Riding the Bubble with Convex Incentives*

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ABSTRACT

We offer a simple portfolio choice-based rationale for the documented failure of institutional investors to trade against mispricing. In a dynamic model where an informed and risk-averse money manager faces convex (i.e., option-like) incentives, we show that the standard mean-variance and hedging components that make up the portfolio often imply opposite stances against mispricing. The first component represents the manager's bets against overpriced securities. By contrast, the manager's hedge against the risk of forfeiting an end-of-period performance fee can result in substantial over-investment in overpriced securities. This "bubble-riding" component is more likely to drive the manager's portfolio as overpricing increases. Although we do not model the informed manager's price impact, our analysis suggests that the incentives of sophisticated investors might lead them to exacerbate security mispricing.

Keywords: Money management, convex incentives, incomplete information, mispricing.

JEL Classification: D81, D82, D83, G11, G23.

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1 Introduction

The role of institutional investors in enhancing financial markets efficiency has come under increased scrutiny following recent episodes of prolonged—perceived—mispricing. Indeed, several empirical studies document that hedge funds, as well as mutual funds and independent investment advisors, were heavily invested in technology stocks during the "tech bubble" of the late 1990s.[1] Such over-investment in overvalued securities, or "bubble-riding", has often been associated with herd behavior resulting from career concerns. Yet, some authors argue that career concerns should be substantially lessened by the use of short-term incentive contracts.[2] Bubble-riding by institutional investors thus seems particularly puzzling in light of the widespread use of pay-for-performance and the high power of the incentives commonly observed in the asset management industry.

In this paper, we show the opposite effect: high-power incentives of money managers can lead to the bubble-riding behavior observed in the literature. Using a dynamic investment model with no career concerns, we show that short-term contracts may induce bubble-riding when they include "convex" incentives, i.e. those that reward good performance more than they penalize poor outcomes. In the model, convex incentives induce large changes in effective risk aversion as a function of the manager's performance relative to a benchmark. Bubble-riding arises in the model as the manager either increases the portfolio exposure to a risky asset ("risk shifting") or mimics the benchmark ("indexing") in response to the changes in effective risk aversion. Whereas risk-shifting and indexing have been previously studied in the literature, we argue that under mispricing both behaviors can precisely imply investing less aggressively against mispriced securities than in the absence of convex incentives. In addition, this effect worsens with the level of mispricing.

Convex incentives are ubiquitous in the money management industry. Among hedge funds, an explicit convexity arises from their typical fee structure. This structure includes a flat management fee plus a "bonus" performance fee—normally, several times larger than the management fee—over profits in excess of a hurdle performance rate or a high-water mark. In the mutual fund industry, an implicit convexity results from the relation between a fund's performance and its clients' share purchases and redemptions. Indeed, an extensive literature (see, e.g., Chevalier and Ellison (1997), and Sirri and Tufano (1998)) documents that mutual fund inflows after good performance largely exceed outflows following poor returns. Since mutual funds' revenue is commonly proportional to their assets under management (AUM), such a convex flow-performance relationship suggests an implicit option-like relation between mutual fund performance and managerial compensation.[3]

[1] See, e.g., Brunnermeier and Nagel (2004), Greenwood and Nagel (2009), and Griffin, Harris, Shu, and Topaloglu (2011).

[2] See, e.g., Scharfstein and Stein (1990) and Dass, Massa, and Patgiri (2008).

[3] A further source of convexity in mutual fund managers' incentives is created by the prevalence of bonus payments in their end-of-year compensation packages (see, e.g., Farnsworth and Taylor (2006), and Ma, Tang,

We incorporate this type of non-linear incentives into the dynamic investment problem of a money manager who trades in potentially mispriced assets over a finite horizon. The money manager in our model is risk averse and subject to convex incentives. This convexity follows from a "bonus" component in the manager's compensation which is contingent on good performance relative to a given benchmark. Rather than adopting arbitrary dynamics for security mispricing, we assume that mispricing arises from the trading of "uninformed" investors in the economy, whose aggregate asset holdings represent the market portfolio. Specifically, we assume that whereas the manager has complete information about asset fundamentals, asset prices are determined by the trading of risk-averse retail investors who do not observe fundamentals. Instead, they learn about these fundamentals from the realization of asset dividends over time. The informational asymmetry between the money manager and retail investors in our setup implies that the manager has superior information, an assumption consistent with the empirical findings in Brunnermeier and Nagel (2004) and Hendershott, Livdan, and Schurhoff (2014). Since the uninformed traders' inference of asset payoffs is subject to errors, their trading strategies can push market prices away from fundamental value. Given the ensuing mispricing dynamics, our main goal is to study the optimal trading policies of the informed money manager in partial equilibrium.

We solve for the price dynamics and for the informed manager's trading strategies in closed-form. Depending on the uninformed traders' up-to-date inferred underlying asset parameters, prices can be higher or lower than fundamental value, i.e. the corresponding prices in a full-information economy—as observed by the informed manager. Thus, time-varying learning by the uninformed traders leads to time-variation in the level of asset mispricing, potentially resulting in periods of large overpricing (as well as underpricing) of securities. As a particular example of mispricing that has received plenty of attention in the literature, we associate episodes of large overpricing with bubbles.[4]

Under these price dynamics, we first address the question of how much an informed direct trader—one who has the same information and risk aversion as the manager but faces no convex incentives—optimally invests in mispriced assets. This case provides us with a standard or, following Basak, Pavlova, and Shapiro (2007), "normal" policy against which we can assess the effects of convex incentives. For the short investment horizon we consider in this paper, we show that (i) the normal portfolio overweights underpriced assets and underweights overpriced assets relative to the market portfolio, (ii) the size of these positions increases with the extent of mispricing and (iii) the normal policy can result in substantial short-sale positions for largely overvalued securities.

Next, we examine the extent to which convex incentives can make the money manager trade

and Gomez (2013)). These payments are contingent on good performance and typically represent a substantial portion of managers' salaries. More generally, this type of compensation packages is prevalent in the financial industry beyond its pervasiveness in the asset management industry.

[4] A detailed characterization of the emergence and dynamics of bubbles is beyond the scope of this paper. See, for example, Brunnermeier and Oehmke (2013) for such a characterization.

more or less aggressively against mispricing than in the absence of these incentives—i.e., under the normal policy. If short-term profit-based contracts are to offset the incentives resulting from career concerns, we would expect convex incentives to induce the manager to trade more aggressively than the normal policy against mispricing.

In contrast with this line of argument, we find that convex incentives can induce substantial bubble-riding behavior by a hedge fund or mutual fund manager. Specifically, the manager's optimal dynamic trading strategy includes a mean-variance component that summarizes the manager's bets against mispricing, but also a hedging component against the risk of under-performing or, equivalently, of forfeiting the performance-linked bonus payment. This hedging component features well-known risk-shifting and indexing behaviors when the manager is under- or outperforming the benchmark, respectively. Our main contribution is to show that, under mispricing, both behaviors (i) can lead the manager to over-invest in overvalued assets relative to the case of no convex incentives, and (ii) can distort the manager's investment policy further as mispricing heightens.

Although convex incentives induce similar distortions on the hedge fund and mutual fund managers' trading against mispricing, the exact mechanisms differ in accord with the different incentives—as stated above, both types of money managers face convex incentives but with different features. In the case of the hedge fund manager, the hedging component can overweight the overpriced assets even more than the portfolio of the uninformed traders, i.e., the market portfolio. This behavior responds to the manager's gambles for large returns—risk-shifting—in response to low effective risk aversion when underperforming. As the manager exceeds the benchmark, the hedging component locks in the interim gain by investing like the benchmark—indexing. For "absolute return strategies"—as the goal of hedge funds is commonly advertised, this benchmark resembles a (scaled) money market account. Thus, a strategy that mimics a money market account can restrict the extent to which the hedge fund manager short sells a highly overpriced stock. This endogenous constraint on short selling limits the manager's bets against overvalued assets even in the absence of explicit portfolio constraints. It also suggests a limited role for sophisticated investors in stabilizing financial markets in situations of large overpricing.

In the case of the mutual fund manager, we show that the indexing component can lead to less aggressive trading against both under- and overpricing than in the absence of convex incentives. Since the implicit benchmark in investors' flows is a stock market index, the mutual fund manager's benchmark can itself be overvalued. Therefore, the attempt to lock in excess performance over the benchmark results in over-investing (respectively, under-investing) in the overvalued (undervalued) asset. Moreover, we show that an informed mutual fund manager facing higher sensitivity of flows to medium and bottom performance trades less aggressively against deviations in market prices from fundamental value at all mispricing levels.

We further show that either type of money manager generally invests more conservatively against mispricing as the information advantage over other market participants heightens. A greater information advantage—equivalently in our model, expected extent of mispricing or overvaluation—translates into a higher probability of outperforming and leads the manager to act more conservatively against mispricing. Following Pastor and Veronesi (2009)'s interpretation of "bubbles" as periods of heightened uncertainty about a technology, this investment pattern provides further rationale for the failure of sophisticated investors' to trade against overpricing during bubble-like episodes. These findings lead us to conclude that, when managers face convex incentives, their optimal investment strategy can in general be inconsistent with the common wisdom that bets against mispricing should build up as mispricing worsens.

An interesting aspect of our analysis is that we can justify some "puzzles" regarding the trading strategy of presumably sophisticated investors without recurring to behavioral arguments, and only using incentives documented in the literature—although not standard in financial models. In particular, we argue that informed hedge funds may find it optimal to invest in overpriced stocks in a proportion even higher than the market portfolio, as documented by Brunnermeier and Nagel (2004). This result is also consistent with evidence from the experimental work in Holmen, Kirchler, and Kleinlercher (2014), who find that trading at inflated prices is rational for subjects with convex incentives. Second, we show how investors' flows can induce excessive holdings of overpriced assets in mutual fund portfolios as found by Greenwood and Nagel (2009). Third, we provide an incentive-based—as opposed to financial constraint-based—explanation for the low short interest during overpricing periods as documented by, e.g., Stein and Lamont (2004).

From a methodological perspective, our paper is closest to the literature on money managers' risk taking in response to incentives. In particular, we build on Basak, Pavlova, and Shapiro (2007) and extend their analysis to a setup in which risky assets can be potentially mispriced due to an information wedge between managers and other investors in the economy. This allows us to interpret the risk-shifting and indexing effects in Basak, Pavlova, and Shapiro (2007) in terms of trading either against or in the direction of mispricing under different levels of over- and undervaluation. Since prices in our model are determined by investors that, except for their incentives and information, are otherwise identical to the money manager, our setup also allows us to attach meaningful probabilities to the different mispricing states and analyze the manager's behavior on an ex-ante basis. In a career concerns model with risk-neutral agents, Makarov and Plantin (2014) show that managers chasing investors' flows can invest in securities with negative expected returns and tail risk. In general equilibrium dynamic asset pricing setups featuring symmetric information, Cuoco and Kaniel (2011), Kaniel and Kondor (2013) and Basak and Pavlova (2013) study the effect of non-linear incentives or benchmarking on the time series and the cross-section of stock returns. Also in general equilibrium models with symmetric information about asset fundamentals, Vayanos and Woolley (2013) and Buffa, Vayanos, and Woolley (2014) find that money managers subject to time-varying investors' flows, or perceiving fees that depend linearly on relative performance, can push prices away from fundamental value. Malamud and Petrov (2014) further study the effects of convex incentives on price informativeness and volatility in a general equilibrium model with asymmetrically informed and risk-neutral managers. Although the asymmetric information structure along with the CRRA preferences in our setup prevents us from attempting a full equilibrium analysis of informed managers' trading strategies, we expect the effects we identify to remain significant in similar general equilibrium extensions of our model.

