The other values are derived using numerical integration.
trade more aggressively as the Market is learning about his type more slowly. Overall $P$ trades a finite quantity as he faces a trade-off between profiting from his superior information and revealing his type too much. This trade-off is fundamentally similar to the one in Kyle (1985), however differently from that setting $P$’s private information is not about the fundamental directly, but about his impact or lack of impact on date 1 price. Theorem 1 is in stark contrast to Proposition 1, because Theorem 1 demonstrates that price-contingent trading is profitable.

Before analyzing the date 1 problem, we can now formally define the two traditional types of price-contingent strategies within the context of our model.

**Definition** $P$’s date 2 strategy is called\(^{29}\)
- trend-following (momentum) for some $y_1$ if $y_1 > 0$ and $h_2^{UP} > 0$, or $y_1 < 0$ and $h_2^{UP} < 0$
- contrarian for some $y_1$ if $y_1 > 0$ and $h_2^{UP} < 0$, or $y_1 < 0$ and $h_2^{UP} > 0$.

Provided the price is monotonic in the order flow (as in all our examples below), it would be equivalent to define $P$’s strategy through date 1 price.

### 4.3 Date 1 problem

So far, we have shown that whenever the date 1 price differs from uninformed $P$’s expectations of the fundamental (drawn from any prior distribution) $P$’s optimal trading strategy is price-contingent. In this section we study the direction of $P$’s trading and focus on the three-point prior, (1). Given the symmetry of the distribution and the fact that $P$ has no superior information at date 1, it is natural to expect that in state $R = U$, the informed trader $K$’s optimal demand is some real number $\tilde{g}_U$ if $\theta = \tilde{\theta}$, zero if $\theta = 0$ and $h_1^{UK} = -\tilde{g}_U$ if $\theta = -\tilde{\theta}$; and uninformed trader $P$ does not trade. In state $R = I$, the total demand by informed traders $K$ and $P$ is a real number $\tilde{g}_I$ if $\theta = \tilde{\theta}$, zero if $\theta = 0$ and $-\tilde{g}_I < 0$ if $\theta = -\tilde{\theta}$. Given these beliefs, we can derive the expressions and main properties of $\mathbb{E}[\theta|y_1, R]$, the price, and $Q_1$ as described by the following lemma.

**Lemma 3** For the equilibrium price and conditional expectations of the fundamental, it holds that

1. The price is given by

$$p_1(y_1) = \frac{\partial M_{n}(y_1) - M_{p}(y_1)}{M_{n}(y_1) + M_{p}(y_1)}$$

\(^{29}\)The words "momentum" and "contrarian" only refer to $P$’s strategy. They should not be confused with positive and negative autocorrelation in returns. By the assumption of efficient markets (2), there is zero autocorrelation by construction. See also Section 4.4.
where $M_n(y_1) \equiv \frac{1-\gamma}{2} (\eta \phi_s(y_1 - \bar{g}_I) + (1 - \eta) \phi_s(y_1 - \bar{g}_U) + \frac{\gamma}{1-\gamma} \phi_s(y_1))$
and $M_p(y_1) \equiv \frac{1-\gamma}{2} (\eta \phi_s(y_1 + \bar{g}_I) + (1 - \eta) \phi_s(y_1 + \bar{g}_U) + \frac{\gamma}{1-\gamma} \phi_s(y_1))$;

2. The conditional expectation of the fundamental is
\[
\mathbb{E}[\theta|y_1, R] = \bar{\theta} \frac{\phi_s(y_1 - \bar{g}_R) - \phi_s(y_1 + \bar{g}_R)}{\phi_s(y_1 - \bar{g}_R) + \phi_s(y_1 + \bar{g}_R) + \frac{2\gamma}{1-\gamma} \phi_s(y_1)};
\] (15)

3. The updated probability of state $R = I$ is
\[
Q_1(y_1) = \Pr(I|y_1) = \frac{\eta (\phi_s(y_1 - \bar{g}_I) \frac{1-\gamma}{2} + \phi_s(y_1) \gamma + \phi_s(y_1 + \bar{g}_I) \frac{1-\gamma}{2})}{M_n(y_1) + M_p(y_1)};
\] (16)

4. The price is increasing in the order flow, i.e., $p_1'(y_1) > 0$;

5. The price is symmetric around zero, i.e., $p_1(y_1) = -p_1(-y_1)$;

6. It holds that $\lim_{y_1 \to -\infty} p_1(y_1) = \bar{\theta}$ and $\lim_{y_1 \to -\infty} p_1(y_1) = -\bar{\theta}$;

7. $\bar{\theta} - p_1(y_1) > 0$ for all (finite) $y_1$;

Proof. See Appendix B. ■

Lemma 3 confirms some reasonable and desirable properties of date 1 price, e.g., the price is increasing in the order flow, symmetric around zero and always between $-\bar{\theta}$ and $\bar{\theta}$. If the state is $R = U$, then the expected profit (4) of $K$ can be written as
\[
\pi_1^{UK} = h_1^{UK} \int_{-\infty}^{\infty} \left( \theta - p(h_1^{UK} + s_1) \right) \phi_s(s_1) \, ds_1.
\] (17)

If the state is $R = I$ then the expected profit (3) of trader $J \in \{K, P\}$ can be written as
\[
\pi_1^{IJ} = h_1^{IJ} \int_{-\infty}^{\infty} \left( \theta - p(h_1^{IK} + h_1^{IP} + s_1) \right) \phi_s(s_1) \, ds_1.
\] (18)

As in Section 4.2, we cannot be immediately sure if the informed trader’s expected profit has a unique maximum in own demand. The reason is that, unlike in the date 2 problem, we cannot be sure that the traders’ objective function is log-concave. Namely, one sufficient condition for this would be that $\theta - p(y_1)$ is log-concave and this only holds for some parameters. However log-concavity is only a sufficient, but not a necessary condition for a unique maximum. What we need is that the trader’s profit is quasi-concave in own demand, i.e., that $-\frac{\partial \pi_1^{RJ}}{\partial s_1}$ is a single crossing function of $h_1^{RJ}$. In Appendix B we prove that this is always the case for $\theta = 0$ and
we identify some conditions where this is also the case for $\theta = \{-\tilde{\theta}, \tilde{\theta}\}$. Intuitively, a sufficient condition is that the expected slope of the price at high order flows is not too flat relative to that same slope at low order flows, or, at a minimum, the slope of the price does not decrease too rapidly in expectation. Provided that the informed trader’s problem has a unique maximum in own demand, we can state

**Proposition 2** There is a pure strategy equilibrium at date 1, where the following holds:

1. Informed traders’ demand is given by
   
   $$h_{1K}^U = \begin{cases} 
   \tilde{g}_U = \sigma_s\mu_U & \text{if } \theta = \tilde{\theta} \\
   0 & \text{if } \theta = 0 \\
   -\tilde{g}_U = -\sigma_s\mu_U & \text{if } \theta = -\tilde{\theta} 
   \end{cases}$$
   
   where $h_{1K}^I = h_{1P}^I = \begin{cases} 
   \frac{\tilde{g}_I}{2} = \sigma_s\mu_I & \text{if } \theta = \tilde{\theta} \\
   0 & \text{if } \theta = 0 \\
   -\frac{\tilde{g}_I}{2} = -\sigma_s\mu_I & \text{if } \theta = -\tilde{\theta} 
   \end{cases}$, and $\mu_U$ and $\mu_I$ only depend on $\eta$ and $\gamma$.

2. Total demand by informed traders in the event of news ($\theta = \tilde{\theta}$ or $\theta = -\tilde{\theta}$) is always higher in absolute value in state $R = I$ compared to state $R = U$, i.e., $g_I > g_U$ (equivalently $\mu_I > \mu_U$).

3. In state $R = U$, the uninformed trader $P$ does not trade at date 1, i.e., $h_{1P}^U = 0$.

**Proof.** See Appendix B. □

Proposition 2 states some intuitive properties of date 1 equilibrium. First, informed traders face the standard trade-off as in Kyle (1985) and Holden and Subrahmanyam (1992). On the one hand, whenever they have private information that indicates $\theta \neq 0$ they earn positive expected profits from trading, so they have an incentive to trade a high volume. On the other hand, they know that due to market impact, trading a high volume reveals information about the fundamental (and also—less importantly for these traders—about the state $R$) to the Market. Therefore, they trade a finite amount and the price will not adjust immediately to equal the fundamental value.

The trading volume is always proportional to the standard deviation of noise trading. This is because informed traders benefit on average at the expense of noise traders and more noise allows them to hide private information more easily. Because the equilibrium price is proportional to the fundamental (see (14)), the magnitude of the fundamental value does not affect the informed trader’s optimal strategy, but clearly profits are higher if $\tilde{\theta}$ is higher. By Proposition 2 we know
that the optimal strategy only depends on two parameters that are between 0 and 1. Figure 2 illustrates these dependences by plotting on the vertical axis $\mu_U$ and $\mu_I$ against $\eta$ (on the left panel, assuming $\gamma = 0$) and against $\gamma$ (on the right panel, assuming $\eta = 0.5$). These plots are qualitatively similar for different values of $\eta$ and $\gamma$. First, if the prior probability of the state with two informed traders $R = I$ (i.e., $\Pr(I) = \eta$) is higher, then the informed traders are trading less aggressively. This is because $\bar{g}_I > \bar{g}_U$ and the Market expects more informed trading and is updating its beliefs faster. This in turn increases the informed traders’ market impact and reduces their willingness to trade aggressively. Second, if the prior probability of "no news" ($\gamma$) is higher, the informed traders trade more aggressively whenever they observe $\theta \neq 0$. This is because by Bayes’ rule the Market is relatively reluctant to update its beliefs toward the more extreme realizations of the fundamental. This reluctance reduces the market impact of the informed traders and gives them incentives to trade more.

The most important part of Proposition 2 is part 3 which states that the total order flow by informed traders is different in the two states, as we can see when comparing the expressions for $p_1$, $\mathbb{E}[\theta|y_1,I]$ and $\mathbb{E}[\theta|y_1,U]$. If the state is $R = U$, then $P$ again obtains superior information exactly because he knows that he did not trade and we can explore the direction of his trade at date 2.

### 4.3.1 Direction of price contingent trading

We start by examining the two-point prior, i.e., for now we set $\gamma = 0$.

\footnote{While the solution does not generally have a nice analytical expression, we can find the benchmark cases corresponding to Kyle (1985) and to the two-traders version of Holden and Subrahmanyam (1992) by extending these to a different prior distribution. Namely if $\gamma = 0$ and $\eta = 0$, it holds that $\mu_U = 1$; if $\gamma = 0$ and $\eta = 1$, it holds that $\mu_I = \sqrt{2}$.}
**Proposition 3** When the prior distribution of the fundamental is a symmetric two-point distribution, it holds that

\[ \mathbb{E}[\theta | y_1, U] < (>) \mathbb{E}[\theta | y_1, I], \text{ for any } y_1 > (0). \]

Whenever \( 0 < \eta < 1 \), the optimal strategy of \( P \) in state \( R = U \) at date 2 is contrarian.

**Proof.** See Appendix B. ■

With a discrete two-point distribution we find that if the true state is \( R = U \), i.e., \( P \) is uninformed, then \( P \)’s optimal strategy at date 2 is always contrarian. Note that when assuming a two-point prior we are focusing on an enviroment where any "news" about the fundamental is either "good" or "bad" and the Market always expects informed traders to trade. Any positive order flow is more likely to be associated with \( \theta = \bar{\theta} \) compared to \( \theta = -\bar{\theta} \). Furthermore, by part 2 of Proposition 2, we know that two informed traders would always trade a larger quantity in absolute value than one informed trader and therefore whenever the order flow is positive it holds that \( \Pr(\bar{\theta}|y_1, I) > \Pr(\bar{\theta}|y_1, U) \) and \( \Pr(-\bar{\theta}|y_1, I) < \Pr(-\bar{\theta}|y_1, U) \).\(^{31}\) As the Market prices the asset considering that both states are possible, it tends to overprice the asset whenever the order flow is positive and the true state is \( R = U \). Effectively the Market tends to underestimate the fact that it may have been a positive noise trading shock rather than the demand of the informed traders that generated a positive order flow.

The above conclusion is specific to a two-point distribution with no mass in the center. With a three-point prior, we can establish some more general properties about the direction of price-contingent trading.

**Proposition 4** When the prior distribution of the fundamental is a symmetric three-point distribution and \( R = U \), then for any \( 0 < \eta < 1 \) the following conditions hold for large and very small order flows

1. For order flows \( |y_1| \geq \left| \frac{\bar{\mu} + \bar{\nu}}{2} \right| \), \( P \) always pursues a contrarian strategy at date 2.

2. For order flows \( y_1 \) in the neighborhood of zero (i.e., for \( y_1 \to 0 \)), \( P \) pursues a trend-following strategy at date 2 iff the following condition holds.

\[
\frac{1 + \exp \left( \frac{\mu_I^2}{2} \right)^{\gamma / (1-\gamma)} \mu_I}{1 + \exp \left( \frac{\mu_U^2}{2} \right)^{\gamma / (1-\gamma)} \mu_U} > \frac{\mu_I}{\mu_U}
\]

\(^{31}\)This is straightforward to verify using part 3 of lemma 3 and the properties of log-concave functions in Appendix A.
3. Provided that (19) holds, there exists a threshold order flow in the interval of \((0, \frac{\bar{g}_U + \bar{g}_I}{2})\) below which \(P\)’s optimal strategy is trend-following and above which it is contrarian.

Proof. See Appendix B. ■

Proposition 4 shows that with a three-point prior both trend-following and contrarian strategies are possible at date 2. We also gain further insights on how the characteristics of the prior distribution drive the direction of price-contingent trading.

Part 1 of Proposition 4 shows that, when the date 1 order flow is large in absolute value, then at date 2 uninformed \(P\) always pursues a contrarian strategy. The reason for this is similar to our argument about the two-point prior. Intuitively, high order flows in state \(R = U\) are relatively more likely to be driven by high noise trading shocks compared to what the Market expects. For example, if the true state is \(R = U\), then any order flow that exceeds \(\bar{g}_U\) must mean that there was a positive noise trading shock, while the Market will still consider order flows between \(\bar{g}_U\) and \(\bar{g}_I\) to be potentially reflecting small or even negative noise trading shocks. And this generates incentives for \(P\) to pursue a contrarian strategy.