Our paper also contributes to the theoretical literature on rational explanations of limits to arbitrage. Allen and Gorton (1993) argue that unskilled fund managers with limited liability buy overvalued assets in order to appear skilled. Our analysis shows that skilled managers too may choose to buy overvalued assets. Shleifer and Vishny (1997) show that managers trade less aggressively than expected in presence of an arbitrage opportunity when they face the risk of investors' capital withdrawals. Liu and Longstaff (2004) show that capital-constrained risk-averse arbitrageurs can trade conservatively in the presence of arbitrage opportunities and even lose money in the process. Stein (2009) suggests that sophisticated investors can buy an overvalued asset due to an unawareness of the aggregate capital involved in eliminating the mispricing. Sato (2009) shows that the synchronization problem identified by Abreu and Brunnermeier (2003) can exacerbate the persistence of bubbles when traders are portfolio managers subject to relative performance concerns. Investors' strategies in our model are not limited by financial constraints, and our simple asset pricing setup leaves out synchronization and "crowded-trade" risks. Therefore, our explanation can complement the existing rationalizations of bubble-riding behavior using only the type of compensation arrangements for money managers commonly observed in practice.

The paper is structured as follows. In Section 2 we describe the economic setting. We derive price dynamics and the optimal trading strategies of the informed money manager in response to these prices in Section 3. In Section 4 we examine the average trading against mispricing of the informed manager. We close the paper with conclusions in Section 5.

2 Economic Setting

We are interested in the effects of convex compensation on the incentives of informed—i.e., sophisticated—institutional investors to trade against security mispricing over short-term periods.[5] We concentrate our analysis on the behavior of hedge funds and mutual funds, for which explicit or implicit option-like compensation structures have been extensively reported in the literature.[6] Rather than adopting arbitrary dynamics for security mispricing, we assume that mispricing arises from the trading of investors that, except for their incentives and access to information, are otherwise identical to the manager.

More precisely, we adopt a partial equilibrium approach in which prices are determined by uninformed retail investors. This assumption potentially gives rise to asset overvaluation (bubble-like prices) and, more generally mispricing, in our model. At the same time, it allows us

[5] See Brunnermeier and Nagel (2004) and Hendershott, Livdan, and Schurhoff (2014) for evidence of superior information on the part of institutional investors.

[6] See references in the introduction.

to derive the investment policies of a money manager in response to mispricing in closed-form.[7] Nevertheless, we expect our results to have implications for the more realistic case in which informed managers can affect equilibrium prices.

2.1 Financial Markets

We consider a pure exchange economy over the finite period $t \in [0, T']$. Financial markets consist of one risk-less asset β paying constant interest rate r per unit of time, and one risky asset S (henceforth, a "stock") which represents a claim to the dividend $D_{T'}$ at $t = T'$. $D_{T'}$ is the terminal value of a dividend process with initial value d_0 and dynamics given by:

$$dD_t = D_t(\rho dt + \delta dB_t), \tag{1}$$

where the dividend's mean growth rate ρ and volatility δ are positive constants, and B is a standard Brownian motion process under the probability measure P that explains the dynamics of this economy. Everyone observes δ; however, as we describe later, only a money manager with superior information observes ρ. The constant ρ is the unobserved realization at $t = 0$ of a random variable with normal distribution $\mathbf{N}(\rho_0, v_0)$, for given constant prior ρ_0 and variance $v_0 \geq 0$.

We assume that the risk-less asset is in zero net supply, while the stock is in unit supply. The stock price satisfies the following dynamics:

$$dS_t = S_t(\mu_t dt + \sigma_t dB_t), \tag{2}$$

with mean rate of return μ_t and volatility $\sigma_t > 0$ to be determined by the strategies of uninformed retail investors.

2.2 Money Manager

The money manager receives a compensation $f_T W_T$, which is the product of a fee rate f_T and of assets under management (AUM) W at the (possibly interim) date $T \leq T'$. Our main analysis focuses on the particular case in which the compensation date T coincides with the investment horizon T', i.e. $T = T'$. However, our conclusions does not depend on this specific assumption and we also examine the robustness of our results to the more general case $T < T'$. For consistency, we then keep the separate notation for the interim compensation date T and the final horizon T' and distinguish between the cases $T = T'$ and $T < T'$ whenever it is appropriate hereafter.

[7] Our approach to the informed manager's trading under mispricing is in the spirit of DeLong, Shleifer, Summers, and Waldman (1990), who analyze the survival of irrational traders in a model in which noise traders do not affect prices. By contrast, we focus on the trading of the informed manager in partial equilibrium when less informed traders determine prices.

The manager maximizes utility from final wealth with relative risk aversion γ. The manager's final wealth consists of the compensation $f_T W_T$ in the case $T = T'$, and is proportional to the terminal AUM, with time-T value $f_T W_T$, in the case $T < T'$.[8] The manager dynamically chooses an investment policy ϕ_t representing the fraction of W_t that is allocated to the stock at time $t \in [0, T']$. Given $W_0 = w_0$, the value of the portfolio (AUM) follows:

$$dW_t = W_t \left(r + \phi_t(\mu_t - r)\right) dt + W_t \phi_t \sigma_t dB_t. \tag{3}$$

The compensation fee f_T is a function of the fund's performance relative to a benchmark index Y (henceforth just "benchmark"). For an arbitrary initial value Y_0, this benchmark represents a long-only fixed-weight portfolio investing a fraction $\phi^Y \in [0, 1]$ of its value in the stock and the remaining fraction in the risk-free asset:[9]

$$dY_t = Y_t \left(r + \phi^Y (\mu_t - r)\right) dt + Y_t \phi^Y \sigma_t dB_t. \tag{4}$$

The fee rate f_T is specified as follows:

$$f_T = k \left(\frac{R_T}{\bar{\zeta} R_T^Y}\right)^{\alpha_1} \mathbb{1}_{\{R_T < \bar{\zeta} R_T^Y\}} + k \left(\frac{R_T}{\bar{\zeta} R_T^Y}\right)^{\alpha_2} \mathbb{1}_{\{R_T \geq \bar{\zeta} R_T^Y\}}, \tag{5}$$

where $k, \bar{\zeta} > 0$, $0 \leq \alpha_1 \leq \alpha_2$, $R_T \equiv W_T / W_0$, and $R_T^Y \equiv Y_T / Y_0$. This specification is a generalization of the function used to represent mutual fund investors' flow-performance relationship by Basak and Makarov (2014), and is similar to the incentive function in Kaniel and Kondor (2013).

The fee rate (5) is positive and (weakly) increasing in the fund's performance relative to the benchmark R_T / R_T^Y. More important for our purpose, it implies an asymmetric relation between relative performance and perceived fees whenever the slope α_1 in the *underperformance* region, $\{R_T < \bar{\zeta} R_T^Y\}$, is smaller than the slope α_2 in the *outperformance* region, $\{R_T \geq \bar{\zeta} R_T^Y\}$. In particular, $\alpha_1 < \alpha_2$ implies that the fees perceived by the manager increase with performance at a faster rate when relative performance R_T / R_T^Y exceeds a threshold $\bar{\zeta}$. This asymmetric relation implies an "option-like" compensation scheme, according to which the manager receives a "bonus" payment conditionally on performing relatively well. This bonus is given by the difference in fee rates between the outperformance and underperformance regions in (5).

We adopt alternative parameterizations of the fee rate to reflect the compensation structure of two types of institutional investors:

Hedge funds: We assume that a hedge fund manager is compensated through a base man-

[8] The proportionality constant in this case depends on the particular type of money manager (hedge fund or mutual fund manager) that we examine, as explained below. In any case, it does not change incentives in the margin nor, therefore, the manager's optimization problem.

[9] Setting the benchmark's initial value to equal $Y_0 \equiv S_0^{\phi^Y} \beta_0^{1-\phi^Y} e^{-\frac{\phi^Y}{2}(1-\phi^Y)(\delta^2 + v_0 T')T'}$, this benchmark is equivalent to a claim to a Cobb-Douglas composite basket $Y_{T'} = S_{T'}^{\phi^Y} \beta_{T'}^{1-\phi^Y}$ to be paid at the end of the period.

agement fee rate proportional to AUM, plus a substantially higher incentive fee rate proportional to the realized profits in excess of a preassigned benchmark. This specification is meant to reflect the typical practice in the hedge fund industry of charging a 1%-2% management fee along with an incentive fee equal to 20% of investment profits beyond a stipulated benchmark performance. Given that most hedge funds' preassigned benchmark is not a market index but a money market rate such as LIBOR plus a spread, we further assume that the benchmark is the risk-free rate r plus (possibly) a positive hurdle rate h.[10] This implies $\phi^Y = 0$ in (4), in which case $Y_T = \beta_0 e^{rT}$. Defining the continuously-compounded rates $r_T \equiv \ln(R_T)/T, r_T^Y \equiv \ln\left(R_T^Y\right)/T = r$, and setting the threshold $\bar{\zeta} = e^{hT}$ for the spread (hurdle rate) $h \geq 0$, for any $\alpha > 0$ we can write:

$$k\left(\frac{R_T}{\bar{\zeta}R_T^Y}\right)^{\alpha} = ke^{\alpha(r_T-(r+h))T}. \tag{6}$$

A first-order approximation of the RHS of (6) around $r_T = r + h$ gives:

$$k\left(\frac{R_T}{\bar{\zeta}R_T^Y}\right)^{\alpha} \approx kT + k\alpha(r_T - (r+h))T, \tag{7}$$

Applying (7) to the two terms in the RHS of (5) and setting $\alpha_1 = 0$ implies a fee rate:

$$f_T \approx kT + kT\alpha_2(r_T - (r+h))^+, \tag{8}$$

where $x^+ \equiv \max(0, x)$. Equation (8) makes it clear how we can parameterize the fee rate (5) to approximate the typical fee structure for hedge funds, consisting of a management fee rate kT plus an option-like incentive fee rate $kT\alpha_2$ on fund profits in excess of the hurdle performance $(r + h)$. We assume that the hedge fund is liquidated and paid back to its owners at $t = T$, after deducting the manager's compensation $f_T W_T$. No additional fund share purchases or redemptions by the hedge fund's investors occur during $[0, T]$. Whenever we set $T < T'$, we assume that during $[T, T']$ the hedge fund manager becomes a standard investor who maximizes utility over final wealth, whose initial value consists of the fees collected at $t = T$.

Mutual funds: A mutual fund manager charges a base management fee rate m in proportion to AUM. At $t = T$, but at no other $0 \leq t < T$, the mutual fund's investors purchase or redeem additional fund shares depending on the manager's performance during $[0, T]$ relative to the benchmark Y according to an exogenously given flow-to-relative performance relationship

[10] This is consistent with hedge funds' goal of delivering "absolute returns" in all market conditions. The fee structure often includes a "high-water mark" stating that following a year in which the fund declined in value, the fund would first have to recover those losses before any incentive fee can be charged. Such provisions may reduce the long-term risk-seeking incentives of a hedge fund manager, as analyzed by Hodder and Jackwerth (2007), Panageas and Westerfield (2009), and Drechsler (2013). Since we focus on the trading behavior of the manager over short-horizons, we follow Buraschi, Kosowski, and Sritrakul (2011) in assuming that the high-water mark is prespecified at the beginning of the period, and allow for differences in high-water marks by varying our spread parameter $\bar{\zeta}$ in (5).

(F-PR) q_T:

$$q_T = \underline{q} \left(\frac{R_T}{\bar{\zeta} R_T^Y} \right)^{\alpha_1} \mathbb{1}_{\{R_T < \bar{\zeta} R_T^Y\}} + \underline{q} \left(\frac{R_T}{\bar{\zeta} R_T^Y} \right)^{\alpha_2} \mathbb{1}_{\{R_T \geq \bar{\zeta} R_T^Y\}}, \tag{9}$$

with $\underline{q} > 0$. Defining $k \equiv m\underline{q}$, we see that the mutual fund manager's fee rate $f_T = mq_T$ follows the specification (5). This functional form allows flows to be sensitive (and potentially locally concave, if $\alpha_1 < 1$) to medium and low relative performance. At the same time, f_T reflects the well-documented convexity in the sensitivity of flows to performance (see, e.g., Chevalier and Ellison (1997), and Sirri and Tufano (1998)) for $\alpha_1 < \alpha_2$, according to which outperforming funds receive a disproportionally high amount of inflows.[11] The F-PR (5) can also capture linear relationships ($\alpha_1 = \alpha_2 = 1$), near-linear relationship ($\alpha_1 = \alpha_2 \neq 1$), as well as no relationship ($\alpha_1 = \alpha_2 = 0$). The particular case of $\alpha_1 = 0, \alpha_2 > 0$ resembles the linear-convex flow relationship in Basak, Pavlova, and Shapiro (2007). In contrast with the case analyzed by these authors, the dynamics of the benchmark for a given weight ϕ^Y in our setup are not entirely arbitrary but reflect the pricing impact of the retail investors. In the analysis of the model, this feature allows us to attach meaningful probabilities to the different mispricing states. Although we present our main results for general benchmarks, we specialize most of our analysis to the case of an all-equity mutual fund for which $\phi^Y = 1$. No additional fund share purchases or redemptions take place until $t = T'$, when the fund is liquidated and paid back to its owners, after deducting the final managerial fee compensation $mW_{T'}$.

2.3 Retail Investors

Retail investors are in unit mass and are initially endowed with one share of the stock. They all have identical constant relative risk aversion (CRRA) preferences, with the same coefficient γ as the money manager. Retail investors are *uninformed*, in the sense that they do not observe the realized value of the dividend growth rate ρ at $t = 0$. We refer to these as U-investors. At each time $t \in [0, T']$, U-investors allocate a fraction ϕ_t^U of their wealth W_t^U to the stock and the remaining fraction to the risk-less asset to maximize utility from final wealth at time T'. Given initial wealth $w_0 = S_0$ their wealth process evolves according to:

$$dW_t^U = W_t^U \left(r + \phi_t^U (\mu_t - r) \right) dt + W_t^U \phi_t^U \sigma_t dB_t. \tag{10}$$

Since they do not observe ρ at $t = 0$, U-investors have to infer it from the observation of dividends during the investment period. We explain the dynamics of the posterior of the expected dividend growth rate in section 3.1.

The sequence of events during $[0, T']$, for $T \leq T'$, is summarized in the timeline of Figure 1. Crucially, the money manager is *better informed* than U-investors in our setup. In particular,

[11] While many empirical studies document no sensitivity of flows to poor relative past performance (e.g. Sirri and Tufano (1998)), many others (e.g. Huang, Wei, and Yan (2007)) find it is positive, although lower than the sensitivity to medium or high relative returns.

the manager observes the realization of the dividend growth rate ρ at $t = 0$, whereas U-investors learn its value from their prior and the observation of the dividend process D_t over the period $t \in [0, T']$.[12] Moreover, we assume $\alpha_2 > \alpha_1$ for either type of money manager (hedge fund or mutual fund), so that the manager's compensation is convex.

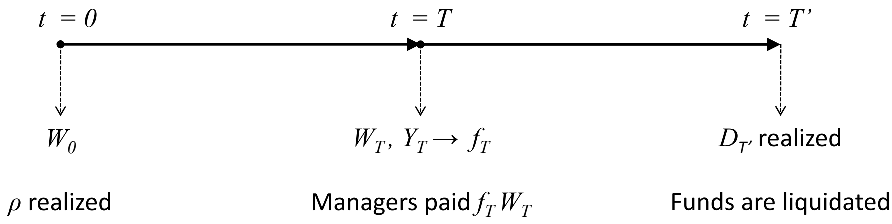

Figure 1: Model timeline

3 Solution

We are ultimately interested in understanding trading under mispricing, i.e. the trading by hedge fund or mutual fund managers—possibly facing convex incentives—who observe that asset prices do not reflect all the information available to them. This requires solving for the dynamics of prices in a first stage, and for the optimal portfolio of an informed investor under these prices in a second stage. All proofs are given in Appendix A.

A solution to our model consists of a set of trading strategies and asset prices such that: (i) U-investors' and the manager's individual investment policies are optimal, and (ii) bond and stock markets clear. Assuming bonds are in zero net supply, market clearing along with the requirement of no-arbitrage implies:[13]

$$W^U_{T'} = S_{T'} = D_{T'}. \tag{11}$$

Based on equation (11), we next provide closed-form expressions for the asset prices in this economy by solving for the optimal strategies of U-investors.

3.1 Price Dynamics

U-investors start the investment period with a prior distribution for ρ and update it over time according to Bayes rule, based on the arrival of information D_t. We denote by $\tilde{P}$ the probability space that describes the dynamics of the dividend process according to U-investors' priors.

[12] Assuming that all retail investors are uninformed makes the model tractable but is not key for our results. Qualitatively, we just need that prices do not reflect all the information available to the money manager.

[13] The condition of absence of arbitrage in the stock market requires that the stock price equals the liquidation dividend at the terminal date: $S_{T'} = D_{T'}$.

Letting $\tilde{E}$ denote the expectation under $\tilde{P}$, the distribution of ρ conditional on D_t is Gaussian, with mean $\tilde{\rho}_t \equiv \tilde{E}\left[\rho|D_t\right]$ and variance $v_t \equiv \tilde{E}\left[(\rho-\tilde{\rho}_t)^2|D_t\right]$ satisfying:[14]

$$\begin{cases} d\tilde{\rho}_t = \frac{v_t}{\delta}\frac{1}{\delta}\left(\frac{dD_t}{D_t} - \tilde{\rho}_t dt\right) \equiv \frac{v_t}{\delta} d\tilde{B}_t, \\ dv_t = -\frac{v_t^2}{\delta^2}dt, \end{cases} \tag{12}$$

for the initial values $\tilde{\rho}_0 = \rho_0$ and v_0. $\tilde{B}_t$ is a standard Brownian motion with respect to the filtration $\mathcal{F}_t^D$ generated by the dividend process D under $\tilde{P}$, with dynamics $d\tilde{B}_t \equiv \frac{1}{\delta}\left(\frac{dD_t}{D_t} - \tilde{\rho}_t dt\right) = dB_t + \frac{\rho-\tilde{\rho}_t}{\delta}dt$. Letting $\tilde{\mu}_t \equiv \tilde{E}\left[\mu_t|D_t\right]$ and $\tilde{\eta}_t \equiv \frac{\tilde{\mu}_t - r}{\sigma_t}$ be U-investors' time-t inferred stock mean rate of return and market price of risk under $\tilde{P}$, U-investors allocate $\hat{\phi}_t^U$ to the stock market to maximize expected utility over terminal wealth:

$$\max_{(\phi_t^U)_{t\in[0,T']}} \tilde{E}_0\left[\frac{(W_{T'}^U)^{1-\gamma}}{1-\gamma}\right], \tag{13}$$

subject to initial wealth w_0 and the restated, in terms of observable variables, self-financing constraint (10):

$$dW_t^U = W_t^U\left(r + \phi_t^U(\tilde{\mu}_t - r)\right)dt + W_t^U\phi_t^U\sigma_t d\tilde{B}_t. \tag{14}$$

From U-investors' perspective, markets are complete with respect to the observable states of the economy (a single risky asset S driven by a single Brownian motion $\tilde{B}$). We can then solve problem (13) as in an equivalent full-information framework. Absent arbitrage opportunities, U-investors see financial markets as driven by a unique state-price deflator (SPD) $\tilde{\pi}$ with dynamics $d\tilde{\pi}_t = -r\tilde{\pi}_t dt - \tilde{\pi}_t\tilde{\eta}_t d\tilde{B}_t$. The dynamic budget constraint (14) can be restated (see e.g. Karatzas and Shreve (1998)) as:

$$\tilde{E}_0\left[\tilde{\pi}_{T'}W_{T'}^U\right] = w_0. \tag{15}$$

Using the martingale/duality approach of Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987), the dynamic optimization problem (13) can be solved as a static problem over final payoffs $W_{T'}^U$. Standard individual optimization results along with the market clearing condition (11) then lead to the following

Proposition 1. *Let $\tau' = T' - t > 0$. Equilibrium stock prices and uninformed investors' SPD are given by:*

$$S_t = D_t \exp\left\{\left(\tilde{\rho}_t - r - \gamma\delta^2 - \left(\gamma - \frac{1}{2}\right)v_t\tau'\right)\tau'\right\}, \tag{16}$$

$$\tilde{\pi}_t = \lambda^{-1}D_t^{-\gamma}\exp\left\{\left(r - \gamma\tilde{\rho}_t + \frac{1+\gamma}{2}\gamma\delta^2 + \frac{\gamma^2}{2}v_t\tau'\right)\tau'\right\}, \tag{17}$$

where $\lambda > 0$ is the Lagrange multiplier of the equivalent static problem and its solution is given in Appendix A. Under the probability $\tilde{P}$, equilibrium stock mean return, volatility and market

[14] See, e.g., Liptser and Shirayayev (2001).

price of risk are time-varying and deterministic, as given by:

$$\tilde{\mu}_t = r + \gamma\sigma_t^2,$$
$$\sigma_t = \delta + \frac{v_t}{\delta}\tau',$$
$$\tilde{\eta}_t = \gamma\sigma_t. \tag{18}$$

The stock price dynamics in equations (16) are affected by investors' estimate of the dividend growth rate $\tilde{\rho}_t$ and by their uncertainty v_t. U-investors' incomplete information can then introduce a wedge between the stock market price in this economy and its *fundamental value*, understood as the stock price S^{CI} that would prevail if all traders in the economy had complete and symmetric information about the dividend growth rate ρ (i.e., $\rho_0 = \rho$ and $v_0 = 0$).