Part 2 of Proposition 4 shows that if the probability of no-news is large enough, then at least for small order flows uninformed \(P\)’s optimal strategy at date 2 is trend-following. Namely, there is a threshold level for \(\gamma\), above which this inequality (19) holds\(^{32}\) and back-of-the-envelope calculations indicate that this threshold is quite low.\(^{33}\) This observation highlights the fact that for trend-following trading there should be at least some mass in the center of the distribution. The intuition for why at small order flows it is optimal for \(P\) to pursue a trend-following strategy again relates to part 2 of Proposition 2. Consider for example a small positive order flow. If the true state is \(R = U\), then the Market is now reluctant to believe that it is driven by informed traders who observed \(\bar{\theta}\) (as it considers the possibility that two informed traders who would trade a much larger quantity, \(\bar{g}_I\) in total, while the actual informed trading could have been at most \(\bar{g}_U\)) and sets the price relatively close to zero. Because uninformed \(P\) knows at date 2 that his trading did not contribute to date 1 order flow, he benefits from trend-following trading on average. What is crucially different in the situation where not just extreme events, but also moderate events regarding the fundamental are possible is that the Market tends to underestimate the probability that there were very good (or bad) news regarding the fundamental when it observes a small order flow, as it would expect two informed traders to always trade much more aggressively than one. This tendency to underestimate the

\(^{32}\)Note that the right hand side of (19) is always bigger than 1 as \(\mu_I > \mu_U\) by point 2 in Proposition 3. The right hand side is 1 if \(\gamma = 0\), strictly increasing in \(\gamma\) and converges to \(\exp(\mu_I^2) / \exp(\mu_U^2)\) when \(\gamma \to 1\). We can also verify that \(\exp(\mu_I^2) / \exp(\mu_U^2) > \mu_I / \mu_U\) at the limit. This is because \(\exp(\mu_R^2) / \mu_R\) is strictly increasing in \(\mu_R\) for any \(\mu_R > 0.5\). Hence (19) will hold at \(\gamma \to 1\) if \(\mu_I > \mu_U > 0.5\). It can also be shown that \(\mu_U\) is at its lowest when \(\eta = 1\) and \(\gamma = 0\), and from Figure (2) that in such case \(\mu_U\) is noticeably higher than 0.5.

\(^{33}\)For example, if \(\eta = 0.5\) then the threshold is around \(\gamma \approx 0.21\).
possibility of big fundamental news is what gives incentives for $P$ to pursue a trend-following strategy.

Figure (3) illustrates the equilibrium difference, $\mathbb{E} [\theta | y_1, U] - p_1$, (vertical axis) for different values of $\gamma$, assuming that $\eta = 0.5$. On the horizontal axis, there is always the date 1 order flow, $y_1$ and the shaded area point out the values of $y_1$ where $P$’s optimal strategy is trend-following. We can see that when $\gamma$ is high enough, then there is a set of order flows around zero where $|\mathbb{E} [\theta | y_1, U]| > |p_1|$ and uninformed $P$’s optimal strategy at date 2 is trend-following. As the informed trading volume is proportional to the standard deviation of noise trading, the values along the horizontal axis reflect the order flows normalized by the standard deviation of noise trading. We can see that at already $\gamma = 0.25$ order flows up to the magnitude of one standard deviation of noise trading will lead to trend-following trading. The magnitude of such order flows doubles if $\gamma = 0.75$. At very high order flows in absolute value, it always holds that $|\mathbb{E} [\theta | y_1, U]| < |p_1|$ and uninformed $P$’s optimal strategy at date 2 is contrarian.

The three-point distribution also allows to derive richer empirical implications. We find that price-contingent traders are likely to react differently when they observe order flows of different magnitude. It is plausible to expect that quantitative traders who typically trade in the direction of past price changes will adjust their behaviour and become contrarian at extreme order flows that are most likely driven by noise trading shocks.
4.4 Predictability of order flow and the effect of price-contingent trading on market efficiency.

Here we point out some natural consequences of equilibrium price-contingent trading under either the semi-strong or weak form of market efficiency.

**Proposition 5** While there is no predictability in returns, the order flow is predictable.

**Proof.** The lack of predictability in returns is immediate and is due to imposing the efficient market condition (2). By construction $p_2 = \mathbb{E}[\theta|y_1,y_2]$ and $p_1 = \mathbb{E}[\theta|y_1]$, and by application of the law of iterated expectations, it is clear that $\mathbb{E}[p_2 - p_1|y_1] = \mathbb{E}[\mathbb{E}[\theta|y_1,y_2]|y_1] - p_1 = \mathbb{E}[\theta|y_1] - p_1 = 0$. At the same time by Theorem 1 we know that if the state is $R = U$ then $P$ trades at date 2 a known amount $\bar{h}_2$. Therefore, $\mathbb{E}[y_2|y_1] = \text{Pr}(I|y_1)\mathbb{E}[y_2|y_1,I] + \text{Pr}(U|y_1)\mathbb{E}[y_2|y_1,U] = Q_1\mathbb{E}[s_2|y_1,I] + (1 - Q_1)\mathbb{E}[\bar{h}_2 + s_2|y_1,U] = (1 - Q_1)\mathbb{E}[\bar{h}_2|y_1,U] \neq 0$. ■

In Kyle (1985) and Holden and Subrahmanyam (1992) and subsequent models that build on their framework, imposing the market efficiency condition implies both the lack of predictability of returns and the lack of predictability of the order flow. As discussed in Section 4.1, there is no profitable and predictable price-contingent trading and future order flow can only reflect unpredictable noise trading and informed trading. Matters differ considerably in our more general setting, because the Market cannot be perfectly sure of whether there is a price-contingent trader $P$ or not, but the Market still knows that if there is one, he will trade in a predictable direction, described in Propositions 3 and 4. For example, if the optimal strategy is trend-following, the Market expects a positive order flow with some probability; if the actual order flow is zero, the prices fall.

It should also be noted that the type of price-contingent trading we analyze as emerging in a fully rational setting without other frictions, on average facilitates price discovery by moving prices closer to the fundamental. In state $R = U$, the best estimate of the fundamental conditional on all the information apart from the fundamental itself is $\mathbb{E}[\theta|y_1,U]$, and not $\mathbb{E}[\theta|y_1]$, so that uninformed $P$’s price-contingent trading on average pushes date 2 price $p_2$ closer to $\mathbb{E}[\theta|y_1,U]$.

Importantly, in our model there is also no sense in which contrarian trading is more stabilizing than momentum trading. For example, it is true that in our setting a rare situation can arise whereby prices change purely because of a noise trading shock and $P$’s optimal trend-following strategy moves prices further away from the fundamental, but similarly there can be a rare situation whereby following some draws of noise trading $P$’s optimal contrarian trading delays information about the fundamental from being reflected into prices. As a result, while both contrarian and momentum trading are on average stabilizing, both can end up pushing prices away from fundamentals.


5 Discussion of Special Cases and Extensions

In this Section we discuss some special cases and alternative assumptions and extensions. In Section 5.1 we allow both traders $P$ and $K$ to trade at date 2, too; in Section 5.2 we examine the number of traders; in Section 5.3 we examine alternative distributions of the fundamental; and in Section 5.4 we discuss other assumptions.

5.1 Allowing both traders $K$ and $P$ to trade at date 2 when they are informed

We have derived our main results in Sections 3 and 4 under the assumption that $K$ and $P$ cannot trade at date 2 if they are informed. This way, we have derived our results in the clearest manner, abstracting from the well-studied incentives of strategic informed traders to split their orders. On the one hand, the assumption that informed traders can trade less frequently could be justified by arguing that simple price-contingent trades are easy to programme and can therefore be implemented much faster than any fundamental information-based trades. On the other hand, it can also be argued that informed traders benefit from algorithms as well, as an appropriate algorithm is capable of quickly breaking up a large informed order into several smaller orders. Whichever view is more realistic, we show in this Section that allowing all traders to trade at both dates does not qualitatively alter our findings.

Namely, we alter the assumptions (3) and (4), and assume instead that if $R = I$, then trader $J = \{K, P\}$ is forward-looking and solves

$$\max_{h_{2}^{I, J}} \pi_{2}^{I, J} = \mathbb{E} \left[ h_{2}^{I, J} (\theta - p_{2}) | y_{1}, \theta, I \right] \text{ at date 2}, \tag{20}$$

$$\max_{h_{1}^{I, J}} \pi_{1}^{I, J} = \mathbb{E} \left[ h_{1}^{I, J} (\theta - p_{1}) + h_{2}^{I, J} (\theta - p_{2}) | \theta, I \right] \text{ at date 1.}$$

If $R = U$, then informed trader $K$ is also forward-looking and solves

$$\max_{h_{2}^{I, K}} \pi_{2}^{U, K} = \mathbb{E} \left[ h_{2}^{U, K} (\theta - p_{2}) | y_{1}, \theta, U \right] \text{ at date 2}, \tag{21}$$

$$\max_{h_{1}^{I, K}} \pi_{1}^{U, K} = \mathbb{E} \left[ h_{1}^{U, K} (\theta - p_{1}) + h_{2}^{U, K} (\theta - p_{2}) | \theta, U \right] \text{ at date 1.}$$

All other assumptions remain unchanged. First, we prove the following result.

\textbf{Lemma 5} Provided that $\eta > 0$, uninformed $P$ trading zero at date 2 can not be an equilibrium strategy.

\textbf{Proof.} See Appendix B. ■
This result is in the spirit of Theorem 1 - uninformed $P$ has incentives to trade at date 2, as he has superior information compared to the Market. As for Theorem 1, this result does not hinge on any particular distribution of the prior.

Next, we examine the direction of trading, and assume a three-point prior, (1). This problem is noticeably more complex than in the basic setting, and does not have a straightforward analytical solution - one complication being that all agents’ trades at date 2 are non-trivial functions of date 1 order flow. We can nevertheless show that the demand of each trader must be proportional to the standard deviation of noise trading, as in the basic setting.

We then derive the expressions that characterize the equilibrium, and focus on the numerical results. Similarly to the basic setting, uninformed $P$ cannot gain from trading at date 1. At date 2, it is immediate from (5) that the trader $P$ gains from buying (selling) the asset, i.e., $h_2^{UP} > (<) 0$, if and only if $E[\theta|y_1, U] - E[p_2|y_1, U] > (<) 0$ in equilibrium. Figure 4 plots these differences for different values of $\gamma$ (the order flow is normalized by dividing it by $\sigma_s, \bar{\theta} = 1$ and $\eta = 0.5$). Comparing Figure (3) and Figure (4), we can see that the results are remarkably similar. It remains true that $P$ benefits from contrarian trading at high values of the order flow as well as in the case in which there is no mass in the centre of the prior distribution of the fundamental (low $\gamma$). If there is enough mass in the centre of distribution, then $P$ benefits from trend-following trading at low absolute values of order flow. What changes is the magnitude - we can see that the difference between $P$'s expectations of the fundamental and $P$'s expectations of the price is smaller than in the basic setting.

In Appendix C we also report the equilibrium demand of traders at date 2 for different realizations of date 1 order flow and $\gamma$. The results are intuitive: 1) informed trading volume is decreasing in date 1 order flow (either because $\theta = \bar{\theta}$ and there is an increasingly lower possibility to buy the asset at a favorable price, or $\theta = \{-\bar{\theta}, 0\}$ and there are increasingly more possibilities to benefit by taking a position against date 1 noise trading shock); 2) as in the basic setting two informed traders demand a higher total quantity than one; 3) trading volume at $\theta = \{-\bar{\theta}, \bar{\theta}\}$ is higher in absolute value if $\gamma$, the probability of "no news", is higher. There is also a Kyle effect - informed traders strategically trade less is absolute value at date 1 than in the basic setting where splitting the orders is assumed away. Table 1 illustrates this by comparing informed trader’s date 1 total demand in the basic setting of Section 3 and 4 with

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34 Technical notes with expressions that characterize the equilibrium, and the MATLAB code that solves the problem are available as a technical supplement.
Due to the presence of informed traders, the volume traded by $P$ at date 2 is smaller than in the basic setting. Nevertheless allowing for the possibility of informed trading does not eliminate the possibility of profitable price-contingent trading.

When does $K$ lose due to the presence of uninformed $P$? While the volume traded by uninformed $P$ is small compared to the volume traded by informed $K$, it is intuitive that $K$’s returns on information must be lower whenever $P$ and $K$ trade in the same direction, i.e., provided that date 1 order flow is positive, this holds for $\theta = \bar{\theta}$ whenever $P$’s optimal strategy is positive-feedback and for $\theta = \{-\bar{\theta}, 0\}$ whenever $P$’s optimal strategy is contrarian. In the case of other outcomes, $K$ benefits not just from trading against noise traders, but also from taking an opposite position to uninformed $P$. If the state is $R = U$, then on average uninformed $P$ gains from price-contingent trading, and therefore $K$ typically earns lower profits than he
would if he were the only strategic trader active at date 2. Nevertheless, competing with $P$ reduces $K$’s profits even more in the state $R = I$, where $P$ is informed.

5.2 Number of traders

We have assumed that in any state there is always an informed trader $K$. The reason was to guarantee that asset prices always reflect fundamental information at least from $K$, which $P$ may learn from prices. The presence of trader $K$ allows for a rich set of effects and generates a rationale for trend-following trading under some conditions. However, we should emphasise that uncertainty about $P$’s type alone is sufficient for rational price-contingent trading to emerge even if there is no fundamental information. Namely, assume that there is no trader $K$, but the Market still does not know $P$’s type as in the baseline model of Section 3. Consider a symmetric prior and normal noise trading.

**Proposition 6** If there is no informed trader $K$, then uninformed $P$ will pursue and profit from a contrarian strategy at date 2.

**Proof.** See Appendix B. ■

Proposition 6 further highlights why profitable price-contingent strategies emerge in a rational setting where traders’ types are not perfectly observable by the average market participants, as it leads to "mistakes" in interpreting the order flow. In this example, prices would never change if the Market knew that $P$ is uninformed. However, the possibility that the order flow may contain fundamental information is enough to make the Market sometimes change prices unnecessarily based on order flows. Naturally $P$’s optimal strategy in this example is always contrarian because uninformed $P$ knows that the best guess about the fundamental is still the prior and would profit from speculating against any changes of prices.

It would be trivial to add more type $K$ and type $P$ traders. All the effects would be the same as long as the number of sophisticated traders of type $K$ and $P$ is finite. The reason why $K$ and $P$ trade finite amounts and earn returns on their information is because they have market impact and they are aware of it. If the number of type $P$ traders were infinite, then they would be indistinguishable from the Market; if the number of type $K$ traders were infinite, then in the limit prices would tend towards strong-form market efficiency, but also towards an information acquisition paradox in the spirit of Grossman and Stiglitz (1980). Second, it would also be possible to add more trading rounds in which uninformed $P$ can trade. This would complicate the model as $P$ would likely have a Kyle’s (1985) type of incentive to split his orders and reveal information more slowly. However, it is intuitive that the Market will then still be imperfectly and slowly learning about the true state until the price eventually converges to $\mathbb{E}[\theta|y_1, R]$. 28
5.3 Alternative distribution of the fundamental and some more general properties

As we discussed in Sections 3, 4.1, and 4.2 above, our results on existence and uniqueness of an optimal price-contingent trading strategy at date 2 are general and do not depend on any particular prior distribution of the fundamental. However, we showed in Section 4.3 that the choice of prior distribution is crucial for determining the qualitative results, and most notably for the direction of price contingent trading.

We focused on the examples of a two-point and a three-point prior. While the former is a common assumption in the literature (e.g., Cho and El Karoui (2000)), the latter allowed us to derive richer and more realistic results. Another common assumption in the literature is a normal prior, used for example in Kyle (1985) and Holden and Subrahmanyam (1992). Given the intuition we have built in Section 4.3 with the three-point distribution, it is intuitive to expect the presence of the force pushing towards trend-following trading, due to the sizable mass in the centre of the normal prior distribution. Furthermore, the normal prior has also infinite support and thin tails, which also implies that very high order flows are not necessarily driven by extreme noise trading, but may be still driven by very high draws of the fundamental. In Appendix D we explore the normal prior and show that in such case the optimal price-contingent strategy is trend-following. The reason is that the tendency for the market to underestimate the fundamental at a given order flow is always present and dominates the tendency to overreact to noise trading shocks. Namely, at any given fundamental two traders will always trade more than one, and it is also true that at a given level of demand by informed traders, the fundamental is always higher if there is one informed trader rather than two. This force leads the Market to setting prices too insensitive to the order flow, which makes trend-following trading profitable. While there is also a secondary effect where the Market underestimates the contribution of noise traders to the total order flow if the true state is \( R = U \), we find that this latter effect never dominates. Appendix D includes the case where \( \theta \) and \( R \) are independent as in our baseline setting, but we also give an example with a particular joint density of \( R \) and \( \theta \) that keeps the date 1 equilibrium linear. Results in both cases are similar, but the latter allows for comparable equations to the aforementioned papers that are a special cases with \( \eta = 0 \) and \( \eta = 1 \) in our setting.

We can further derive some more general properties of our model with other prior distributions. For clarity, assume that the fundamental, \( \theta \), is a continuous variable;\(^{35}\) that noise trading is also continuous in the interval \([\bar{s}, \bar{s}]\); and \( f_s(s) \) is log-concave and symmetric. Provided that the price is increasing in the order flow and the date 1 problem is quasi-concave (with interior

\(^{35}\) Similar arguments hold for a discrete fundamental.
maximum)\textsuperscript{36}, there exists a pure strategy equilibrium where a set of properties are true, and we can derive some properties using the tools from the monotone comparative statics literature. Namely, we establish

**Proposition 7** If date 1 price is increasing in the order flow and the informed traders’ problem is quasi-concave (with interior maximum), then:

1. The total demand of informed traders, $g_R(\theta)$ in state $R \in \{I, U\}$ is strictly increasing in $\theta$.
2. It holds that $g_I(\theta) > (\frac{\partial}{\partial \theta}) g_U(\theta)$ for any $\theta > (\frac{\partial}{\partial \theta}) 0$.

**Proof.** See Appendix B. ■

The most important implication of Proposition 7 is that, holding fixed any fundamental, two informed traders trade a higher quantity in equilibrium than one informed trader. This implies that conditional on date 1 order flow $y_1$, the state $R$ and the fundamental $x$ are not independent. In particular, $\Pr(U | \theta, y_1) = \frac{f_s(y_{1-n} - g_U(\theta)) \Pr(U | \theta)}{f_s(y_{1-n} - g_U(\theta)) \Pr(U | \theta) + f_s(y_{1-n} - g_I(\theta)) \Pr(I | \theta)}$ is generally a function of $\theta$. This allows to conclude that in general there will be a difference between uninformed $P$’s and the markets expectations. Namely, it holds that

$$\mathbb{E} [\theta | y_1, U] - p_1 = \frac{\mathbb{E} [\theta \cdot \Pr(U | \theta, y_1) | y_1]}{\Pr(U | y_1)} - \mathbb{E} [\theta | y_1] = \frac{\text{Cov} (\theta, \Pr(U | \theta, y_1))}{\Pr(U | y_1)},$$

where we used the market efficiency condition (2) and Bayes’ rule $f(\theta | y_1, U) = \frac{f(\theta | y_1) \Pr(U | \theta, y_1)}{\Pr(U | y_1)}$. As we argued that $\Pr(U | \theta, y_1)$ is a function of $\theta$, it is clear that $\theta$ and $\Pr(U | \theta, y_1)$ are not independent, which means that the covariance will generally not be zero. Therefore, our main result that price-contingent trading is profitable in a setting where there is uncertainty about traders’ types is very general.

Also, given Proposition 7, the forces that affect the direction of the difference $\mathbb{E} [\theta | y_1, U] - p_1$ are also present more generally. As an example, assume that both noise trading and the fundamental $\theta$ are continuous and have support in $(-\infty, \infty)$ and noise trading has symmetric log-concave density. In such case, by Proposition 7 part 1, we know that $g_R(.)$ is an invertible function. Therefore, in state $R$ we obtain the following signal from the order flow only: given $y_1$ and unknown noise trading shock $s_1^*$, the fundamental is $\theta^* = g_R^{-1}(y_1 - s_1^*)$. The cumulative distribution function (c.d.f.) of $\theta$ conditional on the order flow only is therefore, $\Pr(\theta^* < \theta) = \Pr(g_R^{-1}(y_1 - s_1^*) < \theta) = 1 - F_s(y_1 - g_R(\theta))$, where $F_s$ is the c.d.f. of noise trading. By part 2 of Proposition 7 and the fact that the c.d.f. is monotonically increasing,

\textsuperscript{36}Sufficient conditions for this are similar to those discussed in Appendix B.
1 - F_s(y_1 - g_U(\theta)) < (>) 1 - F_s(y_1 - g_I(\theta)) for any g_I(\theta) > (<) g_U(\theta) \iff \theta > (<) 0. This confirms that the distribution of \theta conditional on the order flow only is more dispersed in state R = U compared to state R = I for any order flow y_1. The probability density function of the fundamental conditional on order flow only is g_R(\theta) f_s(y_1 - g_R(\theta)). We find that the expected value conditional on the order flow only is then \int_{-\infty}^{\infty} \theta g_R(\theta) f_s(y_1 - g_R(\theta)) d\theta = \int_{-\infty}^{\infty} g_R^{-1}(y_\theta) f_s(y_1 - y_\theta) dy_\theta = \int_{0}^{\infty} g_R^{-1}(y_\theta) (f_s(y_1 - y_\theta) - f_s(y_1 + y_\theta)) dy_\theta. It then holds that the difference in expectations is \int_{0}^{\infty} (g_R^{-1}(y_\theta) - g_I^{-1}(y_\theta)) (f_s(y_1 - y_\theta) - f_s(y_1 + y_\theta)) dy_\theta > (<) 0 if y_1 > (<) 0. The inequality follows from (27) and Lemma A.1 in Appendix A. Therefore, if y_1 > 0, then in state R = U the order flow signal has a higher mean (which pushes toward trend-following trading) and a more dispersed distribution (which pushes toward contrarian trading) than in state R = I. Any Bayesian updating trades off these two effects, and which one dominates depends on the prior. If, additionally, the distributions are bounded then the same effects are present and there are additional effects due to these bounds - most importantly, large order flows in absolute value are always more likely in state R = I.  

5.4 Normal noise trading and other assumptions

As discussed in Section 3, we view the noise traders in our model as capturing a large number of traders who trade for idiosyncratic reasons outside the main focus of our model. Therefore, the main argument for assuming normally distributed noise trading stems from the central limit theorem. However, technically, many realistic properties of our model rely on the less restrictive assumption of log-concave noise trading. Indeed, a log-concave distribution guarantees that the Market updates at date 2 in the "correct direction" - that is, in state R = U, if trader P submits a positive quantity in equilibrium, then higher order flows at date 2 always signal a higher posterior probability that the state is indeed R = U. It also guarantees that the expected value \mathbb{E}[\theta|y_1, R] is increasing in date 1 order flow, which in turn often implies that also the price is increasing in order flow. Both of these properties hold because log-concavity implies the monotone likelihood ratio property (MLRP). These properties are realistic in the context of financial markets and guarantee that sophisticated large traders in our model face a meaningful trade-off in the spirit of Kyle (1985). Namely, an informed trader (either with superior information about the fundamental directly or indirectly due to superior knowledge of his own past actions) benefits from trading a higher volume due to positive expected returns,  

\textsuperscript{37}If the prior has bounded support \([-\bar{\theta}, \bar{\theta}]\) and noise trading support is noticeably wider (such that any order flow can be generated by a noise trading shock), then conditional on the order flow only in state R the cdf of \theta is \frac{F_s(y_1 + g_R(\theta)) - F_s(y_1 - g_R(\theta))}{F_s(y_1 + g_R(\theta)) - F_s(y_1 - g_R(\theta))} and the probability density function is \frac{g_R(\theta) F_s(y_1 - g_R(\theta)) - F_s(y_1 - g_R(\theta))}{F_s(y_1 + g_R(\theta)) - F_s(y_1 - g_R(\theta))}. With some algebra we can then identify the same effects, and find that informed trading at the highest fundamental, g_R(\bar{\theta}) in R = \{I, U\} affect both the mean and the dispersion.
but trading a higher volume is costly due to market impact as it reveals more about his private information—whether about the fundamental or about his own type.

In our proofs we frequently relied only on log-concavity rather than on the explicit form of the normal density. For example, the determinants of the direction of price-contingent trading in the case of two- and three-point prior distribution hold for any (symmetric) log-concave noise trading.

It should also be noted that to avoid a situation whereby $P$ at date 2 has an incentive to trade an infinite amount—which cannot occur in equilibrium—we need at least that the noise trading distribution, $f_s(.)$, is such that the likelihood ratio $r(y_2) = \frac{f_s(y_2-h_2)}{f_s(y_2)}$ is unbounded, which is true for some, but not all log-concave densities.38

6 Empirical Implications

We have presented a theory of algorithmic trading as an automated system under constant human supervision. In section 6.1 we discuss how our equilibrium helps understand basic features of algorithmic trading in the real world, and in section 6.2 we discuss the impact of algorithmic trading on market stability and market crashes.

6.1 Understanding Algorithmic Trading

Algorithmic trading is systematically profitable. While hedge funds and quantitative traders are shrouded in secrecy and systematic data is thus hard to come by, it is becoming increasingly evident that quantitative trading with algorithms generates large profits on a regular basis.39 These regular profits are hard to reconcile with a view of algorithmic trading as mere implementation of standard portfolio selection models, and do suggest the need to examine the micro-foundation of algorithmic trading strategies.

Our paper does offer such a micro-foundation of algorithmic trading. In our equilibrium, when trader $P$ ends up uninformed about the fundamental, he trades a systematic non-zero quantity based on past prices, whose direction—trend-following or contrarian—depends on the magnitude of the order flow. By contrast, when $P$ ends up informed (e.g., rumors of a takeover bid), he trades on that information and disregards or override the algorithm (e.g., see the

38By unbounded, we mean that $\lim_{y_2 \rightarrow \hat{s}} r(y_2) \rightarrow \infty$ for $0 < \hat{h}_2 < \hat{s}$. For example, not just the normal, but also the $Beta(\alpha, \beta)$ distribution with parameters $\alpha, \beta > 1$ is strictly log-concave and has an unbounded likelihood ratio. However, $Gamma(k, \theta)$ with $k > 0$ is strictly log-concave, but the likelihood ratio is not unbounded (Gamma is also not symmetric around zero).

39Quantitative hedge funds such as Citadel, CQS, Renaissance Technologies, and others, which implement multiple trading strategies with a strong emphasis on directional trading, feature regularly among the top performing hedge funds, e.g., see http://media.bloomberg.com/bb/avfile/rMz9ZuocMhKo.
illustration based on rumors about the Merrill Lynch and Bank of America merger in Narang (2013, p.15-16)). Thus, in our equilibrium automated algorithmic trading needs constant human supervision: it is the very possibility of submitting an informed order at some point that makes algorithmic trading systematically profitable.\footnote{Alternatively, one can also think of quantitative trading as being a portfolio of various trading strategies, some of which based on prices, such as trend following or contrarian strategies, and others based on fundamentals, such as for example 'value' or 'growth' strategies (albeit not based on publicly observed 'factors' but on proprietary research, e.g., see Kissell (2014) and Narang (2013)).} Interestingly, this view of automated trading under human supervision is consistent with the accounts of the source of hedge funds’ profits that are typically found in the press.\footnote{For example, Fortune Magazine (16 July, 2003) features an article on the success of Barclays Global Investors, which describes the fund’s strategy as “They don’t follow hunches; they follow computer models. /.../ This is modern quantitative investing. It falls somewhere between stock picking as it has been practiced for generations and the computer-driven "black box" techniques”.}

Given the mechanics of automated trading under human supervision and its interplay with fundamental trading described above, our model most closely rationalizes price-contingent strategies at short horizons by Commodity Trading Advisors (CTAs) in futures markets and by various hedge funds in equity markets.\footnote{Some large and prominent funds that are known to use systematic trend-following and contrarian strategies are AQR, Renaissance Technologies, D.E. Shaw, Citadel, Barclays Global Investors.} CTAs are popular recent investment vehicles that execute profitable trend-following strategies in futures markets at daily, weekly, and monthly frequencies (e.g., see Clenow (2013), and Baltas and Kosowski (2014)). By contrast, various hedge funds execute profitable contrarian strategies in equities at weekly (Lehmann (1990)) and monthly (Jegadeesh (1990)) frequency. Our model can account for both types of strategies, by noticing that futures markets are typically characterized by a distribution of the fundamental with substantial mass in the center, similar to our normal case examined in Section 5.2, or the three-point distribution with enough probability of no news of Section 4; and that equity markets at short horizons are rarely characterized by informed trading, so that they resemble the case examined in Proposition 6 without trader $K$. Along similar lines, in any financial market at such high frequencies that the likelihood of informed trading is essentially zero, our model predicts that any price-contingent trading should be contrarian.

More generally, one way to take our model to the data could be to see whether the direction of algorithmic trading correlates with the magnitude of total order flow (potentially proxied by trading volume). In light of our model, trend-following trading should be more likely when total order flow is smaller; and contrarian trading more likely when total order flow is larger.

Finally, one testable implication of our model is that the order flow should be predictable from past information even when returns are not. The reason is that, even if the Market does not know if trader $P$ is informed, still the Market knows the direction of $P$’s trade when uninformed. Therefore, we expect a non-zero autocorrelation of the order flow, and we expect its sign to go
hand in hand with the direction of price-contingent trading in equilibrium: contrarian trading should imply a negative autocorrelation, and trend-following trading should imply a positive autocorrelation. Importantly, these predictions differ from those of Barberis et al. (1998), Daniel et al. (1998), and Hong and Stein (1999), as in these models order flow predictability is a direct consequence of return predictability, which is ruled out in our model.

6.2 Market Quality and Crashes

In terms of the impact of the introduction of quantitative trading on various aspects of market quality such as volatility and liquidity, we find that quantitative trading is on average stabilizing, in the sense that price contingent trading typically moves prices closer to the fundamental, consistent with the empirical evidence of Hendershott et al. (2011) and the practitioners’ accounts in Kissell (2014), Durenard (2013), and Narang (2013). However, there is a concern that in particular circumstances quantitative trading can propagate adverse negative shocks and generate instability, as in the Quant Meltdown of August 2007, and the Flash Crash of May 6, 2010. For example, the report on the "events of May 6," (CFTC and SEC (2010)) stated that a large ‘mistaken’ sell order triggered algorithms to start selling; soon after the volume of sell orders increased, and algorithms started to buy. Eventually, many algorithms incurred large losses and just stopped trading, so that the mismatch of supply and demand became so large that the entire system went to a halt for a few minutes.