Arguably, any situation in which $S \neq S^{CI}$ would be perceived as stock *mispricing* by fully informed investors such as the money manager in our setup. Hence, we measure the extent of stock overvaluation as of time $t < T'$ by the quantity $OV_t \equiv \left(S_t/S_t^{CI}\right)^{1/\tau'} - 1$, and the extent of mispricing by the quantity $MP_t \equiv |OV_t|$.[15] We say that stock mispricing reflects *overvaluation* or *overpricing* (respectively, *undervaluation* or *underpricing*) whenever $OV_t > 0$ ($OV_t < 0$). Since by no-arbitrage $S_{T'} = D_{T'} = S_{T'}^{CI}$, stock mispricing equals zero at the terminal date T'. The following Lemma characterizes the stock fundamental value and the mispricing and overvaluation as perceived by the informed manager at any interim period:

Corollary 1. *Under complete information for the retail investors in the economy, stock prices are:*

$$S_t^{CI} = D_t \exp\left\{\left(\rho - r - \gamma\delta^2\right)\tau'\right\}. \tag{19}$$

The stock mean return, volatility and market price of risk are:

$$\mu^{CI} = r + \gamma\left(\sigma^{CI}\right)^2,$$
$$\sigma^{CI} = \delta,$$
$$\eta^{CI} = \gamma\sigma^{CI}. \tag{20}$$

The time-t stock overvaluation OV_t, as perceived by a fully-informed agent, is:

$$OV_t = \exp\left\{\left(\tilde{\rho}_t - \rho - \left(\gamma - \frac{1}{2}\right)v_t\tau'\right)\right\} - 1. \tag{21}$$

As is intuitive, we see from equation (21) that an over-estimation of the mean dividend growth rate by U-investors, $\tilde{\rho}_t > \rho$, will typically lead to stock overvaluation.[16] Moreover, the

[15] As can be seen from the proof of Corollary 1 below, the ratio S_t/S_t^{CI} depends on the time to maturity τ'. The definitions of OV_t and MP_t then make the extent of mispricing at different dates comparable.

[16] The stock will be overvalued as long as U-investors over-estimate the dividend growth rate by a large enough margin: $\tilde{\rho}_t - \rho > (\gamma - .5)v_t\tau' > 0$. This implies that a low enough over-estimation of fundamentals, $(\gamma - .5)v_t\tau' > \tilde{\rho}_t - \rho > 0$, is still consistent with an undervalued stock in our economy with sufficiently risk-averse U-traders.

extent of overvaluation OV_t is increasing in the over-estimation margin $\tilde{\rho}_t - \rho$, consistent with the intuition that better perceived dividend growth prospects leads U-investors to push prices further up.

We note that in our setup there is no uncertainty about the date at which prices converge to fundamental value, since by construction $S_{T'} = S_{T'}^{CI}$ with probability one, even though the path to convergence is random. This rules out bubble-riding behavior due to "synchronization" risk as originally studied by Abreu and Brunnermeier (2003) and generalized to a delegated portfolio management context by Sato (2009).

We further note that, due to the equilibrium restriction (11), U-investors are fully invested in the stock at all times, i.e. $\phi^U \equiv \hat{\phi}_t^U = 1$ for all $t \in [0, T']$. This leads naturally to the interpretation of the stock price S (equivalently, U-investors' wealth W^U) as the value of the *market portfolio* in our model.[17] We will draw on this interpretation later on as we assess the trading policies of an informed investor in this economy. We expand on the relation between over-estimation of fundamentals, stock overpricing and an informed investor's investment decisions in the next subsection.

3.2 Optimal Investment Strategy of the Informed Money Manager

The manager observes the true value of the dividend growth rate ρ, and sees the stock price dynamics under the true probability P as given by:

$$\begin{aligned} dS_t =& S_t \left(\tilde{\mu}_t dt + \sigma_t \left(\frac{\rho - \tilde{\rho}_t}{\delta} dt + dB_t \right) \right) \\ =& S_t \left(\left(\tilde{\mu}_t + \sigma_t \frac{\rho - \tilde{\rho}_t}{\delta} \right) dt + \sigma_t dB_t \right), \end{aligned} \tag{22}$$

with the following dynamics for U-investors' estimated growth rate $\tilde{\rho}_t$

$$d\tilde{\rho}_t = \frac{v_t}{\delta^2}(\rho - \tilde{\rho}_t)dt + \frac{v_t}{\delta} dB_t. \tag{23}$$

Under the true probability P, the stock mean return $\mu_t \equiv \tilde{\mu}_t + \sigma_t \frac{\rho - \tilde{\rho}_t}{\delta}$ is *stochastic* and follows a *mean-reverting* process: higher inferred dividend growth by U-investors leads *indirectly* to higher expected returns over the next instant $t + dt$ (via the negative impact of $\tilde{\rho}_t$ on $d\tilde{\rho}_t$ in (23)), but *directly* to lower mean stock returns today (via the contemporaneous negative impact of $\tilde{\rho}_t$ on μ_t). The market price of risk under P is also stochastic and consists of the sum of the market price of risk under $\tilde{P}$ and a term proportional to the uninformed investors' estimation error $\rho - \tilde{\rho}_t$: $\eta_t = \tilde{\eta}_t + \frac{\rho - \tilde{\rho}_t}{\delta}$.[18]

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[17] As noted in footnote 12, this assumption is for tractability only.

[18] In fact, the market price of risk under the true probability P follows a first-order autoregressive process, as given by:

$$d\eta_t = -\frac{v_t}{\delta^2}(\eta_t dt + dB_t),$$

where $0 < v_t/\delta^2 < 1$.

Markets are complete for the money manager, who sees financial markets as driven by a unique SPD π with dynamics $d\pi_t = -r\pi_t dt - \pi_t \eta_t dB_t$, i.e.:

$$\begin{aligned}\pi_t =& e^{-rt-\frac{1}{2}\int_0^t \eta_s^2 ds - \int_0^t \eta_s dB_s}\\ =& \tilde{\pi}_t \xi_t,\end{aligned} \tag{24}$$

where $\xi_t \equiv \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{\rho-\tilde{\rho}_s}{\delta}\right)^2 ds - \int_0^t \frac{\rho-\tilde{\rho}_s}{\delta} dB_s\right\}$ is the likelihood process (a P-martingale) for the measure transformation from P to $\tilde{P}$: $\xi_t = \frac{d\tilde{P}}{dP}$ on $\mathcal{F}_t$. Therefore, the extent to which the manager's SPD, π_t, differs from the SPD of retail investors, $\tilde{\pi}_t$, depends on the size and sign of the estimation error $\rho - \tilde{\rho}_t$.

According to equation (21), the manager observes that time-varying learning by U-investors induces time-variation in the level of asset mispricing, potentially resulting in sustained periods of stock overpricing or underpricing. An important particular case of stock overpricing is the price appreciation arising during bubbles. Although fully characterizing the emergence and dynamics of a bubble is beyond the scope of this paper, a main goal is to analyze the behavior of money managers under the asset overvaluation typical of bubbles. The behavior of money managers in the presence of bubbles has been extensively documented in the empirical literature, including Brunnermeier and Nagel (2004), Greenwood and Nagel (2009), and Griffin, Harris, Shu, and Topaloglu (2011).

We first address the question of how much an informed direct trader—one who has the same information and risk aversion as the manager but does not face convex incentives—optimally invests in the mispriced stock.

3.2.1 Benchmark Case: Investment Policy without Convex Incentives

In order to single out the effects of convex incentives on the manager's trading strategy, we first examine the dynamic policy of a hypothetical retail investor with superior information—but without this type of incentives. We follow Basak, Pavlova, and Shapiro (2007) in referring to this standard (default) policy as the *normal* (N) policy.

For an arbitrary coefficient of relative risk aversion $\tilde{\gamma} > 1$, we define $\phi^N_{\tilde{\gamma},t}$ as the time-t ($t \in [0, T']$) normal trading in the stock of an investor with RRA coefficient $\tilde{\gamma}$. Proposition 2 characterizes $\phi^N_{\tilde{\gamma},\cdot}$ along with the associated portfolio value process $W^N_{\tilde{\gamma},\cdot}$:

Proposition 2. *For $t \in [0, T']$, the normal trading strategy $\phi^N_{\tilde{\gamma},t}$ and associated portfolio value process $W^N_{\tilde{\gamma},t}$ are:*

$$\phi^N_{\tilde{\gamma},t} \equiv \frac{\delta^2 + v_t\tau'}{\delta^2 + \frac{v_t}{\tilde{\gamma}}\tau'} \frac{\eta_t}{\tilde{\gamma}\sigma_t}, \tag{25}$$

$$\hat{W}^N_{\tilde{\gamma},t} = (\lambda_N \pi_t)^{-\frac{1}{\tilde{\gamma}}} Z_{1-\frac{1}{\tilde{\gamma}},t,T'}, \tag{26}$$

where for any $\psi \in (0,1)$ *and* $0 \le t \le t' \le T'$

$$Z_{\psi,t,t'} \equiv \delta^{\psi}\sqrt{\frac{(\delta^2 + v_t(t'-t))^{1-\psi}}{\delta^2 + (1-\psi)v_t(t'-t)}}\exp\left\{-\psi r(t'-t) - \frac{\psi(1-\psi)\delta^2(t'-t)}{\delta^2 + (1-\psi)v_t(t'-t)}\frac{\eta_t^2}{2}\right\}, \tag{27}$$

and $\lambda_N = \left(\frac{Z_{1-\frac{1}{\tilde{\gamma}},0,T'}}{w_0}\right)^{\tilde{\gamma}}$.