Remarkably, while not specifically designed to describe these events, our model does capture some of their key features. First, algorithms did not trigger either episode—the trigger was a noise trading shock such as the ‘mistake’ by a large investor in 2010; and a series of large trades on the news of problems with subprime mortgages in 2007. Second, and consistent with our model, the initial response of algorithms in both cases was trend-following trading, as long as total order flow was ‘small enough’. Third, and again consistent with our model, when total order flow became larger, algorithms started pursuing contrarian strategies. On the other hand, by its very design our model does not capture the failures of market efficiency that occurred when many algorithms just stopped trading and prices could no longer equate supply and demand. Most important, though, the events of August 2007 and May 2010 underscore a key feature of our model: quantitative trading through algorithms is profitable on average, as it is better able to chase information than the rest of the market, but it can occasionally end up chasing noise trading shocks, thereby incurring losses.  

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7 Concluding Remarks

We have presented a theory of quantitative trading as an automated system under human supervision. We establish that price-contingent trading is the optimal strategy of large rational agents in a setting in which there is uncertainty about whether large traders are informed about the fundamental. We provide conditions under which price-contingent trading is trend-following (momentum) or contrarian in equilibrium. A robust implication of our results is that the order flow is predictable from current prices even if the market is semi-strong efficient and future returns are thus unpredictable.

Our model explains why hedge funds and other large financial institutions who engage in automated trading with algorithms are systematically profitable; and it explains why the secrecy of their algorithms, trading portfolios, and exposures is key to their success. By having market impact and by being relatively less known than other agents, hedge funds can learn any information that is reflected into prices better than any other investor who does not perfectly know their trading strategies and portfolios. As a result, hedge funds can successfully implement a broader range of strategies, such as trend-following and contrarian trading, than individual and retail investors without market impact that would lose money from those same strategies.

Quantitative trading with algorithms has recently come under attack in the popular press, and the profitability of these strategies has been occasionally attributed to illegal practices such as front-running.\(^{44}\) We have demonstrated that the superior systematic performance of trend-following and contrarian strategies needs not stem from illegal practices, as the simple and perfectly legitimate market impact of trades, together with uncertainty about access to fundamental information, is sufficient to generate systematic profits.

Of course, in the real world quantitative strategies can be a lot more sophisticated than our simple equilibrium momentum and contrarian strategies, and can use as input an array of quantifiable public information in addition to prices and order flows. One robust insight of our model is that quantifiable information can arise from superior knowledge of market participants’ trading styles rather than economic fundamentals as traditionally thought. Extending our model to capture the additional nuances of real-world quantitative strategies would seem to be an interesting area for future research.

\(^{44}\)Even though such critique is most applicable to the subset of trading strategies at the highest frequencies in the millisecond environment, it is the full universe of quantitative trading that has lately come under scrutiny from the popular press.
A Background theorems and lemmas

Lemma A.1 If $f_s(\cdot)$ is strictly log-concave, then it holds that

$$\frac{f_s(x_2 - c_2)}{f_s(x_2 - c_1)} > \frac{f_s(x_1 - c_2)}{f_s(x_1 - c_1)} \text{ for any } x_2 > x_1 \text{ and } c_2 > c_1. \quad (22)$$

Proof. By definition of log-concavity it must hold that

$$\alpha \ln (f_s(x_1 - c_2)) + (1 - \alpha) \ln (f_s(x_2 - c_1)) < \ln (f_s(\alpha (x_1 - c_2) + (1 - \alpha) (x_2 - c_1)))$$
and

$$(1 - \alpha) \ln (f_s(x_1 - c_2)) + \alpha \ln (f_s(x_2 - c_1)) < \ln (f_s((1 - \alpha) (x_1 - c_2) + \alpha (x_2 - c_1))) \text{ for any } 0 < \alpha < 1.$$

Let $\alpha = \frac{x_2 - x_1}{x_2 - x_1 + c_2 - c_1}$. Then

$$\ln (f_s(x_1 - c_2)) + \ln (f_s(x_2 - c_1)) = \ln (f_s(x_1 - c_1))$$
and

$$\ln (f_s((1 - \alpha) (x_1 - c_2) + \alpha (x_2 - c_1))) = \ln (f_s(x_2 - c_2)).$$

Adding up the inequalities, we obtain that

$$\ln (f_s(x_1 - c_2)) + \ln (f_s(x_2 - c_1)) < \ln (f_s(x_1 - c_1)) + \ln (f_s(x_2 - c_2)).$$

Exponentiating both sides and rearranging, we obtain (22). \hfill \blacksquare

Note that, in probability theory, this implies that if we interpret $c$ as a signal about some random variable such that $x = c + s$, where the density $f_s(s)$ is strictly log-concave, then the conditional distribution of $f(x|c) = f_s(x - c)$ satisfies the strict monotone likelihood ratio property (MLRP).

Corollary A.1.1 If $f_s(\cdot)$ is strictly log-concave and symmetric ($f_s(s) = f_s(-s)$), then for any $x > 0$, it holds that

$$f_s(x - c) > (<) f_s(x + c) \text{ for any } c > (<) 0$$

Proof. For the case $c > 0$, let $x_2 = x$, $x_1 = -x$ and $c_2 = c_1 = 0$. By (22) $\frac{f_s(x - c)}{f_s(x)} > \frac{f_s(x + c)}{f_s(x)} \Rightarrow f_s(x - c) > f_s(x + c)$. For the case $c < 0$, let $x_2 = x$, $x_1 = -x$ and $c = c_1 < c_2 = 0$ to obtain that $\frac{f_s(x)}{f_s(x - c)} > \frac{f_s(x)}{f_s(x - c)} \Rightarrow f_s(x + c) > f_s(x - c).$ \hfill \blacksquare

For the next Lemma, assume that the prior distribution in state $R \in \{I, U\}$ is $f(\theta|R) \equiv f_{0R}(\theta) = f_{0R}(-\theta)$ in between $-\tilde{\theta}$ and $\tilde{\theta}$ (the case $\tilde{\theta} = \infty$ is easy to incorporate in this framework). We consider a continuous prior, but a similar argument applies to a discrete prior. Assume that the informed trader’s total demand at date 1 is symmetric around zero and strictly increasing, i.e., $h_I^{UK} = g_U(\theta)$ and $h_I^{IK} + h_I^{IP} = g_I(\theta)$, where $g_R(\theta) = g_R(-\theta)$ and $g_R'(\theta) > 0$ for $R \in \{0, 1\}$. Assume that uninformed $P$’s demand at date 1 is some constant $\bar{h}_{1U}$.

Lemma A.2 If the noise trading distribution is log-concave and symmetric, $f_s(s_1)$, it holds for any prior distribution that $\mathbb{E}[\theta|\tilde{y}_1, R] - \mathbb{E}[\theta|y_1, R] > 0$ for any $\tilde{y}_1 > y_1$. It also holds that $\mathbb{E}[\theta|y_1, I] = \mathbb{E}[\theta|(-y_1), I]$ and $\mathbb{E}[\theta|(y_1 - \bar{h}_{1U}), U] = -\mathbb{E}[\theta|(-y_1 + \bar{h}_{1U}), U]$.

Proof. The proof uses Milgrom (1981) and log-concavity of $f_s(s)$. Define the observable part of the order flow in state $R$ as follows: $y_{1R} = y_1$ if $R = I$ and $y_{1R} = y_1 - \bar{h}_{1U}$ if $R = U$. It then
holds that \( y_{1R} = g_R(\theta) + s_1 \) for \( R = \{I, U\} \). We first show that \( \mathbb{E} [\theta|\tilde{y}_1, R] - \mathbb{E} [\theta|y_1, R] > 0 \) for any \( \tilde{y}_1 > y_{1R} \). It is well known that \( \mathbb{E} [\theta|\tilde{y}_1, R] > \mathbb{E} [\theta|y_{1R}, R] \) if the cumulative distribution \( F(\theta|\tilde{y}_1, R) \) dominates \( F(\theta|y_{1R}, R) \) in the sense of first order stochastic dominance, i.e., \( F(\theta|\tilde{y}_1, R) \leq F(\theta|y_{1R}, R) \) for all \( \theta \), with strict inequality for some \( \theta \). Given that the order flow \( y_{1R} = g_R(\theta) + s_1, g_R(.) \) is increasing, we know from Lemma A.1 that log-concavity of \( f_s(.) \) implies that \( f_s \left( \tilde{y}_1 - g_R(\hat{\theta}) \right) f_s \left( y_{1R} - g_R(\hat{\theta}) \right) > f_s \left( \tilde{y}_1 - g_R(\hat{\theta}) \right) f_s \left( y_{1R} - g_R(\hat{\theta}) \right) \) for every \( \hat{\theta} > \tilde{\theta} \). We can equivalently write this inequality as \( f \left( \tilde{y}_1|\tilde{\theta}, R \right) f \left( y_{1R}|\tilde{\theta}, R \right) > f \left( \tilde{y}_1|\tilde{\theta}, R \right) f \left( y_{1R}|\tilde{\theta}, R \right) \) for every \( \tilde{\theta} > \tilde{\theta} \). The fact that the latter inequality implies first order stochastic dominance for any prior density \( f(\theta|R) \) is Proposition 1 in Milgrom (1981). As in state \( R = I \), \( y_{1R} = y_1 \), this immediately proves that \( E[\theta|\tilde{y}_1, I] - E[\theta|y_1, I] > 0 \) for any \( \tilde{y}_1 > y_1 \). For the state \( R = U \), notice that because \( \tilde{h}_{1U} \) is known, conditioning on \( y_1 \) is equivalent to conditioning on \( y_{1U} = y_1 - \tilde{h}_{1U} \) and it must always hold that \( E[\theta|y_1, R] = E[\theta|y_{1R}, R] \) so we know that \( E[\theta|\tilde{y}_1, U] - E[\theta|y_1, U] > 0 \) for any \( \tilde{y}_1 - \tilde{h}_{1U} > y_1 - \tilde{h}_{1U} \Leftrightarrow \tilde{y}_1 > y_1 \). For the second part, note that we can express that \( \mathbb{E} [\theta|y_{1R}, R] = \int_{\tilde{\theta}}^{\theta} f (\theta|y_{1R}, R) d\theta = \frac{\int_{\tilde{\theta}}^{\theta} f_s(y_{1R} - \tilde{g}_R(\theta))f_{\theta R}(\theta)d\theta}{\int_{\tilde{\theta}}^{\theta} f_s(y_{1R} - \tilde{g}_R(\theta))f_{\theta R}(\theta)d\theta} \). Using the symmetry of \( f_s(.) \) and \( f_{\theta R}(\theta) \), we then find that \( \mathbb{E} [\theta|y_{1R}, R] = \int_{\tilde{\theta}}^{\theta} f (\theta - y_{1R}, R) d\theta = \frac{\int_{\tilde{\theta}}^{\theta} f_s(y_{1R} - \tilde{g}_R(\theta))f_{\theta R}(\theta)d\theta}{\int_{\tilde{\theta}}^{\theta} f_s(y_{1R} - \tilde{g}_R(\theta))f_{\theta R}(\theta)d\theta} = -\mathbb{E} [\theta|y_{1R}, R] \). Using then the definition of \( y_{1R} \) proves the lemma. 

Theorem A.3 (Prékopa (1973) Theorem 6) Let \( f(x, y) \) be a function of \( n + m \) variables where \( x \) is an \( n \)-component and \( y \) is an \( m \)-component vector. Suppose that \( f \) is logarithmic concave in \( \mathbb{R}^{n+m} \) and let \( A \) be a convex subset of \( \mathbb{R}^m \). Then the function of the variable \( x \):

\[
\int_A f(x, y)dy
\]

is logarithmic concave in the entire space \( \mathbb{R}^n \).

Theorem A.4 (Chebyshev’s integral inequality) Let \( g, h : [a, b] \to \mathbb{R} \) and \( f : [a, b] \to \mathbb{R} \) be a probability density function. Suppose that \( g \) is monotonically increasing. Define \( H_F : (a, b) \to \mathbb{R}, H_F(t) = \int_{t}^{b} h(x)f(x)dx \int_{a}^{b} f(x)dx \). If \( H_F(t) \leq H_F(b) \) for all \( t \in (a, b) \) then

\[
\int_{a}^{b} g(x) h(x) f(x) ds \geq \int_{a}^{b} g(x) f(x) dx \int_{a}^{b} h(x) f(x) dx
\]

Proof. See for example Theorem 1 in Wagener (2006). Also see Mitrović, Pečarić, and Fink (1993).
B Proofs

Proof of Proposition 1

As this statement does not rely on any specific distribution of the fundamental, we prove a more general case. Therefore, assume more generally that the prior distribution is symmetric (and prior mean exists) and noise trading distribution is log-concave. Assume that informed trader $K$’s optimal strategy is given by $g_U(\theta)$ and it holds $g_U(\tilde{\theta}) > g_U(\theta)$ for any $\tilde{\theta} > \theta$ and $g_U(\theta) = -g_U(-\theta)$.$^\text{45}$

By a straightforward application of Bayes’ rule, it holds that $\eta = 0 \implies Q_1 = 0$ and $Q_2 = 0$. Also, notice that in this case date 2 order flow is not informative about the fundamental, i.e., \( \mathbb{E}[\theta|y_1, y_2, U] = \mathbb{E}[\theta|y_1, U] \). From (7), we then find that $p_1 = \mathbb{E}[\theta|y_1] = \mathbb{E}[\theta|y_1, U] = p_2$. From (5), $P$’s expected date 2 profits are $\mathbb{E}[h_2^{UP}(\theta - p_2) | y_1, U] = h_2^{UP}(\mathbb{E}[\theta|y_1, U] - p_1) = 0$. Therefore, $P$ earns zero profit at date 2 irrespective of the quantity he trades. For date 1, suppose that the Market sets the price under the belief that $P$ trades some known quantity $\hat{h}_{1U}$ at date 1. Given these beliefs, $P$ chooses $h_1^{UP}$ that corresponds to an order flow (see (6)) $y_1 = h_1^{UP} + s_1 + g_U(\theta)$. The order flow is uncertain at the time of $P$’s date 1 trading decision due to the presence of $s_1$ and $\theta$. Using that $P$’s profit at date 2 is always zero and $\mathbb{E}[\theta|U] = 0$, (5) can be written as $\pi_1^{UP} = -h_1^{UP}\mathbb{E}[p_1|U]$. Denoting the prior distribution of the fundamental with $f_\theta(\theta)$, we find that $\mathbb{E}[p_1|U] = \int_s \int_\theta p_1(h_1^{UP} + s_1 + g_U(\theta)) f_s(s_1) f_\theta(\theta) ds_1 d\theta.$ $^\text{46}$ The first derivative of the profit is $\partial \pi_1^{UP} / \partial h_1^{UP} = -\mathbb{E}[p_1|U] - h_1^{UP}(\partial \mathbb{E}[p_1|U] / \partial h_1^{UP})$. Because $p_1 = \mathbb{E}[\theta|y_1, U]$, by Lemma A.2 in Appendix A, the price is increasing in the order flow, and it holds that $-h_1^{UP}(\partial \mathbb{E}[p_1|U] / \partial h_1^{UP}) < (>) 0$ for any $h_1^{UP} > (>) 0$. Due to the symmetry of distributions and the fact that prices increase in the order flow, it also holds that $-\mathbb{E}[p_1|U] < (>) 0$ for any $h_1^{UP} > (>) 0$. $^\text{47}$ Therefore, $P$’s profit is maximized at $h_1^{UP} = 0$. In equilibrium, the beliefs of the Market must be consistent with $P$’s optimal strategy, i.e., it holds that $h_1^{UP} = \hat{h}_{1U} = 0$. In such a case, also $\pi_1^{UP} = 0$.

Proof of the remaining parts of Theorem 1

Assume that $\mathbb{E}[\theta|y_1, U] - p_1 > 0$. It is clear from (9) that the optimal demand $h_2^{UP}$ cannot be negative. Because by Lemma 2 $P$’s problem at date 2 is log-concave, it is sufficient to

$^\text{45}$Proposition 6 in Section 5 proves that the total demand by informed traders must be increasing in $\theta$ due to supermodularity of the problem. Symmetry follows from the symmetry of all distributions.

$^\text{46}$A similar argument holds for a discrete distribution of the fundamental and/or of noise trading.

$^\text{47}$From Lemma A.2 in Appendix A it also holds that $p_1(y_1 - \hat{h}_{1U}) = -p_1(-y_1 + \hat{h}_{1U})$. Therefore, $\mathbb{E}[p_1|U] = \int_{\theta > 0} \int_{s_1 > 0} P_1(h_1^{UP}, s_1, \theta, \hat{h}_{1U}) f_s(s_1) f_\theta(\theta) ds_1 d\theta$, where $P_1(h_1^{UP}, s_1, \theta, \hat{h}_{1U}) = p_1(h_1^{UP} + s_1 + g_U(\theta) + \hat{h}_{1U} - \hat{h}_{1U}) - p_1(-h_1^{UP} + s_1 - g_U(\theta) - \hat{h}_{1U} + \hat{h}_{1U}) + p_1(h_1^{UP} + s_1 - g_U(\theta) + \hat{h}_{1U} - \hat{h}_{1U}) - p_1(-h_1^{UP} + s_1 + g_U(\theta) - \hat{h}_{1U} + \hat{h}_{1U})$. As prices are increasing in the order flow, it holds that $P_1(h_1^{UP}, s_1, \theta, \hat{h}_{1U})$ is always non-negative and strictly positive for some values of $s_1, \theta > 0$ iff $h_1^{UP} > 0$. 

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explore the first order condition. Using (9), (10), (11), (12) and noticing that \( \frac{\partial \varphi_s(y_2-h_{2}^{UP})}{\partial h_{2}^{UP}} = \frac{y_2-h_{2}^{UP}}{\sigma_s^2} \varphi_s(y_2-h_{2}^{UP}) \), we obtain that

\[
\frac{\partial \pi_{2}^{UP}}{\partial h_{2}^{UP}} = \int_{-\infty}^{\infty} \frac{Q_1 \varphi_s(y_2)}{Q_1 \varphi_s(y_2) + (1-Q_1) \varphi_s(y_2-h_2)} \left( 1 - \frac{(h_{2}^{UP})^2}{\sigma_s^2} + \frac{h_{2}^{UP} y_2}{\sigma_s^2} \right) \varphi_s(y_2-h_{2}^{UP}) \, dy_2 \tag{23}
\]

Define \( \kappa \equiv \frac{h_{2}^{UP} \sigma_s}{y_2} \) and \( z \equiv \frac{y_2}{\sigma_s} \), where \( dy_2 = \sigma_s \, dz \), which implies that \( \varphi_s(y_2-h_2) = \frac{1}{\sigma_s} \phi(z-\kappa) \) and \( \varphi_s(y_2) = \frac{1}{\sigma_s} \phi(z) \), where \( \phi(.) \) is the p.d.f. of a standard normal. The optimal demand must solve \( \frac{\partial \pi_{2}^{UP}}{\partial h_{2}^{UP}} = 0 \) and it must hold in equilibrium that optimal demand \( (h_{2}^{UP})^* = \tilde{h}_{2} = \kappa \sigma_s \).

Using all this, in (23), we obtain that \( \kappa \) is the positive solution of

\[
\int_{-\infty}^{\infty} \frac{Q_1 \phi(z)}{Q_1 \phi(z) + (1-Q_1) \phi(z-\kappa)} \left( 1 - \kappa^2 + \kappa z \right) \phi(z-\kappa) \, dz = 0, \tag{24}
\]

which we know to be unique by Lemma 2. Because \( \sigma_s \) does not enter in (24), it also proves that \( P \)'s demand is proportional to \( \sigma_s \) and only depends on \( Q_1 \). The proof for the case \( \mathbb{E} [\theta | y_1, U] - p_1 < 0 \) is similar and in such a case we need the unique negative solution of (24). It is easy to verify that if \( \kappa > 0 \) solves (24), then also \( -\kappa > 0 \) solves (24).

Next let us prove that \( \kappa > 1 \) by contradiction. Suppose instead that \( 0 < \kappa < 1 \) solves (24). From (24), it must then be the case that \( \kappa \int_{-\infty}^{\infty} z \frac{Q_1 \phi(z)}{Q_1 \phi(z)+(1-Q_1) \phi(z-\kappa)} \phi(z-\kappa) \, dz < 0 \). Using that \( \phi(.) \) is an even function, we can rewrite this as

\[
\kappa \int_{0}^{\infty} z Q_1 \phi(z) \left( \frac{1}{Q_1 \phi(z-\kappa) + (1-Q_1) \phi(z)} - \frac{1}{Q_1 \phi(z+\kappa) + (1-Q_1)} \right) \, dz < 0.
\]

Because \( \phi(.) \) is log-concave, it holds that \( \phi(z-\kappa) > \phi(z+\kappa) \) for all \( z, \kappa > 0 \) by Corollary A.1.1 from Appendix A. This implies that \( \frac{1}{Q_1 \phi(z-\kappa) + (1-Q_1)} > \frac{1}{Q_1 \phi(z+\kappa) + (1-Q_1)} \). So all terms inside the integral are non-negative for all \( z \geq 0 \) (with strict inequality for \( z > 0 \)), which leads to a contradiction and therefore \( 0 < \kappa < 1 \) does not hold.

**Proof of Lemma 3**

For parts 1-3 note that (7) implies that, \( p_1(y_1) = Q_1 \mathbb{E} [\theta | y_1, I] + (1-Q_1) \mathbb{E} [\theta | y_1, U] \). By the law of total expectations \( \mathbb{E} [\theta | y_1, R] = \tilde{\theta} \mathbb{E} (\theta = \tilde{\theta} | y_1, R) + \tilde{\theta} \mathbb{E} (\theta = -\tilde{\theta} | y_1, R) \) and by Bayes’ rule \( \mathbb{P}(\theta | y_1, R) = \frac{1-\gamma}{2} f(y_1 | \theta, R) / f(y_1 | R) \), where \( f(y_1 | \theta = \tilde{\theta}, R) = \varphi_s(y_1 - \tilde{g}_R) \); \( f(y_1 | \theta = 0, R) = \varphi_s(y_1) \); \( f(y_1 | \theta = -\tilde{\theta}, R) = \varphi_s(y_1 + \tilde{g}_R) \) and \( f(y_1 | R) = f(y_1 | \theta = \tilde{\theta}, R) + f(y_1 | \theta = 0, R) + f(y_1 | \theta = -\tilde{\theta}, R) \). By Bayes’ rule \( Q_1(y_1) = \frac{\eta f(y_1 | I)}{\eta f(y_1 | I) + (1-\eta) f(y_1 | U)} \). Combining all this proves parts 1-3. For part 4, note that \( \partial \varphi_s(y_1-c) / \partial y_1 = -\frac{y_1-c}{\sigma^2_s} \varphi_s(y_1-c) \) for any constant \( c \), and therefore \( M_n(y_1) = -\frac{y_1}{\sigma^2_s} M_n(y_1) + M_{ng}(y_1) \) and \( M_p(y_1) = -\frac{y_1}{\sigma^2_s} M_p(y_1) - M_{pg}(y_1) \), where
term is clearly positive. The second term can be written as
\[ \frac{1-\gamma}{2} \left( \eta \frac{\partial}{\partial \sigma^2} \varphi_s(y_1 - \tilde{g}_I) + (1-\eta) \frac{\partial}{\partial \sigma^2} \varphi_s(y_1 - \tilde{g}_U) \right) > 0 \] and \( M_{pg}(y_1) \equiv \frac{1-\gamma}{2} \left( \eta \frac{\partial}{\partial \sigma^2} \varphi_s(y_1 + \tilde{g}_I) + (1-\eta) \frac{\partial}{\partial \sigma^2} \varphi_s(y_1 + \tilde{g}_U) \right) > 0. \) Using the above and differentiating, \( p'_1(y_1) = 2\theta \frac{M_{sg}(y_1)M_p(y_1) + M_{pg}(y_1)M_n(y_1)}{(M_n(y_1) + M_p(y_1))^2} > 0. \) Parts 5-7 are straightforward when using (14), the expression of the normal density, and the fact that \( \varphi_s(.) \) is an even function that is always non-negative and positive at finite values.

**Quasiconcavity of informed traders’ date 1 problem.**

Let us focus on the state \( R = U. \) If \( \theta = 0 \) then from (17), we obtain the first order condition
\[\frac{\partial \pi_{UK}}{\partial h_{UK}} = \int_{-\infty}^{\infty} p_1(h_{UK} + s_1) + h_{UK} p'_1(h_{UK} + s_1) \varphi_s(s_1) ds_1 = \int_{-\infty}^{\infty} p(y_1) + h_{UK} p'(y_1) \varphi_s(y_1 - h_{UK}) dy_1 = 0. \] It is clear that \( h_{UK} = 0 \) satisfies the first order condition as \( p(y_1) \varphi_s(y_1) \) is an odd function of \( y_1 \) and therefore \( \frac{\partial \pi_{UK}}{\partial h_{UK}} |_{h_{UK} = 0} = 0. \) Furthermore, the negative of the first derivative can be expressed as
\[\frac{\partial \pi_{UK}}{\partial h_{UK}} = \int_{0}^{\infty} p(y_1) \left( \varphi_s(y_1 - h_{UK}) - \varphi_s(y_1 + h_{UK}) \right) dy_1 + h_{UK} \int_{-\infty}^{\infty} p'(y_1) \varphi_s(y_1 - h_{UK}) dy_1. \] Note that \( \varphi_s(.) \geq (>) 0, \) for all (some) \( y_1 \) and \( h_{UK}, \) \( p(y_1) \geq (>) 0 \) for all (some) \( y_1 \geq 0 \) and \( p'(y_1) \geq 0 \) for all (some) \( y_1. \) Furthermore, from Corollary A.1.1 in Appendix A, we know that \( \varphi_s(y_1 - h_{UK}) > \varphi_s(y_1 + h_{UK}) \) if and only if \( h_{UK} > 0 \) and \( y_1 > 0. \) Therefore, \( -\frac{\partial \pi_{UK}}{\partial h_{UK}} \) is strictly single-crossing, which proves that the objective function (17) is quasiconcave and achieves the maximum at \( h_{UK} = 0. \)

If \( \theta = \tilde{\theta}, \) then it is clear from (17) that \( h_{UK} < 0 \) cannot be the best response as it leads to negative expected profits, and there would be a profitable deviation to \( h_{UK} = 0. \) The negative of the first derivative is now \( -\pi'(h_{UK}) = -\frac{\partial \pi_{UK}}{\partial h_{UK}} = \int_{-\infty}^{\infty} p_1(h_{UK} + s_1) + h_{UK} p'_1(h_{UK} + s_1) - \tilde{\theta} \varphi_s(s_1) ds_1 = \int_{-\infty}^{\infty} p(y_1) + h_{UK} p'(y_1 - \tilde{\theta}) \varphi_s(y_1 - h_{UK}) dy_1. \) The solution on \( -\pi'(h_{UK}) = 0 \) is a unique maximum if \( -\pi'(h_{UK}) \) is a strictly single crossing function—that is \( -\pi'(h) \geq 0 \) implies that \( -\pi'(h) > 0 \) for any \( 0 < h < \tilde{h}. \) Using the expression for \( -\pi'(h_{UK}) \) we require that \( (\tilde{h} - \tilde{h}) \int_{-\infty}^{\infty} p'(y_1) \varphi_s(y_1 - \tilde{h}) ds_1 + \int_{-\infty}^{\infty} (h p'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - \tilde{h}) dy_1 > 0. \) The first term is clearly positive. The second term can be written as
\[\int_{-\infty}^{\infty} (h p'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - \tilde{h}) \varphi_s(y_1 - h) dy_1, \] where \( \frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} \) is increasing in \( y_1 \) due to log-concavity (see Lemma A.1).

Notice that if \( h p'(y_1) + p(y_1) - \theta \) is single crossing in \( y_1, \) we can prove that this integral is non-negative similarly to Lemma 5 and Extension to Lemma 5 in Athey (2002). Namely, suppose that \( h p'(y_1) + p(y_1) - \theta \) is single crossing in \( y_1 \) then there exists \( y_1 = \tilde{y} \) such that \( h p'(y_1) + p(y_1) - \theta < (>) 0 \) for any \( y_1 < (>) \tilde{y}. \) Furthermore, it is clear that \( \frac{\varphi_s(y_1 - \tilde{h})}{\varphi_s(y_1 - h)} < \)
where the first inequality follows from the monotonicity of \( \frac{\varphi_s(y_1 - h)}{\varphi_s(y_1 - \tilde{h})} \). Namely, using Theorem A.5 from Appendix A, it holds that the sufficient condition for \( hp'(y_1) + p(y_1) - \theta \) to be single crossing in \( y_1 \) is that \( \frac{\theta - p(y_1)}{p'(y_1)} \) is decreasing in \( y_1 \), i.e., \( \theta - p(y_1) \) is log-concave.\(^{48}\)

We can also identify a somewhat more general sufficient condition for the term \( \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1 - h)}{\varphi_s(y_1 - \tilde{h})} \varphi_s(y_1 - h) \, dy_1 \) to be non-negative using Chebyshev’s integral inequality. Namely, using Theorem A.5 from Appendix A, it holds that the sufficient condition for \( \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \frac{\varphi_s(y_1 - h)}{\varphi_s(y_1 - \tilde{h})} \varphi_s(y_1 - h) \, dy_1 \geq \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h) \, dy_1 = -\pi'(h) \) is that for every \( t \)

\[
\frac{\int_{-\infty}^{t} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h) \, dy_1}{\int_{-\infty}^{t} \varphi_s(y_1 - h) \, dy_1} \geq \int_{-\infty}^{\infty} (hp'(y_1) + p(y_1) - \theta) \varphi_s(y_1 - h) \, dy_1.
\]

This condition can also be written as

\[
\mathbb{E} [hp'(h + s_1) + p(h + s_1) | s_1 \leq t - h] \geq \mathbb{E} [hp'(h + s_1) + p(h + s_1) | s_1 > t - h].
\]

As \( \mathbb{E} [p(h + s_1) | s_1 \leq t - h] \geq \mathbb{E} [p(h + s_1) | s_1 > t - h] \) due to the fact that the price is increasing in the order flow, this condition essentially requires that the slope of \( p(y_1) \) at high order flows is not too small compared to the slope at small order flows and is less restrictive than requiring \( hp'(y_1) + p(y_1) - \theta \) to be single crossing.