Comparing the portfolio weight in the stock of the U-investors (the market portfolio), ϕ^U, to the normal policy of an equally risk-averse ($\tilde{\gamma} = \gamma$) investor, $\phi^N_{\gamma,t}$, we obtain the following

Corollary 2. *For* $t \in [0, T']$*, the normal excess holding of the stock relative to the market is:*

$$\phi^N_{\gamma,t} - \phi^U = -\frac{1}{\gamma}\frac{\tilde{\rho}_t - \rho - (\gamma-1)v_t\tau'}{\delta^2 + \frac{v_t}{\gamma}\tau'}. \tag{28}$$

Thus, the normal portfolio implies a lower stock holding than the market, $\phi^N_{\gamma,t} < \phi^U$*, iff:*

$$\begin{aligned} \tilde{\rho}_t >& \rho + (\gamma-1)v_t\tau' \\ \Leftrightarrow OV_t >& \exp\left\{-\frac{1}{2}v_t\tau'\right\} - 1. \end{aligned} \tag{29}$$

The normal portfolio implies a higher stock holding than the market if the converse of (29) *holds; both holdings are the same when* (29) *holds as an equality.*

Table 1 summarizes the relationship between over-estimation of fundamentals, stock overpricing and the normal policy. Except for a typically low-probability range of underpricing, $OV_t \in \left(\exp\left\{-\frac{1}{2}v_t\tau'\right\} - 1, 0\right)$ (corresponding to an estimation error $(\tilde{\rho}_t - \rho) \in ((\gamma-1)v_t\tau', (\gamma-0.5)v_t\tau')$),[19] we see that for an *overpriced* stock, $S_t > S_t^{CI}$, the normal portfolio *underweights* the stock relative to the market ($\phi^N_{\gamma,t} < \phi^U$), and conversely for an *underpriced* security ($\phi^N_{\gamma,t} > \phi^U$ for $S_t < S_t^{CI}$). Moreover, rewriting equation (28) as:

$$\phi^N_{\gamma,t} - \phi^U = -\frac{1}{\gamma}\frac{\ln(1+OV_t) + \frac{1}{2}v_t\tau'}{\delta^2 + \frac{v_t}{\gamma}\tau'}, \tag{30}$$

we see that a higher overvaluation leads to larger stock underweighting in the normal portfolio relative to the market, potentially resulting in sizable short positions in the stock for high levels of overpricing.

We emphasize that even though the informed agent underweights the stock relative to the market, the allocation to the stock can still be positive in our setting. This is because the local expected return μ can be greater than the risk-free rate r despite the fact that the stock is

[19] In these states, the normal portfolio underweights slightly underpriced securities. This occurs because positive but small enough over-estimation of the dividend growth rate by U-investors, $0 \le (\gamma-1)v_t\tau' < \tilde{\rho}_t - \rho < (\gamma-.5)v_t\tau'$, does not translate into stock overpricing (see Section 3.1) but still leads to below-normal stock holdings. However, these states occur with low probability for short enough investment horizons T'.

overvalued. The local expected return is determined by retail investors, who eventually learn the true dynamics of the dividend process. However, they learn only gradually over a horizon that largely exceeds the evaluation period of the money manager. In this case, a risk-averse informed investor will optimally hold a long position in the stock for diversification purposes. Therefore the puzzle in practice is not that money managers invest in overvalued stocks, but rather that they invest more than the market portfolio, as documented by Brunnermeier and Nagel (2004).

	$\tilde{\rho}_t - \rho$			
	$(-\infty, 0]$	$(0, (\gamma-1)v_t\tau')$	$((\gamma-1)v_t\tau', (\gamma-0.5)v_t\tau')$	$((\gamma-0.5)v_t\tau', +\infty)$
$sgn(OV_t)$	$-$	$-$	$-$	$+$
$sgn(\phi^N_{\gamma,t} - \phi^U)$	$+$	$+$	$-$	$-$

Table 1: Relation Between Fundamentals, Overpricing and the Normal Portfolio

Summing up, for the short investment horizon we consider in this paper we have shown that the normal policy is consistent with the commonly expected behavior of an informed trader under efficient financial markets. In particular, (i) the informed trader overweights underpriced assets and underweights overpriced assets relative to the market portfolio, (ii) the size of the informed trader's bets against mispricing increases with the extent of mispricing and (iii) can even result in substantial short-sale positions for largely overvalued securities. In the next section we contrast this with the behavior of an informed trader under convex incentives—the money manager.

3.2.2 The Informed Money Manager

Previous authors have suggested that high-powered incentives can alleviate the bubble-riding behavior associated with career concerns of money managers (see, e.g., Scharfstein and Stein (1990), Dass, Massa, and Patgiri (2008)). The argument is based on the intuition that a high enough weight on managers' short-term performance should offset the negative longer-term effects of a loss of reputation, making managers more willing to deviate from the herd and away from overvalued assets. In this subsection and the next we assess the validity of this argument under the convex managerial incentives that we describe in section 2.

Within our setup, a simple low-powered compensation arrangement consists in setting the fee rate equal to an arbitrarily small positive constant c, i.e. $f_T = c$. Under this arrangement, it is straightforward to show that the manager's optimal trading strategy equals the normal policy at all times. If high-powered compensation arrangements such as option-like performance fees are to offset potential incentives to over-invest in overpriced assets in a more general setup, we would expect the investment policy under performance fees to induce a trading against mispricing at least as aggressive as the normal policy.

We start by analyzing whether this is indeed the case in the simplest possible setup, corresponding to the one-period investment problem in which the manager collects the compensation fees only at the end of the investment horizon: $T = T'$. This assumption may seem restrictive, as it implies that market prices S converge to fundamental value S^{CI} with certainty at the same time as the manager receives the compensation. However, the main intuition and qualitative results carry over to the two-period case in which $T < T'$.

One-Period Investment Problem ($T = T'$): This case allows us to isolate the effects of profit-based incentives on the manager's trading against mispricing, beyond any effect potentially induced by career concerns. In our general setting with two periods, the manager's decisions in the first period affect the compensation over the second period. This induces longer-term considerations that resemble career concerns. For this reason, we present most of our results by reference to the one-period problem, and assess the robustness of these results in the case $T < T'$ later on in the paper.

Whenever possible, we relate the manager's optimal policy to the portfolios introduced in previous sections ϕ^Y, ϕ^U and $\phi^N_{\tilde{\gamma},\cdot}$: allocations in the stock of the benchmark portfolio, the market (U-investors') portfolio and the normal policy (for a given RRA coefficient $\tilde{\gamma}$). As before, we let $\tau' = T' - t$ denote the time remaining to horizon, and we introduce the following additional notation: $\zeta = \bar{\zeta} W_0/Y_0$ is the normalized performance threshold, $\gamma_i \equiv \gamma + \alpha_i(\gamma - 1)$, for $i = 1, 2$, represents the manager's *effective* RRA in the *underperformance* and *outperformance* regions, respectively, of section 2.2. The interpretation of γ_i follows from computing the RRA coefficient corresponding to the manager's utility function when wealth is augmented by the fee rate (5) and implies that, for $\alpha_2 > \alpha_1 > 0$, the manager's effective RRA in the outperformance region increases by $\gamma_2 - \gamma_1 > 0$ relative to the underperformance region. This is the consequence of the higher sensitivity of expected fees to relative performance when the manager's funds do better than the benchmark.[20] Our main result is stated in the following:

Proposition 3. *When $T = T'$, the informed manager's optimal trading in the stock during $[0, T']$ is:*

$$\begin{aligned}\hat{\phi}_t =& \omega_t \phi^N_{\gamma_1,t} + (1-\omega_t)\phi^N_{\gamma_2,t} + \left[\omega_t \left(1 - \frac{\gamma}{\gamma_1}\right)\frac{\delta}{\underline{\sigma}_{\gamma_1,t}} + (1-\omega_t)\left(1 - \frac{\gamma}{\gamma_2}\right)\frac{\delta}{\underline{\sigma}_{\gamma_2,t}}\right]\phi^Y \\ &+ \frac{1}{\sqrt{\sigma_t \tau'}}\left[\omega_t \Phi_{1,t} + (1-\omega_t)\Phi_{2,t}\right] \end{aligned} \tag{31}$$

and the optimal wealth process is:

$$\hat{W}_t = f_{1,t} + f_{2,t}, \tag{32}$$

[20] To see this, note that changes in *actual* wealth are augmented by a fee rate $k\left(W_T/\zeta W_T^Y\right)^{\alpha_2}$ in the outperformance region, but only by a flow rate $k\left(W_T/\zeta W_T^Y\right)^{\alpha_1} < k\left(W_T/\zeta W_T^Y\right)^{\alpha_2}$ for $\alpha_2 > \alpha_1$ outside of it. The fee charged by a top performer is increasing in wealth at a higher rate. Therefore, effective wealth fluctuates more in response to the same change in actual wealth in this region than in the underperformance region, raising manager's effective risk aversion.

where, for $i = 1, 2,$:

$$f_{i,t} \equiv \left(\frac{1+\alpha_i}{\lambda_M \xi_t}\right)^{\frac{1}{\gamma_i}} \zeta^{1-\frac{\gamma}{\gamma_i}} \left(\frac{\delta}{\lambda}\right)^{1-\frac{1}{\gamma_i}} \sqrt{\frac{(\delta^2 + v_t \tau')^{\frac{1}{\gamma_i}}}{\delta^2 + \frac{v_t}{\gamma_i}\tau'}} \beta_t^{\left(1-\frac{\gamma}{\gamma_i}\right)(1-\phi^Y)} D_t^{\left(1-\frac{\gamma}{\gamma_i}\right)\phi^Y - \left(1-\frac{1}{\gamma_i}\right)\gamma} \tilde{\pi}_t^{-1}$$
$$\times \exp\left\{\left[\left(1-\frac{\gamma}{\gamma_i}\right)(1-\phi^Y) r - \left(1-\frac{1}{\gamma_i}\right)\left(k_i\left(\rho - ((\gamma_i - 1)k_i + 1)\frac{\delta^2}{2}\right)\right.\right.\right.$$
$$\left.\left.\left. + \frac{1}{2\gamma_i}\frac{\left(\rho - \tilde{\rho}_t - (\gamma_i - 1)k_i\delta^2\right)^2}{\delta^2 + \frac{v_t}{\gamma_i}\tau'}\right)\right]\tau'\right\}\Pi_{i,t}, \tag{33}$$

$$\Phi_{i,t} \equiv \frac{1}{\underline{\sigma}_{\gamma_i,t}} \frac{\mathcal{N}'(\underline{d}_{i,t}) - \mathcal{N}'(\bar{d}_{i,t})}{\Pi_{i,t}}, \tag{34}$$

$$\underline{d}_{i,t} \equiv \frac{\gamma\delta^2 \underline{\sigma}_{\gamma,t}}{v_t\sqrt{\tau'}}\sqrt{\frac{\delta^2 + v_t\tau'}{\delta^2 + \frac{v_t}{\gamma_i}\tau'}}\left(\phi^N_{\gamma,t} - \phi^Y\right) - \frac{\sigma_t}{v_t\sqrt{\tau'}}\sqrt{\frac{\delta^2 + \frac{v_t}{\gamma_i}\tau'}{\delta^2 + v_t\tau'}}\Gamma, \quad \bar{d}_{i,t} \equiv \underline{d}_{i,t} + 2\frac{\sigma_t}{v_t\sqrt{\tau'}}\sqrt{\frac{\delta^2 + \frac{v_t}{\gamma_i}\tau'}{\delta^2 + v_t\tau'}}\Gamma, \tag{35}$$

λ_M is the Lagrange multiplier that solves $\hat{W}_0 = w_0$, $\mathcal{N}(.)$ is the standard normal cumulative distribution function, $\Pi_{1,t} \equiv \mathcal{N}(\underline{d}_{1,t}) + 1 - \mathcal{N}(\bar{d}_{1,t})$, $\Pi_{2,t} \equiv \mathcal{N}(\bar{d}_{2,t}) - \mathcal{N}(\underline{d}_{2,t})$, $\omega_t \equiv \frac{f_{1,t}}{\hat{W}_t}$, $0 \le \omega_t \le 1$, $\underline{\sigma}_{\tilde{\gamma},t} \equiv \delta + \frac{v_t}{\tilde{\gamma}\delta}\tau'$, $k_i = \gamma - \frac{\gamma_i - \gamma}{\gamma_i - 1}\phi^Y > 0$ $(i = 1, 2)$, and $\Gamma \ge 0$ is as given in Appendix A.