While numerically both sufficient conditions clearly hold for a wide set of parameters, to the best of our knowledge there are no more mathematical results that we can apply to our setting.

\(^{48}\)Using the results from Lemma 3, we can prove that this is indeed the case for \( \eta \) close to 0 or 1 and \( \gamma = 0. \)
to derive further analytical results. Overall, the necessary (and the least restrictive) condition for quasiconcavity is that if \( -\pi'(h) > 0 \) then
\[
(\hat{h} - h) \int_{-\infty}^{\infty} p'(y_1) \varphi_s(y_1 - \hat{h}) \; ds_1 + \int_{-\infty}^{\infty} (h p'(y_1) + p(y_1)) \left( \frac{\varphi_s(y_1 - \hat{h})}{\varphi_s(y_1 - h)} - 1 \right) \varphi_s(y_1 - h) \; dy_1 > 0,
\]
which appears to always hold, at least numerically.

As the problem is symmetric, similar arguments apply for \( \theta = -\tilde{\theta} \) as well as for the quasiconcavity in own demand in the state \( R = I \).

**Proof of Proposition 2**

We already know from the previous part that when \( \theta = 0 \), the unique solution is \( h_{1{UK}} = h_{1{IK}} = h_{1{IP}} = 0 \). So let us focus on the case \( \theta = \tilde{\theta} \). Provided that the trader’s problem has a unique maximum in own demand, we focus on the first order conditions.

\[
- \frac{\partial \pi_{1{UK}}}{\partial h_{1{UK}}} = \int_{-\infty}^{\infty} (p(y_1) + h_{1{UK}} p'(y_1) - \tilde{\theta}) \varphi_s(y_1 - h_{1{UK}}) \; dy_1 = 0
\]

\[
- \frac{\partial \pi_{1{IP}}}{\partial h_{1{IP}}} = \int_{-\infty}^{\infty} (p(y_1) + h_{1{IP}} p'(y_1) - \tilde{\theta}) \varphi_s(y_1 - h_{1{IP}} - h_{1{IP}}) \; dy_1 = 0
\]

which by integration by parts can be also expressed as

\[
- \frac{\partial \pi_{1{UK}}}{\partial h_{1{UK}}} = \int_{-\infty}^{\infty} (\tilde{\theta} - p(y_1)) \left( \frac{h_{1{UK}} (h_{1{UK}} - y_1)}{\sigma^2} - 1 \right) \varphi_s(y_1 - h_{1{UK}}) \; dy_1 = 0
\]

(25)

\[
- \frac{\partial \pi_{1{IP}}}{\partial h_{1{IP}}} = \int_{-\infty}^{\infty} (\tilde{\theta} - p(y_1)) \left( \frac{h_{1{IP}} (h_{1{IP}} + y_1)}{\sigma^2} - 1 \right) \varphi_s(y_1 - h_{1{IP}} - h_{1{IP}}) \; dy_1 = 0
\]

It is straightforward to prove that \( K \) and \( P \) and must trade the same quantity in equilibrium in state \( R = I \) and that the solution is symmetric for \( \theta = \tilde{\theta} \) and \( \theta = -\tilde{\theta} \). In equilibrium the Market’s beliefs must be consistent with optimal strategies, i.e., it must hold that \( h_{1{IK}} = h_{1{IP}} = \bar{q}_R \) and \( h_{1{UK}} = \bar{g}_U \). Define \( \mu_R \equiv \frac{\bar{q}_R}{\sigma_s} \) for \( R \in \{I, U\} \) and \( z \equiv \frac{y_R}{\sigma_z} \). Using the expression for normal density we can express \( \varphi_s(y_2 - \bar{g}_R) = \frac{1}{\sigma_s} \phi(z - \mu_R) \), \( \varphi_s(y_2) = \frac{1}{\sigma_s} \phi(z) \) and \( \varphi_s(y_2 + \bar{g}_R) = \frac{1}{\sigma_s} \phi(z + \mu) \), where \( \phi \) is the p.d.f. of a standard normal. Using (14), we then find that

\[
\bar{p}(z) \equiv p_1(z \sigma_s) = \tilde{\theta} \frac{1}{\sigma_z} \frac{\eta \phi(z - \mu) + (1 - \eta) \phi(z - \mu_U) - \eta \phi(z + \mu) + (1 - \eta) \phi(z + \mu_U)}{\eta \phi(z - \mu) + (1 - \eta) \phi(z - \mu_U) + \eta \phi(z + \mu) + (1 - \eta) \phi(z + \mu_U)} + \frac{2\gamma}{1 - \gamma} \phi(z)
\]

that clearly does not depend on \( \sigma_s \) and it holds that \( p_1(z \sigma_s) = -p_1(-z \sigma_s) \). Using these in (25) and equating \( \frac{\partial \pi_{1{IP}}}{\partial h_{1{IP}}} = 0 \) for \( J \in \{K, P\} \); \( \frac{\partial \pi_{1{UK}}}{\partial h_{1{UK}}} = 0 \), we find that \( \mu_I \) and \( \mu_U \) are the positive
solutions of
\[
\frac{\partial \pi_1^J}{\partial h_1^J} \bigg|_{h_1^J = \hat{g}_t} = - \left( 1 - \frac{\mu_I^2}{2} \right) \int_{-\infty}^{\infty} (\bar{\theta} - \tilde{p}(z)) \phi(z - \mu_I) \, dz + \frac{\mu_I}{2} \int_{-\infty}^{\infty} (\bar{\theta} - \tilde{p}(z)) z\phi(z - \mu_I) \, dz = 0
\]
(26)

\[
\frac{\partial \pi_U^K}{\partial h_1^K} \bigg|_{h_1^K = \hat{g}_U} = - \left( 1 - \frac{\mu_U^2}{2} \right) \int_{-\infty}^{\infty} (\bar{\theta} - \tilde{p}(z)) \phi(z - \mu_U) \, dz + \mu_U \int_{-\infty}^{\infty} (\bar{\theta} - \tilde{p}(z)) z\phi(z - \mu_U) \, dz = 0
\]

For part 2 notice that from (26), we can express the first order condition of trader \( J \in \{ K, P \} \) in state \( R = I \) as
\[
\frac{\partial \pi_1^J}{\partial h_1^J} \bigg|_{h_1^J = \hat{g}_t} = -\frac{1}{2} \int_{-\infty}^{\infty} (\bar{\theta} - \tilde{p}(z)) \phi(z - \mu_I) \, dz - \frac{1}{2} \frac{\partial \pi_U^K}{\partial h_1^K} \bigg|_{h_1^K = \hat{g}_U} = 0
\]

By part 6 of Lemma 3 \((\bar{\theta} - p_1(y_1)) > 0\) for all finite \( y_1 \). Therefore, also \((\bar{\theta} - \tilde{p}(z)) > 0\) for all finite \( z \) and \((\bar{\theta} - \tilde{p}(z)) \phi(z - \mu_I) \geq 0\) with strict inequality for some \( z \). This implies that it must hold that
\[-\frac{\partial \pi_U^K}{\partial h_1^K} \bigg|_{h_1^K = \hat{g}_U} > 0.
\]
Because \(-\frac{\partial \pi_U^K}{\partial h_1^K}\) is a single-crossing function and \(\frac{\partial \pi_1^J}{\partial h_1^J} \bigg|_{h_1^J = \hat{g}_U} = 0\), it then follows that \(\tilde{g}_t > \hat{g}_U\).

For the uninformed trader’s strategy, we need to verify that it is optimal for him to trade zero. We now verify that the first order condition of his problem indeed holds at zero. Define \(\Delta(y_1) \equiv \mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I]\) and \(Q_{1U}(y_1) \equiv \mathbb{E}[Q_2|y_1, U]\). By (15) in Lemma 3 (and also by Lemma A.1 in Appendix A), it holds that \(\Delta(y_1) = -\Delta(-y_1)\). Also, it is clear from (16) that it holds that \(Q_1(y_1) = Q_1(-y_1)\). Using this in (10) and (12) we confirm that (12) \(Q_{1U}(y_1) = Q_{1U}(-y_1)\).

Recalling uninformed \(P\)’s optimal trading strategy at date 2 from (13) in Theorem 1 and using (7) and (9), we can then find \(P\)’s expected profit at date 2 conditional on \(y_1\) as
\[
\pi_2^{UP} = \begin{cases} 
\sigma_s K Q_{1U}(y_1) \Delta(y_1) & \text{if } \Delta(y_1) > 0 \\
-\sigma_s K Q_{1U}(y_1) \Delta(y_1) & \text{if } \Delta(y_1) < 0 
\end{cases}
\]

Suppose that at date 1, uninformed \(P\) trades \(h_1^{UP}\), then he also knows that the distribution of the total order flow is \(f_s(y_1 - h_1^{UP} - \hat{g}_U)\) if \(\theta = \bar{\theta}; f_s(y_1 - h_1^{UP})\) if \(\theta = 0\) and \(f_s(y_1 - h_1^{UP} + \hat{g}_U)\) if \(\theta = -\bar{\theta}\). Using all this, \(\mathbb{E}[\theta|U] = 0\), and we can use the law of iterated expectations to express
the expected profit of uninformed $P$ before date 1 trading as

$$
\pi_1^{UP} = -h_1^{UP} \int_{-\infty}^{\infty} p_1(y_1) \left( m(y_1 - h_1^{UP}) - m(y_1 + h_1^{UP}) \right) dy_1 + \\
\int_{\Delta(y_1) > 0} \sigma_s \kappa_Q(y_1) \Delta(y_1) \left( m(y_1 - h_1^{UP}) + m(y_1 + h_1^{UP}) \right) dy_1,
$$

where $m(x) \equiv \frac{1-\gamma}{2} \varphi_s(x - \tilde{g}U) + \gamma \varphi_s(x) + \frac{1-\gamma}{2} \varphi_s(x + \tilde{g}U)$. Because of symmetry $\varphi_s(\cdot)$, it holds that $m(x) = m(-x)$ and $m'(x) = -m'(-x)$.

The first derivative of the profit is

$$
\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}} = -\int_{0}^{\infty} p_1(y_1) \left( m(y_1 - h_1^{UP}) - m(y_1 + h_1^{UP}) \right) dy_1 \\
+ h_1^{UP} \int_{0}^{\infty} p_1(y_1) \left( m'(y_1 - h_1^{UP}) + m'(y_1 + h_1^{UP}) \right) dy_1 \\
- \int_{\Delta(y_1) < 0} \sigma_s \kappa_Q(y_1) \Delta(y_1) \left( m(y_1 - h_1^{UP}) - m'(y_1 + h_1^{UP}) \right) dy_1
$$

Replacing in $h_1^{UP} = 0$, we can now verify that $\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}|_{h_1^{UP}=0} = 0$. For the intuition that $h_1^{UP} = 0$ is also a global maximum, notice that $-\int_{0}^{\infty} p_1(y_1) \left( m(y_1 - h_1^{UP}) - m(y_1 + h_1^{UP}) \right) dy_1 = \\
-\int_{-\infty}^{\infty} \left( p_1(s_1 + h_1^{UP}) - p_1(s_1 - h_1^{UP}) \right) m(s_1) ds_1$. Due to increasing prices, the first term is negative iff $h_1^{UP} > 0$. Also, using integration by parts, the second term is

$$
h_1^{UP} \int_{0}^{\infty} p_1(y_1) \left( m'(y_1 - h_1^{UP}) + m'(y_1 + h_1^{UP}) \right) dy_1 = -h_1^{UP} \int_{0}^{\infty} p_1'(y_1) \left( m(y_1 - h_1^{UP}) + m(y_1 + h_1^{UP}) \right) dy_1
$$

and negative iff $h_1^{UP} > 0$. Both of these effects alone would guarantee that $-\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}$ is strictly single crossing at 0 as any trading by $P$ at date 1 would lead to short term losses in expectations. The sign of the last term is ambiguous and reflects the fact that by deviating to a non-zero demand at date 1, $P$ could affect the probability he expects the market to assign on him being informed at date 2 and the area where $P$ would pursue different price-contingent strategies. However, it can be verified that this term is relatively small compared to the first two terms and $-\frac{\partial \pi_1^{UP}}{\partial h_1^{UP}}$ remains single crossing at 0. This is true at least as long as $\eta$ is not too close to one.

**Proof of Proposition 3**

Assuming $\gamma = 0$, we obtain from (15) that $\text{sgn} \left( \mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I] \right) = \\
\text{sgn} \left( \frac{\varphi_s(y_1 - \tilde{g}U)}{\varphi_s(y_1 + \tilde{g}U)} -\frac{\varphi_s(y_1 - \tilde{g}I)}{\varphi_s(y_1 + \tilde{g}I)} \right) = \text{sgn} \left( \frac{\varphi_s(y_1 - \tilde{g}U)}{\varphi_s(-y_1 - \tilde{g}U)} -\frac{\varphi_s(y_1 - \tilde{g}I)}{\varphi_s(-y_1 - \tilde{g}I)} \right)$. Because $\varphi_s(\cdot)$ is log-concave and $\tilde{g}_I > \tilde{g}U$ by part 2 in Proposition 2, it holds by the property of log-concave distributions in Lemma A.1 in Appendix A that $\text{sgn} \left( \frac{\varphi_s(y_1 - \tilde{g}U)}{\varphi_s(-y_1 - \tilde{g}U)} -\frac{\varphi_s(y_1 - \tilde{g}I)}{\varphi_s(-y_1 - \tilde{g}I)} \right) = -1$ if $y_1 > -y_1 \Leftrightarrow y_1 > 0$ and
We can then prove that the necessary and sufficient condition for \( \text{sgn} \) is that

\[
B(y_1) \equiv \varphi_s(y_1 - \bar{g}_U) - \varphi_s(y_1 + \bar{g}_U) - \varphi_s(y_1 - \bar{g}_I) + \varphi_s(y_1 + \bar{g}_I).
\]

Consider \( y_1 > 0 \) and let us focus on the sign of \( B(y_1) \). Because \( \varphi_s(.) \) has a maximum at zero and is decreasing for any positive values, it also holds for any \( y_1 > 0 \) and \( \bar{g}_I > \bar{g}_U \) that \(-\varphi_s(y_1 + \bar{g}_U) + \varphi_s(y_1 + \bar{g}_I) < 0\).