The manager's optimal portfolio (31) consists of the sum of three components:

1. A *mean-variance* component $\omega_t \phi^N_{\gamma_1,t} + (1 - \omega_t)\phi^N_{\gamma_2,t}$. This component equals the weighted average of the normal portfolios $\phi^N_{\gamma_1,t}$ and $\phi^N_{\gamma_2,t}$, corresponding to the manager's effective RRA coefficients γ_1 in the underperformance region and γ_2 in the outperformance region. This component has the same sign as the normal policy $\phi^N_{\gamma,t}$ but a smaller absolute value. From equation (25) we note that each portfolio $\phi^N_{\gamma_i,t}$, $(i \in \{1, 2\})$ consists of the standard mean-variance component adjusted by a coefficient that reflects the conditional uncertainty of the retail investors.

2. An *indexing* component, scaling down the benchmark weight in the stock ϕ^Y by the factor $\left[\omega_t (1 - \gamma/\gamma_1)\, \delta/\underline{\sigma}_{\gamma_1,t} + (1 - \omega_t)(1 - \gamma/\gamma_2)\, \delta/\underline{\sigma}_{\gamma_2,t}\right] \in (0, 1)$. Since we examine long-only benchmarks $(0 \le \phi^Y \le 1)$, and the scaling factor takes values in the interval $(0, 1)$, this component represents either long or zero positions in both the stock and the risk-free asset at all times. As expected, this term is increasing in the manager's risk aversion.

3. An additional component, proportional to the sum $\omega_t \Phi_{1,t} + (1 - \omega_t)\Phi_{2,t}$. Portfolios $\Phi_{1,t}$ and $\Phi_{2,t}$ are non-linear functions of the difference between the normal and benchmark portfolios $\phi^N_{\gamma,t} - \phi^Y$ and can reflect large long or short positions in the stock. We refer to this as the *risk-shifting* component.

As usual, we can interpret the difference between the manager's portfolio (31) and its mean-variance component as the manager's *hedging demand*. The manager in our model hedges against the risk of underperforming or, equivalently, of not receiving the performance fee in the outperformance region.

Given the decomposition above, the manager's hedging demand is captured by the indexing and risk-shifting components. The weight ω_t is a key parameter in the determination of the relative weights of the indexing and risk-shifting components in the manager's portfolio, as well as the relative weights of the two pieces in each component. In turn, ω_t is an increasing function of the manager-assessed time-t probability of *underperforming* at T': as the odds of outperforming improve, the manager puts more weight in the second piece of both the indexing and risk shifting components. To see how ω_t depends on the manager's relative performance, we note from Proposition 3 that ω_t represents the fraction of current AUM that can be attributed to the manager's underperformance. This fraction is proportional to the conditional probability of underperforming at the end of the period, $\Pi_{1,t}$.

Within the indexing component, we see that the manager tilts the portfolios towards the sub-portfolio $(1-\gamma/\gamma_2)\delta/\underline{\sigma}_{\gamma_2,t}\phi^Y$ as the probability $1-\omega_t$ of outperforming increases. Since $0 \leq (1-\gamma/\gamma_1)\delta/\underline{\sigma}_{\gamma_1,t}\phi^Y < (1-\gamma/\gamma_2)\delta/\underline{\sigma}_{\gamma_2,t}\phi^Y \leq \phi^Y$, an outperforming manager invests more like the benchmark. This behavior reflects the well-known *lock-in* effect according to which an outperformer (the "winner" in the tournaments literature) and risk-averse manager prefers to secure an interim relative gain by investing like the benchmark. In our setup, this behavior intensifies with the fee-performance sensitivity α_2, and results in the manager investing more than the normal policy $\phi^N_{\gamma,t}$ in an overvalued stock whenever the benchmark also invests more in this stock than the normal policy. This situation is evidently more likely to arise for benchmarks representing high stock holdings, like those employed by most all-equity mutual funds in practice. Importantly, the indexing component leads an informed manager to trade "less aggressively than normal" against mispricing when the benchmark is the market portfolio, $\phi^Y = \phi^U$: in absolute value, the manager's holdings are higher than the normal portfolio when the stock is overvalued, and lower when the stock is undervalued. We expand on this point in our analysis of informed mutual funds below.

Turning to the risk-shifting portfolio, we assess the direction in which this component deviates the manager's trading from the normal policy in the following:

Corollary 3. *At any interim state of the economy as of time $t \in [0, T')$, the sign of the risk-shifting component in the manager's portfolio (31) equals the sign of $(\phi^N_{\gamma,t} - \phi^Y)$:*

$$\operatorname{sgn}\left(\frac{1}{\sqrt{\sigma_t \tau'}}(\omega_t \Phi_{1,t} + (1-\omega_t)\Phi_{2,t})\right) = \operatorname{sgn}(\phi^N_{\gamma,t} - \phi^Y). \tag{36}$$

Thus, whether the risk-shifting component represents a long or a short position in the stock depends on the sign of the difference in the allocations to the stock of the normal and the benchmark portfolios, as first pointed out by Basak, Pavlova, and Shapiro (2007). As indicated by equation (30), the manager's normal policy in our setup is not constant but varies with the (state-dependent) extent of stock mispricing. Thus, the same manager may over- or under-invest in mispriced securities with respect to the normal portfolio at different points in time, unlike in the setting of Basak, Pavlova, and Shapiro (2007). Notably, risk-shifting in our setup can lead to over-investment, even with respect to the market portfolio, in overpriced securities when $\phi^Y < \phi^N_{\gamma,t} < \phi^U$.[21] We argued above that hedge funds' benchmarks in practice can differ substantially from the market portfolio. Therefore, we expect the risk-shifting portfolio to especially distort an informed hedge fund manager's trading against mispricing. We verify this intuition in our analysis below.

Hedge Fund and Mutual Fund Managers' Trading Against Mispricing: We now turn our attention to how the asset price dynamics described by Proposition 1 affect the trading of hedge fund and mutual fund managers in Proposition 3 for a typical parameterization of our model. We show that stock mispricing can potentially induce large deviations in the manager's policies from the normal trading.

Figure 2 illustrates the allocation in the risky asset of a hedge fund (Panel 2(a)) and a mutual fund (Panel 2(b)) informed managers (solid blue line) at a given point in time ($t = (3/4)T'$) across different states of overvaluation OV_t. Managers' policies correspond to the incentives we detail in Section 2.2, with $\phi^Y = 0$ for hedge funds and $\phi^Y = 1$ for mutual funds.[22] These benchmarks are meant to reflect the "absolute return" goal of hedge fund strategies, and the all-equity investment objective that is popular among mutual funds. The figures assume an initial stock overvaluation of 4%, corresponding to a realized dividend growth rate ρ one standard deviation lower than U-investors' prior belief ρ_0.

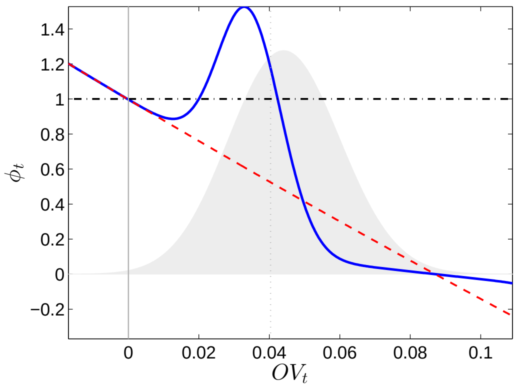
We see that the trading strategies of both the hedge fund and the mutual fund managers are highly non-linear in the extent of overpricing, including situations of both above- and below-normal exposure to the stock. This behavior is not monotonic, and both small and large stock overvaluations can lead the managers to trade *less aggressively than normal against the mispriced security*. Moreover, for this realization of initial overpricing this is the most likely behavior for both types of managers.

The results are stronger for the hedge fund manager of Fig. 2(a), who overweights the overpriced stock by a large margin with high probability. At $t = (3/4)T'$, the stock holdings of the hedge fund manager for a similar level of overpricing as at $t = 0$ can be not only higher than the normal portfolio, but also as much as 50% *higher than the market portfolio*. To understand this, note that hedge funds' absolute performance condition is equivalent to a (scaled) money market benchmark for which $\phi^Y = 0$. This implies that the indexing component in the hedge fund manager's portfolio consists of a risk-free asset-only position. Thus, the over-investment in the overvalued stock in excess of the market portfolio is due exclusively to the risk-shifting component.

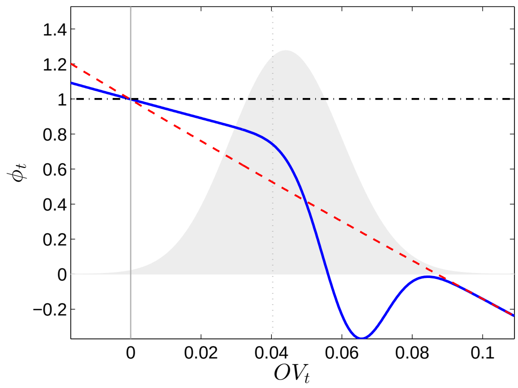
We explain the intuition behind the risk-shifting component with Figure 3, which illustrates

[21] This can be seen by reference to equations (30) and (36). According to (30), an overvalued stock ($OV_t > 0$) calls for a reduction in the normal stock holding, $\phi^N_{\gamma,t}$, relative to the market's, ϕ^U. According to equation (36), though, the risk-shifting component represents a long position in the stock. Under standard parameterizations of the model, this long position can result in a long overall position $\hat{\phi}_t$ that exceeds ϕ^U (and thus $\phi^N_{\gamma,t}$) by a large margin.

[22] The rest of the parameters of the fee structure f_T in the figure are set to capture a typical fee structure ($\alpha_1 = 0, \alpha_2 = 5, r + h = .05$) in the hedge fund industry, or a typical F-PR ($\alpha_1 = 0, \alpha_2 = 1.5, \bar{\zeta} = .94$) in the mutual fund industry, as parameterized by previous authors (see, e.g., Basak, Pavlova, and Shapiro (2007)).



(a) Hedge Fund



(b) Mutual Fund

Figure 2: **Interim Portfolio Weight in a Mispriced Stock**

The solid blue, dashed red and dash-and-dot black lines represent the informed manager's ($\hat{\phi}_t$), normal ($\phi^N_{\gamma,t}$) and market (ϕ^U) portfolio weights, respectively, in the mispriced stock for different degrees of overvaluation OV_t as of $t = (3/4)T'$. The grey area depicts the time-0 actual probability of the corresponding overvaluation state at $t = (3/4)T'$. Results obtain from a time-0 overvaluation of 4% (following a realized dividend growth rate 1 std. dev. below the prior ρ_0), as marked by the vertical dotted line. For hedge funds, we assume: $\alpha_1 = 0, \alpha_2 = 5, \bar{\zeta} = 1.05, \phi^Y = 0$. For mutual funds, we set: $\alpha_1 = 0, \alpha_2 = 1.5, \bar{\zeta} = 0.94, \phi^Y = 1$. The rest of the parameters are as follows: $T' = 1, r = 1.5\%, \delta = .0128, v_0 = 0.05^2, \gamma = 5$.

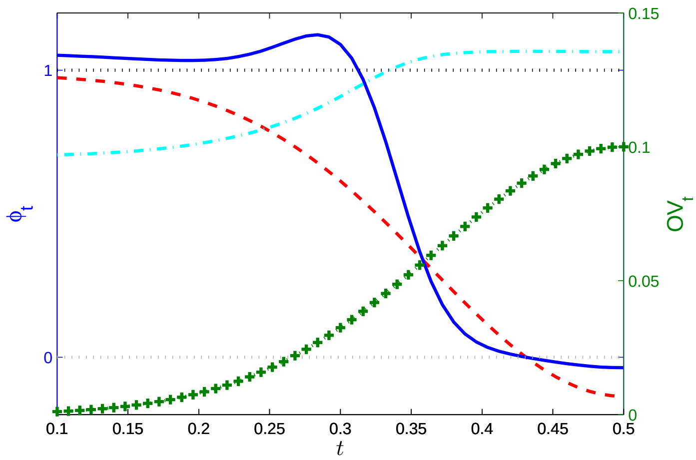


Figure 3: **Hedge Fund's Investment in the Stock during an Overvaluation Path**

The solid blue, dashed red and dotted black lines on the left axis represent the informed manager's ($\hat{\phi}_t$), normal ($\phi^N_{\gamma,t}$) and market (ϕ^U) portfolio weights, respectively, in the stock for an overvaluation path OV_t (green crossed line, right axis). The cyan dash-and-dot line represents the manager's relative performance $R_t/(\bar{\zeta}R^Y_t)$ on the left axis. Results obtain from a realized dividend growth rate 1 std. dev. below the prior ρ_0. We assume: $\alpha_1 = 0, \alpha_2 = 5, \bar{\zeta} = 1.05, \phi^Y = 0, T' = 1, r = 1.5\%, \delta = .0128, v_0 = 0.05^2, \gamma = 5$.

the hedge fund manager's portfolio for an initial trajectory of stock overvaluation, under the same parameterization as in Fig. 2(a). This trajectory is meant to resemble the dynamics of security overpricing in the initial phase of a bubble. At the beginning of the period the option implied by the manager's incentive fee is "out of the money". In other words, the manager starts off below the performance threshold necessary to receive the incentive fee. In order to reach this threshold, the risk-shifting component in the manager's portfolio increases the weight in the overpriced stock (solid blue line) over the weights in both the normal (dashed red line) and market (dash-and-dot black line) portfolios. This happens as long as the manager *underperforms* ($R_t < \bar{\zeta} R_t^Y$). The overinvestment in the stock follows from Corollary 3. As noted above, hedge funds' benchmark puts a zero weight ϕ^Y in the stock, which is lower than the weight $\phi^N_{\gamma,\cdot}$ preferred by a manager with finite risk aversion.[23] Following Corollary 3, the manager shifts risk by aggressively overweighting the overpriced stock in the portfolio, as can be seen in both Fig. 2(a) and Fig. 3.[24]

This investment pattern by the informed hedge fund manager in our model is consistent with the bubble-riding behavior documented empirically by Brunnermeier and Nagel (2004) for hedge funds during the build-up of the tech bubble in the late 1990s. These authors show that several hedge funds overweighted, relative to the market, highly overpriced technology stocks in their portfolios before the bubble burst.

As prices keep rising in Fig. 3, this strategy eventually pays off and the manager exceeds the performance threshold. As an *outperformer* ($R_t > \bar{\zeta} R_t^Y$), the manager's effective risk aversion increases. In an attempt to lock-in the interim outperformance that ensures them the performance fee payment, the optimal policy becomes more conservative and tilts the portfolio towards the indexing component. This results in a substitution of the risk-free security for the overpriced stock as overpricing peaks. At this point, the normal policy sells the overpriced stock short, whereas the indexing component limits the extent of short selling. This benchmark-induced conservative behavior contrasts with the common view of hedge funds as absolute-return investment vehicles.

As can be seen in Fig. 2(a), benchmarking concerns can induce the hedge fund manager to short-sell substantially less of the mispriced stock than the normal policy for large levels of overvaluation OV_t. Convex incentives in our model then lead to endogenous *short-sale restrictions* in the absence of explicit portfolio constraints. This behavior agrees with the decline in short interest in NASDAQ stocks during the tech bubble documented by Stein and Lamont (2004). Arguably, the incentive-based limits to short-selling that we suggest could hamper the role of sophisticated investors in stabilizing the stock market in the same fashion that explicit short-sale constraints limit pessimistic investors' trading against overvaluation in models of disagreement (see, e.g., Hong and Stein (2007)).

Figure 2(b) shows that a flow-concerned mutual fund manager can also over-invest (under-invest) in overpriced (underpriced) stocks relative to the normal portfolio. Since we set the mutual fund's benchmark equal to the market portfolio, $\phi^Y = \phi^U$, we know from Corollary 3

[23] More precisely, this occurs as long as $\ln(1 + OV_t) < \gamma\delta^2 + .5 v_t \tau'$ according to equation (30).

[24] We expect this intuition to survive a multi-asset extension of our setup, where the overpriced security is just one out of N risky assets in which the manager can trade. The reason is that, as long as the overvalued asset has a positive risk premium and provides some diversification value, the normal portfolio will include a positive holding in this asset. Then, a manager levering up the normal portfolio component could indirectly lever up the overpriced asset as well. The exact extent to which the manager levers up this asset relative to all others is left for future research.

that the risk-shifting component always represents a more aggressive stance against mispricing than the normal policy. This implies that the overall less-aggressive-than-normal trading against mispricing is explained exclusively by the indexing component. From our analysis above, this component mimics the benchmark in the outperformance region. Stock overvaluation in the model can then, simultaneously, increase the probability that the benchmark is overpriced *and* the informed mutual fund manager outperforms. This combination leads the mutual fund manager to invest more heavily in the overpriced stock. We will see in the next section that the higher the information advantage of a manager over U-investors, the higher the probability that the manager reaches the outperformance region and that, concerned by the benchmarks, trades too conservatively against mispricing.[25]

These results agree with the empirical findings of Griffin, Harris, Shu, and Topaloglu (2011), who show that both hedge funds and mutual funds invested heavily in tech stocks during the NASDAQ bubble. In addition, these authors find that hedge funds over-invested more aggressively than mutual funds on tech stocks, a situation that is plausible in our model according to a comparison of panels 2(a) and 2(b).

We highlight that in our analysis so far the manager perceives the fees at the same time T' as the stock price converges to its fundamental value. Therefore, our results do not hinge on the manager's conservatism in anticipation of potential losses triggered by further widening in mispricing, as originally suggested by Shleifer and Vishny (1997).

Two-Period Investment Problem ($T < T'$): In this case, the manager maximizes expected utility over final wealth after the second period $[T, T']$ only. For the hedge fund manager, this wealth consists of the compensation fees collected at $t = T$, and optimally invested in the financial markets throughout the second period. For the mutual fund manager, final wealth comprises the fees over the AUM at the end of the second period, which are augmented by investors' money flows at the end of the first period. Since the hedge fund's fees and the mutual fund's flows depend on relative performance during the first period, in either case final wealth depends on performance not only during the second period but in the first period as well. Thus, by the date that the hedge fund manager's fees are paid and investors' flows occur ($t = T$), asset prices S_T may (and most likely will) diverge from fundamental value S_T^{CI}. Our goal here is to derive the optimal policies that we use in our robustness analysis later on. Specifically, we verify in Section 4 that our results do not depend on the assumption we have adopted so far that prices converge to fundamental value with certainty at $t = T$.

From the timeline in Figure 1 and our description of compensation fees in Section 2.2, we can think of the two-period problem as a two-step process. From $t = 0$ to $t = T$, the manager's compensation is affected by time-T fees f_T. From $t = T$ to $t = T'$, the manager starts off with initial wealth $f_T W_T$ and receives no extra fees or income until the end of the period T'. We

[25] The indexing effect that induces bubble-riding in our model is similar to the "relative wealth concerns" identified by DeMarzo, Kaniel, and Kremer (2008), in an equilibrium asset pricing model, as a potential source of bubbles.

assume no intermediate consumption throughout the investment period.

We solve the manager's investment problem recursively. First, we solve for the manager's optimal investment, AUM and indirect utility function during the period $[T, T']$. Second, we solve for the manager's problem during the period $[0, T]$ as a maximization of the indirect utility at $t = T$.

During the period $[T, T']$, our assumptions above imply that either type of manager invests as a standard CRRA investor with superior information:

Corollary 4. *For $t \in [T, T']$, the optimal trading strategy $\hat{\phi}_t$ and wealth $\hat{W}_t$ of the informed manager is:*

$$\hat{\phi}_t = \phi^N_{\gamma,t}, \tag{37}$$

$$\hat{W}_t = \left(\lambda'_M \pi_t\right)^{-\frac{1}{\gamma}} Z_{1-\frac{1}{\gamma},t,T'}, \tag{38}$$

where $\lambda'_M = \pi_T^{\gamma-1}\left(\frac{Z_{1-\frac{1}{\gamma},T,T'}}{w_T}\right)^{\gamma}$, and w_T are the AUM as of T.

As expected, the manager's policy during the second period $[T, T']$ is the normal policy, and AUM differ from the normal portfolio value process (26) only by a constant.

We next characterize the optimal policy during the first period $[0, T]$. For brevity of exposition, we present only the manager's optimal time-T wealth profile in the following:

Proposition 4. *When $T < T'$, the informed manager's optimal time-T AUM are:*

$$\hat{W}_T = \begin{cases} (1+\alpha_1)^{\frac{1}{\gamma_1}} Z^{\frac{\gamma}{\gamma_1}}_{1-\frac{1}{\gamma},T,T'} \left(\zeta Y_T\right)^{\frac{\gamma_1-\gamma}{\gamma_1}} \left(\lambda_M \pi_T\right)^{-\frac{1}{\gamma_1}}, & \text{if } \lambda_M \pi_T > b\left(\zeta Y_T / Z_{1-\frac{1}{\gamma},T,T'}\right), \\ (1+\alpha_2)^{\frac{1}{\gamma_2}} Z^{\frac{\gamma}{\gamma_2}}_{1-\frac{1}{\gamma},T,T'} \left(\zeta Y_T\right)^{\frac{\gamma_2-\gamma}{\gamma_2}} \left(\lambda_M \pi_T\right)^{-\frac{1}{\gamma_2}}, & \text{if } \lambda_M \pi_T \le b\left(\zeta Y_T / Z_{1-\frac{1}{\gamma},T,T'}\right), \end{cases} \tag{39}$$

where $\zeta \equiv \bar{\zeta} W_0 / Y_0$, λ_M is the Lagrange multiplier that solves $E_0[\pi_T \hat{W}_T] = w_0$ and the function $b(.)$ is as given in Appendix A.