We can then prove that the necessary and sufficient condition for \( \varphi_s(y_1 - \bar{g}_U) - \varphi_s(y_1 - \bar{g}_I) \leq 0 \) is that \( y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2} \). Namely, defining \( b \equiv y_1 - \frac{\bar{g}_I + \bar{g}_U}{2} \), it holds that \( \varphi_s(y_1 - \bar{g}_U) - \varphi_s(y_1 - \bar{g}_I) = \varphi_s(b + \frac{\bar{g}_I + \bar{g}_U}{2}) - \varphi_s(b - \frac{\bar{g}_I + \bar{g}_U}{2}) \), which is indeed non-positive if and only if \( b \geq 0 \) (see Corollary A.1.1 in Appendix A and recall that \( \bar{g}_I > \bar{g}_U \)). Therefore, for any \( y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2} \) it holds that \( B(y_1) < 0 \). From the proof of Proposition 3 (and Lemma A.1 in Appendix A), we already know that \( \varphi_s(y_1 - \bar{g}_U) < \varphi_s(y_1 - \bar{g}_I) \) for any \( y_1 > 0 \). Therefore, \( \text{sgn} \left( \mathbb{E} [\theta | y_1, U] - \mathbb{E} [\theta | y_1, I] \right) = -1 \) for any \( y_1 \geq \frac{\bar{g}_I + \bar{g}_U}{2} \) and \( 0 < \eta < 1 \). The definition of contrarian strategy in Section 4.2 completes this part of the proof. The proof for \( y_1 \leq -\frac{\bar{g}_I + \bar{g}_U}{2} \) is similar due to symmetry.

To prove part 2, notice that the function determining the sign of \( \mathbb{E} [\theta | y_1, U] - \mathbb{E} [\theta | y_1, I] \) can be expressed as

\[
S(y_1) = \left( \exp \left( \frac{2\bar{g}_U y_1}{\sigma^2} \right) - 1 \right) \left( 1 + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1 + \bar{g}_U)} - \left( \frac{\varphi_s(y_1 - \bar{g}_U)}{\varphi_s(y_1 + \bar{g}_I)} - 1 \right) \left( 1 + \frac{\gamma f(y_1)}{(1-\gamma)\varphi_s(y_1 + \bar{g}_I)} \right) \right),
\]

which using the expression for the normal density becomes

\[
S(y_1) = \left( \exp \left( \frac{2\bar{g}_U y_1}{\sigma^2} \right) - 1 \right) \left( 1 + \frac{\gamma}{1-\gamma} \exp \left( \frac{2\bar{g}_I y_1 + \bar{g}_U^2}{2\sigma^2} \right) \right) - \left( \exp \left( \frac{2\bar{g}_U y_1}{\sigma^2} \right) - 1 \right) \left( 1 + \frac{\gamma}{1-\gamma} \exp \left( \frac{2\bar{g}_I y_1 + \bar{g}_U^2}{2\sigma^2} \right) \right) - \left( \exp \left( \frac{2\bar{g}_I y_1 + \bar{g}_U^2}{2\sigma^2} \right) - 1 \right) \left( 1 + \frac{\gamma}{1-\gamma} \exp \left( \frac{2\bar{g}_I y_1 + \bar{g}_U^2}{2\sigma^2} \right) \right).
\]

It is clear that \( S(0) = 0 \). Let us consider \( \varepsilon \) arbitrarily close to zero. By Taylor approximation, we find that \( S(\varepsilon) = S'(0) \varepsilon \), where \( S'(0) = \frac{2\bar{g}_U}{\sigma^2} \left( 1 + \frac{\gamma}{1-\gamma} \exp \left( \frac{\bar{g}_I^2}{2\sigma^2} \right) \right) - \frac{2\bar{g}_I}{\sigma^2} \left( 1 + \frac{\gamma}{1-\gamma} \exp \left( \frac{\bar{g}_U^2}{2\sigma^2} \right) \right) \).

Using then \( \bar{g}_R = \mu R \sigma_s \), it is clear that \( S'(0) > 0 \) if (19) holds, which by \( \text{sgn} \left( \mathbb{E} [\theta | \varepsilon, U] - \mathbb{E} [\theta | \varepsilon, I] \right) = \text{sgn} \left( S(\varepsilon) \right) = 1 \) \(-1\) if \( \varepsilon > (\varepsilon) 0 \) and definition from Section 4.2 implies a trend-following strategy.

If the condition (19) hold, then \( S(\varepsilon) > 0 \) for some small \( \varepsilon > 0 \), while \( S\left( \frac{\bar{g}_I + \bar{g}_U}{2} \right) < 0 \). As \( S(y_1) \) is a continuous function of the order flow \( y_1 \), there must exist an order flow in the interval of \((\varepsilon, \frac{\bar{g}_I + \bar{g}_U}{2})\), where \( S(y_1) \) changes its sign.
Proof of Lemma 5
Assume that the prior distribution of the fundamental is \( f(\theta) \), with support \([-\bar{\theta}, \bar{\theta}]\).\(^49\) We prove the lemma by contradiction. Suppose that \( P \) does not trade at date 2 in state \( R = U \). It is first useful to prove the following Claim

**Claim 1** The total volume traded by informed traders at date 2 differs across states.

**Proof.** Let us differentiate (20) and (21) wrt, \( h_2^{UK}, h_2^{IK} \) and \( h_2^{IP} \). We can then define the total informed order flow at date 2, in state \( R = U \) as \( g_{2U}(\theta, y_1) = c \) and in state \( R = U \) as \( g_{2I}(\theta, y_1) = h_2^{IK} + h_2^{IP} \), we obtain that the following first order conditions must hold in equilibrium

\[
\begin{align*}
\int_{-\infty}^{\infty} (\theta - p_2(y_2)) \varphi_s (y_2 - g_{2I}(\theta, y_1)) \left( 2 + \frac{g_{2I}(\theta, y_1) (y_2 - g_{2I}(\theta, y_1))}{\sigma_s^2} \right) dy_2 &= 0 \\
\int_{-\infty}^{\infty} (\theta - p_2(y_2)) \varphi_s (y_2 - g_{2U}(\theta, y_1)) \left( 1 + \frac{g_{2U}(\theta, y_1) (y_2 - g_{2U}(\theta, y_1))}{\sigma_s^2} \right) dy_2 &= 0
\end{align*}
\]

and \( g_{2I}(\theta, y_1), \) and \( g_{2U}(\theta, y_1) \) are generally non-zero. Suppose that informed trading volume is the same across states: \( g_{2I}(\theta, y_1) = g_{2U}(\theta, y_1) = g_2(\theta, y_1) \). From above we obtain then that it must hold that \( \int_{-\infty}^{\infty} (\theta - p_2(y_2)) \varphi_s (y_1 - g(\theta, y_1)) dy_2 = 0 \) for all \( \theta \). But this leads to a contradiction, as it would imply that informed traders obtain zero profits when trading optimal quantity. Hence, it must be the case that \( g_{2I}(\theta, y_1) \neq g_{2U}(\theta, y_1) \) ■

Let us then consider uninformed \( P \)'s date 1 problem. For uninformed \( P \) not to have incentives to deviate from \( h_2^{UP} = 0 \), it must hold that the first derivative of his date 2 problem (5) \( \frac{\partial \pi_2^{UP}}{\partial h_2^{UP}} |_{h_2^{UP}=0} = E \left[ (\theta - p_2) | y_1, U \right] + \frac{\partial E[(\theta - p_2) | y_1, U]}{\partial h_2^{UP}} |_{h_2^{UP}=0} = 0 \). This implies that

\[
\begin{align*}
E \left[ (\theta - p_2) | y_1, U \right] &= E \left[ E \left[ (\theta - p_2) | \theta, y_1, U \right] | y_1, U \right] = \\
&= \frac{1 - \eta}{1 - Q_1} \int_{-\theta}^{\theta} \int_{-\infty}^{\infty} (\theta - p_2) \varphi_s (y_1 - g_{2U}(\theta, y_1)) \varphi_s (y_1 - g_{1U}(\theta)) f(\theta) dy_2 d\theta = 0,
\end{align*}
\]

where \( g_{1U}(\theta) \) is date 1 total order flow by informed trader \( K \). Denoting the date 1 order flow by informed traders in state \( R = I \), with \( g_{1I}(\theta) \),

\[
\begin{align*}
E \left[ (\theta - p_2) | y_1, I \right] &= E \left[ E \left[ (\theta - p_2) | \theta, y_1, I \right] | y_1, I \right] = \\
&= \frac{\eta}{Q_1} \int_{-\theta}^{\theta} \int_{-\infty}^{\infty} (\theta - p_2) \varphi_s (y_1 - g_{2I}(\theta, y_1)) \varphi_s (y_1 - g_{1I}(\theta)) f(\theta) dy_2 d\theta \neq 0,
\end{align*}
\]

\(^{49}\)We consider continuous distribution, but it is straightforward to extend the proof also to discrete distributions.
as it is sufficient that $\eta > 0$ and $g_{2I}(\theta, y_1) \neq g_{2U}(\theta, y_1)$. It must therefore also hold that

$$
\mathbb{E}[(\theta - p_2) | y_1] = Q_1 \mathbb{E}[(\theta - p_2) | y_1, I] + (1 - Q_1) \mathbb{E}[(\theta - p_2) | y_1, I] = Q_1 \mathbb{E}[(\theta - p_2) | y_1, I] \neq 0
$$

However, this leads to a contradiction because it violates the market efficiency condition that must hold in any equilibrium, i.e., by market efficiency conditions and law of iterated expectations, $\mathbb{E}[p_2|y_1] = \mathbb{E}[\mathbb{E}[\theta|y_1, y_2]|y_1] = \mathbb{E}[-\theta|y_1] \iff \mathbb{E}[(\theta - p_2) | y_1] = 0$. Therefore $h_{2P}^U = 0$ cannot be an equilibrium strategy (for all realizations of $y_1$).

**Proof of Proposition 6**

Recall from (13) and the definition of strategies in Section 4.2 that $P$ will optimally pursue and profit from contrarian strategy if $\mathbb{E}[\theta|y_1, U] - p_1$ has an opposite sign compared to $y_1$. Then from (7), it holds that $\mathbb{E}[\theta|y_1, U] - p_1 = Q_1(\mathbb{E}[\theta|y_1, U] - \mathbb{E}[\theta|y_1, I])$. It is clear that without the presence of any informed traders at date 1, no fundamental information can be contained in date 1 order flow in state $R = U$, i.e., with any symmetric prior $\mathbb{E}[\theta|y_1, U] = \mathbb{E}[\theta|U] = 0$. Therefore, $\mathbb{E}[\theta|y_1, U] - p_1 = -p_1 = -Q_1 \mathbb{E}[\theta|y_1, I]$. Also, similarly to the proof of Proposition 7, we know that $g(\theta)$ must be increasing in $\theta$ and symmetric $g(\theta) = g(-\theta)$. These facts and the log-concavity of noise trading together with Lemma A.2 in Appendix A ensure that $\mathbb{E}[\theta|y_1, I]$ is an odd function that is increasing in $y_1$ and the sign of $\mathbb{E}[\theta|y_1, U] - p_1$ is opposite to the sign of $y$. Using Bayes’ rule $Q_1 = \frac{\eta f(y_1 | I)}{f(y_1 | I) + (1 - \eta) f(y_1 | U)} = \frac{\eta}{\eta + (1 - \eta) \int f(y_1 | I) d\theta}$, where $f(y_1 | U) = \int \nu_y(\theta) f(\theta) d\theta$. As $\frac{f(y_1 | I)}{f(y_1 | U)}$ is finite, it is clear that $\eta > 0$ is sufficient for $Q_1 > 0$.

**Proof of Proposition 7**

Part 1 follows from results in the monotone comparative statics literature that we can use to explore informed traders’ profits (4) and (3). Denote in state $R = U$ trader $K$’s expected price when demanding $h_{1K}^U$ as $p_E(h_{1K}^U) \equiv \int p_1(h_{1K}^U + s_1) f_1(s_1) ds_1$. From Milgrom and Shannon (1994) it is known that $g_U(\theta) = \arg \max_{h_{1K}} h_{1K}^U (\theta - p_E(h_{1K}^U))$ is weakly increasing in $\theta$ if the trader’s problem has increasing differences (which also implies the payoff is supermodular) in $h_{1K}^U$ and $\theta$. This is indeed true as for any $\tilde{\theta} > \theta$ and $\tilde{h}_{1K}^U > h_{1K}^U$, it holds that $\tilde{h}_{1K}^U (\tilde{\theta} - p_E(\tilde{h}_{1K}^U)) - \tilde{h}_{1K}^U (\theta - p_E(\tilde{h}_{1K}^U)) > h_{1K}^U (\tilde{\theta} - p_E(h_{1K}^U)) - h_{1K}^U (\theta - p_E(h_{1K}^U)) \iff (\tilde{h}_{1K}^U - h_{1K}^U)(\tilde{\theta} - \theta) > 0$. From Edlin and Shannon (1998), it is also known that $g_U(\theta)$ is strictly increasing if the first derivative of the payoff (profit) is strictly increasing in $\theta$, which is also true in our model, as $\partial h_{1K}^U (\theta - p_E(h_{1K}^U)) / \partial h_{1K}^U = \theta - p_E(h_{1K}^U) - h_{1K}^U p_E(h_{1K}^U)$ is clearly

\[^{50}\]Without explicitly solving, we expect and impose in this expression that $P$ does not trade in date 1 equilibrium if uninformed. Similar arguments would also hold if there existed an equilibrium where $P$ trades at date 1.
increasing in $\theta$. The proof is similar for the state $R = I$, where the same monotone comparative statics establish that $K$’s and $P$’s individual demand is increasing in $\theta$, and so is the sum of their demands.

For part 2, notice that given the above assumptions, it is enough to only look at the first order conditions to find the unique equilibrium demands by all informed traders. Also, it is easy to verify that given $\theta$, both $K$ and $P$ demand the same quantity in state $R = I$. We find that the equilibrium total informed demand $g_R(\theta)$ in state $R$ solves

$$
\theta = \int_{-\delta}^{\delta} \left( p_1(s_1 + g_U(\theta)) + g_U(\theta) p_1'(s_1 + g_U(\theta)) \right) f_s(s_1) \, ds_1
$$

$$
\theta = \int_{-\delta}^{\delta} \left( p_1(s_1 + g_I(\theta)) + \frac{g_I(\theta)}{2} p_1'(s_1 + g_I(\theta)) \right) f_s(s_1) \, ds_1
$$

As by part 1 $g_R(\theta)$ is invertible, it must also hold that

$$
g_U^{-1}(y_\theta) = \int_{-\delta}^{\delta} \left( p_1(s_1 + y_\theta) + y_\theta p_1'(s_1 + y_\theta) \right) f_s(s_1) \, ds_1
$$

$$
g_I^{-1}(y_\theta) = \int_{-\delta}^{\delta} \left( p_1(s_1 + y_\theta) + \frac{y_\theta}{2} p_1'(s_1 + y_\theta) \right) f_s(s_1) \, ds_1,
$$

where an order flow $y_\theta \equiv g_R(\theta)$. It is straightforward to verify that $g_R(\theta) = -g_R(-\theta)$ and clearly $y_\theta > 0$ iff $\theta > 0$. Because date 1 equilibrium price is increasing in the order flow, it holds that $g_U^{-1}(y_\theta) - g_I^{-1}(y_\theta) = \frac{y_\theta}{2} \int_{-\delta}^{\delta} p_1'(s_1 + y_\theta) f_s(s_1) \, ds_1 > (>) 0$ for any $y_\theta > (>) 0$. Taken $y_\theta = g_U(\theta) > 0$, we find that $g_U^{-1}(y_\theta) > g_I^{-1}(y_\theta) \iff \theta > g_I^{-1}(g_U(\theta)) \iff g_I(\theta) > g_U(\theta)$ for any $\theta > 0$. The case $y_\theta < 0$ is immediate by symmetry.
C Date 2 demand by strategic traders in the model with forward looking informed traders

Date 2 demand as a function of date 1 order flow. The order flow is normalized by dividing it with the standard deviation of noise trading. We assume \( \tilde{\theta} = 1 \) and \( \eta = 0.5 \).
References


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D Supplementary Appendix (For Online Publication Only)

D.1 Normal prior

We consider two cases. First, we explore a particular dependence structure of $R$ and $\theta$ that guarantees the linearity of date 1 equilibrium and is thus directly comparable to the benchmark of Kyle (1985). Second, we explore a normal prior independent of the state that is comparable with our baseline setting.

D.1.1 Normal prior with a particular dependence structure

Assume that

$$ f_{\theta R}(\theta) = \text{Pr}(R|\theta) f(\theta) = \text{Pr}(R) f(\theta|R) = $$

$$ = \frac{1}{\sqrt{2\pi}} \left( \frac{\eta}{\sigma_I} \exp \left( - \frac{\theta^2}{2\sigma_I^2} \right) \right) \tilde{I}_\theta \left( \frac{1 - \eta}{\sigma_U} \exp \left( - \frac{\theta^2}{2\sigma_U^2} \right) \right)^{1 - \tilde{I}_\theta}, $$

where $\tilde{I}_\theta$ is an indicator function that takes values $\tilde{I}_\theta = 1$ if $R = I$ and $\tilde{I}_\theta = 0$ if $R = U$ and where $\sigma_I$ and $\sigma_U$ are the standard deviations of the prior distribution in state $R = I$ and $R = U$ respectively. Furthermore, assume that $\sigma_I = \frac{3}{4} \sigma_U$. As shown shortly, this assumption will guarantee that date 1 order flow does not allow updating the types and therefore preserves linearity.

We solve this problem using the standard technique. We conjecture and later verify that there is a rational expectations equilibrium where $P$ does not trade in date 1 and that date 1 price is linear in the order flow, i.e.,

$$ p_1 = \lambda_1 y_1, $$

where $\lambda_1$ is a constant to be solved for in the equilibrium.\footnote{The problem can also be solved without immediately imposing this conjecture (see, e.g., Cho and El Karoui, (2000)).}

Lemma C.1 When the prior is given by (28), then there exists a rational expectations equilibrium where the following holds.

1. Informed traders’ optimal demand in date 1 is

$$ h_{1U}^I = \frac{\theta}{2\lambda_1}; \quad h_{1K}^I = h_{1P}^I = \frac{\theta}{3\lambda_1} $$

$$ g_U(\theta) \equiv h_{1U}^I = \frac{\theta}{2\lambda_1} \text{ and } g_I(\theta) \equiv h_{1K}^I + h_{1P}^I = \frac{2\theta}{3\lambda_1} $$
2. There is no updating about $P$’s type in the first trading round, i.e.,

$$Q_1 = \Pr(I|y_1) = \eta.$$

3. Equilibrium price is given by

$$p_1 = \lambda_1 y_1,$$

where $\lambda_1 = \frac{\sigma_U}{2\sigma_s}\sqrt{\frac{(2-\eta)}{2}}$ and it holds that

$$\mathbb{E}[\theta|y_1, I] = \frac{3}{4-\eta}\lambda_1 y_1 < p_1 < \frac{4}{4-\eta}\lambda_1 y_1 = \mathbb{E}[\theta|y_1, U].$$

4. Uninformed $P$ does not trade at date 1, $h_1^{UP} = 0$.

**Proof.** Part 1: Given the conjectured price (29) and the total order flow (6), we find that in state $R = U$, the informed trader’s expected profit (4) is given by $\mathbb{E}[h_2^{UK}(\theta - p_1)] = h_2^{UK}(\theta - \lambda_1 h_2^{UK})$ and $K$’s optimal demand is $h_1^{UK} = \frac{\theta}{2\lambda_1}$. From (29) and (6), trader $J$’s expected profit is $\mathbb{E}[h_2^{IJ}(\theta - p_1)] = h_2^{IJ}(\theta - \lambda_1 h_2^{IJ} - \lambda_1 h_2^{IJ})$, where $J, \bar{J} \in \{K, P\}$ and $J \neq \bar{J}$ and we find that the optimal demand is the same for $K$ and $P$, and $h_1^{IK} = h_1^{IP} = \theta/3\lambda_1$. Part 2: The total order flow at date 1 in state $R = U$ is $y_2 = \theta/2\lambda_1 + s_1$. As $\theta|U \sim N(0, \sigma_U^2)$ it holds that $y_2|U \sim N(0, \sigma_U^2/4\lambda_1^2 + \sigma_s^2)$. The total order flow in state $R = I$ is $y_2 = \frac{2\theta}{3\lambda_1} + s_1$. Using that $\sigma_I = \frac{3}{4}\sigma_U$, $\theta|I \sim N(0, \sigma_U^2/9) \sim N(0, 9\sigma_U^2/16)$ and $y_2|I \sim N(0, 4\sigma_U^2/9\lambda_1^2 + \sigma_s^2) \sim N(0, \sigma_U^2/4\lambda_1^2 + \sigma_s^2)$. Clearly $f(y_1|U) = f(y_1|I)$, so by Bayes’ rule $Q_1 = \Pr(I|y_1) = \frac{\eta f(y_1|I)}{\eta f(y_1|I) + (1-\eta)f(y_1|U)} = \eta$. Part 3: If $R = U$, then the signal that the Market obtains from the order flow is $2\lambda_1 y_2 = \theta + 2\lambda_1 s_1$, where $2\lambda_1 y_2 \sim N(\theta, 4\lambda_1^2 \sigma_s^2)$. As well known in the case of normally distributed prior and signals, the posterior is a precision-weighted average of the signals, hence we can simplify $\mathbb{E}[\theta|y_1, U] = \frac{\sigma_U^2/4\lambda_1^2 + \sigma_s^2}{2\lambda_1 y_2}$. If $R = I$ then the signal that the Market obtains from the order flow is $\frac{3}{2}\lambda_1 y_2 = \theta + \frac{3}{2}\lambda_1 s_1$, where $\frac{3}{2}\lambda_1 y_2 \sim N(\theta, \frac{9}{4}\lambda_1^2 \sigma_s^2)$. Using $\sigma_I = \frac{3}{4}\sigma_U$ and simplifying, we find $\mathbb{E}[\theta|y_1, I] = \frac{\sigma_U^2/4\lambda_1^2 + \sigma_s^2}{3\lambda_1 y_2}$. Given this and $Q_1 = \eta$, (7), we obtain that $p_1 = \frac{\sigma_U^2/4\lambda_1^2 + \sigma_s^2}{2\lambda_1 y_2}$. Equating coefficients with those in the conjectured prices (29), we find that $\lambda_1^2 = \frac{\sigma_U^2}{4\lambda_1^2 + \sigma_s^2} (2 - \eta) / 8\sigma_s^2$ and the positive solution of this proves the first part of the proposition. We then use the equilibrium value of $\lambda_1$ in the expressions of $\mathbb{E}[\theta|y_1, U]$ and $\mathbb{E}[\theta|y_1, I]$. For part 4, it is easy to verify that $h_1^{UP} = 0$ satisfies the first order condition. ■

We find from Lemma C.3. that while uninformed $P$ optimally does not trade at date 1, the first trading round generates his information advantage for date 2 trading. We can see that for any non-zero order flow, there will be a difference between the expected value conditional on knowing the state, $\mathbb{E}[\theta|y_1, R]$, and date 1 equilibrium price. Furthermore, by the definition of $P$’s strategy in Section 4.2 and Theorem 1, we find that in this setting $P$ always pursues a
trend-following trading strategy in date 2, i.e., he buys at date 2 if date 1 order flow is positive (or equivalently, if the price has increased at date 1) and sells when date 1 order flow is negative (or equivalently, if the price has decreased).\footnote{When we refer to price changes at date 1, we adopt the convention that date 0 price equals the prior $p_0 = \mathbb{E}[\theta] = 0$, which is consistent with our assumption of market efficiency.}

Under this prior, uninformed $P$ always perceives the price at date 2 to be too close to the prior mean. Indeed, the sensitivity of prices to order flow, $\lambda_1$, in Theorem 1 is too low given that the true state is $R = U$. (If the Market knew the state, he would set $\lambda_1 = \frac{\sigma_y}{2\sigma_p}$ as in Kyle (1985)). This is the outcome of two competing effects. The first effect is related to the value of the fundamental revealed by the order flow alone. The fundamental is the inverse of total informed trader’s demand at $y_1 - s_1 \theta = g_{i1}^{-1}(y_1 - s_1) = 2\lambda_1(y_1 - s_1)$ and $\theta = g_{i1}^{-1}(y_1 - s_1) = \frac{2}{3}\lambda_1(y_1 - s_1)$. It is clear that, conditional on the order flow alone, the expected value of the fundamental is higher in state $R = U$. This is true because the expected value $2\lambda_1 > \frac{3}{2}\lambda_1$. This effect tends to make the Market set the price too low at date 1. The second effect relates to the dispersion of the fundamental conditional on the order flow alone, which is higher in $R = U$ (because the variance $4\lambda_1^2\sigma_p^2 > \frac{9}{4}\lambda_1^2\sigma_p^2$). Because the Market uses Bayes’ rule to set the price and puts higher weight on signals that are less noisy, this effect tends to make the Market set the price too high at date 1. With a normal prior given by (28), the first effect always dominates.

D.1.2 Normal prior independent of the state

We now consider the prior density of the fundamental $f_\theta(\theta) = \frac{1}{\sqrt{2\pi\sigma_p}} \exp\left(-\frac{\theta^2}{2\sigma_p^2}\right)$ and assume that the fundamental is independent of the state, i.e., $f(\theta|R) = f_\theta(\theta)$ for $R \in \{I, U\}$. We can no longer conjecture that the price is linear in the order flow, because the Market will learn about the state from the order flow. Instead, we conjecture that the Market believes that total informed trading in state $R \in \{I, U\}$ is $g_R(\theta) = -g_R(\theta)$ and $P$ does not trade at date one if the state is $R = U$. Conditional on the state, $\mathbb{E}[\theta|y_1, R] = \int_{-\infty}^{\infty} \theta f(\theta|y_1, R) d\theta$ as we know the distributions and by Bayes’ rule it holds that $f(\theta|y_1, R) = \frac{f(y_1|\theta, R) f(\theta|R)}{f(y_1|R)} = \frac{\varphi_s(y_1 - g_R(\theta)) f_\theta(\theta)}{\int_{y_1|R} \varphi_s(y_1 - g_R(\theta)) f_\theta(\theta) d\theta}$ and $f(y_1|R) = \int_{-\infty}^{\infty} \varphi_s(y_1 - g_R(\theta)) f_\theta(\theta) d\theta$. Also by Bayes’s rule $Q_1 = \Pr(I|y_1) = \frac{\eta f(y_1|I)}{\eta f(y_1|U) + \eta f(y_1|I)}$. Using this in (7), we find that date one price is

$$p_1(y_1) = \frac{\int_{-\infty}^{\theta} (\eta \varphi_s(y_1 - g_I(\theta)) + (1 - \eta) \varphi_s(y_1 - g_U(\theta))) f_\theta(\theta) d\theta}{\int_{-\infty}^{\theta} (\eta \varphi_s(y_1 - g_I(\theta)) + (1 - \eta) \varphi_s(y_1 - g_U(\theta))) f_\theta(\theta) d\theta}.$$ (30)

It is easy to verify that $p_1(y_1) = -p_1(-y_1)$.

To characterize the equilibrium, we also need the first order conditions of the trader’s prob-
lem.\textsuperscript{53} After taking the first order conditions in (4) and (3), it is easy to verify that it must hold that in state $R = I$, both informed traders trade the same optimal quantity $h^I_{IK*} = h^U_{1P*}$. Imposing then that the equilibrium beliefs must be consistent with the actual trades, we find after changing variables and simplifying that $g_U(\theta)$ and $g_I(\theta)$ solve

$$
\theta = \int_{-\infty}^{\infty} p_1(y_1) \left( 1 - \frac{g^2_U(\theta)}{\sigma_s^2} + \frac{g_U(\theta) y_1}{\sigma_s^2} \right) \varphi_s(y_1 - g_U(\theta)) \, d\theta \quad (31)
$$

and it is straightforward to verify the symmetry of strategies: $g_R(\theta) = -g_R(-\theta)$. Furthermore, Proposition 2 applies and it must hold that $g_I(\theta) > g_U(\theta)$ for any $\theta > 0$. Equations (30) and (31) characterize the functions that determine equilibrium strategies and price. To derive the analytical solution, we can approximate $g_R(\theta)$ with a polynomial, derive the price (30) and change the constants in the polynomial until (31) holds.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{plots.png}
\caption{Date 1 price $p_1$, updated probability $Q_1 = \Pr(I|y_1)$, and the difference in expectations $\mathbb{E}[\theta|y_1,U] - \mathbb{E}[\theta|y_1,U]$ as a function of the order flow.}
\end{figure}

For the numerical exercise assume that $\eta = 0.5$, which is the case in which there is most updating about the state $R$ and hence the solution should in principle be most non-linear. Without loss of generality assume $\sigma_s = 1$ and $\sigma_p = 1$ (note that similarly to other settings in this paper, it can be verified that informed trading is proportional to the noise trading variance $\sigma_s$). It turns out that informed trader’s strategies do not need to be approximated with a high order polynomial, but are already very well approximated by a linear function, namely $g_U(\theta) \approx 1.0284 \cdot \theta$ and $g_U(\theta) \approx 1.3712 \cdot \theta$. Figure (D.1.2) presents the relevant results. The

\textsuperscript{53}In particular, it will be shown shortly that the price and equilibrium strategies are almost linear, and therefore quasiconcavity of the trader’s problem is trivial to verify ex post.
reason why trader’s strategies are close to linear is that the price under linear strategies is "almost linear," i.e., the north-west panel of Figure (D.1.2) shows that it is hard to notice nonlinearity of price (the $R^2$ of the trendline is effectively 1) - only when we zoom in and show the difference between the price and a linear trendline (south-west panel), do we see that it is slightly non-linear. Because informed traders care about the expected price that they do not know when they submit their orders, these small nonlinearities have very little effect on their optimal strategies. On the north-east panel we see that there is some but limited updating of trader’s types. Because two informed traders jointly trade more than one in absolute value, larger order flows in absolute value tend to signal a higher probability that the state is $R = I$. At small order flows, the Market tends to believe that the state is $R = U$, but even at zero, there is not much learning about $P$’s type and therefore $P$’s trading opportunities remain. Finally on the south-east panel, we see that the direction of $P$’s trading at date 2 is the same as in Section 4.3 - trend-following. For any $y_1 > (<) 0$, it holds that $\mathbb{E}[\theta|y_1, U] > (<) \mathbb{E}[\theta|y_1, I] \Leftrightarrow \mathbb{E}[\theta|y_1, U] > (<) p_1$. 