We use the informed manager's optimal wealth profile (39) in the next section to gain insight about the manager's average investment in a mispriced stock in the two-period case.

4 Average Trading of the Informed Manager

Our analysis of Section 3.2 focused on a particular realization of the dividend growth rate ρ at the beginning of the period. In this section we want to verify that the intuition we derived is robust and not specific to this particular realization of ρ. More precisely, we derive predictions about the informed manager's *average* investment against mispricing, i.e. the expected trading *before* the value of ρ is realized. To accomplish this, we examine the impact of the two parameters driving mispricing in our model: U-investors initial estimation error, $\rho_0 - \rho$, and the manager's degree of initial information advantage over U-investors, v_0. Whereas the difference $\rho_0 - \rho$ controls the degree of initial overpricing, we show below that v_0 drives the extent of the expected mispricing in the economy. In this section we consider the average mispricing and portfolio strategies of the money manager for different values of ρ that we generate from the prior distribution $\mathbf{N}(\rho_0, v_0)$ of the uninformed investors.

Our model parameterization assumes that the manager receives the compensation fees at the end of the year, i.e. $T = 1$. All agents' RRA coefficient is set to $\gamma = 5$. Similar to Brennan and Xia (2001) we assume a prior dividend growth rate $\rho_0 = 0.0238$, with associated standard error $\sqrt{v_0} = 0.037$. The dividend volatility is set to $\delta = 0.129$, and the risk-free rate is set to $r = 0.015$.

We choose our parameterization of the fee rate f_T to reflect either the typical fee structure in the hedge fund industry, or the typical flow-performance relationships in the mutual fund industry. In the case of a hedge fund, we consider the following parameterization: $\alpha_1 = 0, \alpha_2 = 5, \phi^Y = 0, r + h = .05$. In the case of a mutual fund, we consider three alternative parameterizations that account for theoretically and empirically motivated differences in the sensitivity of mutual fund flows to bottom and mid-range performance.[26] In all three scenarios, we assume a moderately high flow sensitivity to top performance, $\alpha_2 = 1.5$; we also assume that the benchmark for mutual funds is the market portfolio, i.e. $\phi^Y = 1$. We consider the following scenarios:

(i) No sensitivity of the mutual fund flows to bottom performance, little sensitivity to medium performance: $\alpha_1 = 0, \bar{\zeta} = 1$

(ii) No sensitivity of the mutual fund flows to bottom performance, high sensitivity to medium performance: $\alpha_1 = 0, \bar{\zeta} = .94$

(iii) Moderate sensitivity of the mutual fund flows to bottom performance, higher sensitivity to medium performance: $\alpha_1 = 0.5, \bar{\zeta} = 1$

4.1 Within-Period Trading in Response to Mispricing

The difference in the trading against mispricing between the informed manager and the normal portfolio during the whole investment period $[0, T]$ presumably depends on the extent of initial mispricing OV_0. In examining this dependence, we draw on the following observations to compute a meaningful distance measure between these two policies. First, we know from our analysis in Section 3.2 that the mutual fund manager overweights the stock relative to the market only if the normal portfolio overweights it as well, and similarly for the manager's decision to underinvest. This implies that for the mutual fund manager $(\hat{\phi} - 1)$ and $(\phi^N - 1)$ always have the same sign: negative for the overvalued stock and positive for the undervalued stock. To differentiate between the possible outcomes, we say that the manager *trades less (respectively, more) aggressively than normal* against mispricing if $|\hat{\phi} - 1| < |\phi^N - 1|$ $(|\hat{\phi} - 1| > |\phi^N - 1|)$.

[26] See, e.g., Berk and Green (2004) and Huang, Wei, and Yan (2007) for models that identify mutual fund characteristics driving differences in the flow relationship. The latter authors provide empirical support for their theoretical predictions.

Second, consider the tracking error volatility relative to the market (simply "tracking error" henceforth) of a portfolio with value process W and weight process ϕ in the stock over the period $[0, T]$, defined as

$$TE \equiv StdDev(\log(R_T) - \log(R_T^S)), \tag{40}$$

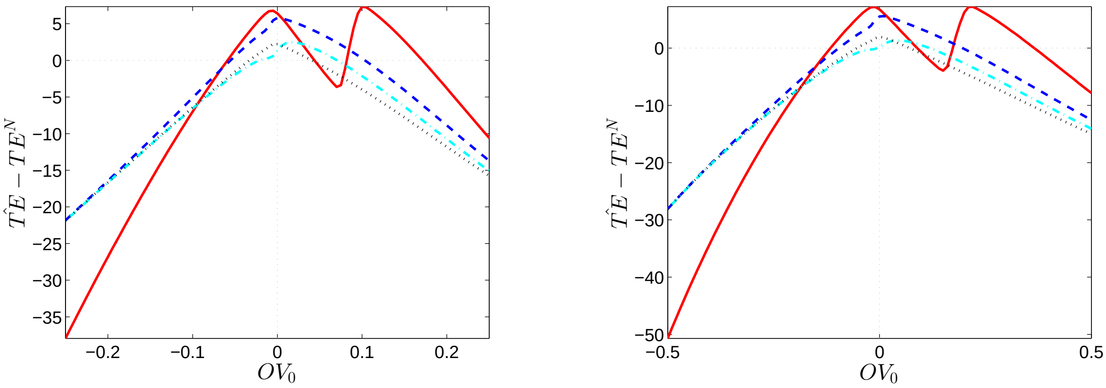
where $R_T \equiv W_T/W_0$. This tracking error is monotonically increasing in the distance $|\phi - 1|$.[27] This implies a link between negative (positive) values of the excess tracking error $\hat{TE} - TE^N$ of the manager compared with that of the normal policy and less-aggressive-than-normal (more-aggressive-than-normal) trading against mispricing.

Third, for the hedge fund (Panel 2(a)) we note that whenever $(\hat{\phi} - 1)$ and $(\phi^N - 1)$ have opposite signs the manager's excess stock holdings—relative to the normal portfolio—have *the same sign as the mispricing*, in the sense that the manager's portfolio either overweights an overpriced stock or underweights an underpriced stock *more than the market portfolio*. In these situations, the hedge fund manager's excess tracking error can be positive as long as $|\hat{\phi} - 1| > |\phi^N - 1|$, even though the manager clearly *fails to trade against the mispricing*. As a consequence, in our model the tracking error overstates the extent of the trading *against* mispricing of the hedge fund manager. Equivalently, it understates then extent to which the hedge fund manager fails to trade against the mispricing.

Figure 4 plots the informed manager's average excess tracking error for different levels of initial mispricing, for both the cases $T = T'$ and $T < T'$ in Section 3.2. As different initial values of mispricing give rise to different distribution of paths $\{OV_t\}_{0 \leq t \leq T'}$, the manager's average trading computed over these distributions varies considerably with initial mispricing.

The large negative values of $\hat{TE} - TE^N < 0$ in Panel 4(a) indicate that for a high enough extent of mispricing—both over- and undervaluation—both types of money managers overweight an initially overpriced stock during the investment period, and conversely for an initially underpriced stock. This pattern does not change when the managers have concerns about compensation in a second period ($[T, T']$), as shown by the managers' trading in Panel 4(b). Thus, the incentives to trade against a given initial mispricing are weakest for highest under- and overvaluation. The economic intuition follows from our analysis in Section 3.2: a larger initial mispricing improves the odds that the manager will outperform early on and lock in the outperformance thereafter. The hedge fund manager locks in the current performance by (partially) mimicking a risk-free asset, which results in either too few holdings of an underpriced stock, or too little trading against a severely overpriced stock. Similarly, the mutual fund manager locks in performance by mimicking the market portfolio, leading to excessive holdings of an overpriced stock or insufficient investment in an underpriced stock. Moreover, the mutual fund manager trades less aggressively against mispricing for all levels of over- and undervaluation as the sensitivity of flows to medium and bottom performance increases. This result makes clear how investors' flows can induce excessive trading in overpriced assets and is consistent with the

[27] Indeed, for the wealth process W in Eq. (3), we have: $TE_t = StdDev_t\,[d\ln(W_t/S_t)] = \sigma_t|\phi_t - 1|dt$.



(a) $T = T'$ (b) $T < T'$

Figure 4: **The Manager's Excess Tracking Error in Response to Initial Mispricing**

Informed manager's average excess tracking error $\hat{TE} - TE^N$ (in %) over $[0, T]$ for different initial levels of overvaluation OV_0. The average is computed over the range of overpricing states as of $t = T$. The solid, dashed, dash-and-dot and dotted lines correspond, respectively, to the policies of a hedge fund manager and of a mutual fund manager subject to the flow relationships (i) to (iii) in Section 4. Panel 4(a) shows results for the case $T = T'$, whereas Panel 4(b) shows results for the case $T < T'$. All results correspond to the following parameterization: $\tilde{\rho}_0 = 0.0238, \sqrt{v_0} = 0.037, T = 1, r = 1.5\%, \delta = .0129, \gamma = 5$.

empirical findings of Brunnermeier and Nagel (2004) and Greenwood and Nagel (2009).

The results in this subsection generalize our observations in Section 3.2. In particular, an informed money managers responding to convex incentives can trade less aggressively against mispricing than would be expected absent such incentives. This problem is particularly severe for sophisticated investors like hedge fund managers relative to mutual fund managers, and for both hedge fund and mutual fund managers in situations of very high over- and underpricing. In the next subsection we show that this is also the case in circumstances of heightened uncertainty, when the manager's information advantage over uninformed traders in the economy widens.

In order to build up intuition for our analysis in the following subsection, Figure 5 plots the hedge fund manager's time-t excess holdings of the stock, relative to the normal portfolio, for different values of the manager's information advantage v_0. To remove the dependence on the realization of ρ, the policies are plotted as a function of the current overvaluation OV_t by averaging the manager's trading across all paths of the inferred growth rate $\tilde{\rho}_t$ that lead to that particular value of overvaluation. If the manager traded more aggressively against the mispricing than the normal policy, the policies should plot in the top-left and bottom-right quadrants of the figure.

As expected from the analysis above, on average the hedge fund manager fails to trade against any levels of undervaluation, as well as against high levels of overvaluation. Crucially, the problem becomes worse in general as (i) up-to-date overvaluation OV_t worsens, and (ii) the manager's initial information advantage v_0 improves. The intuition underlying (ii) is similar to the intuition we provided above for result (i): as the manager's advantage over retail traders widens, the probability of outperforming the benchmark rises and the indexing component dominates the manager's portfolio. This behavior leads to an overly conservative stance

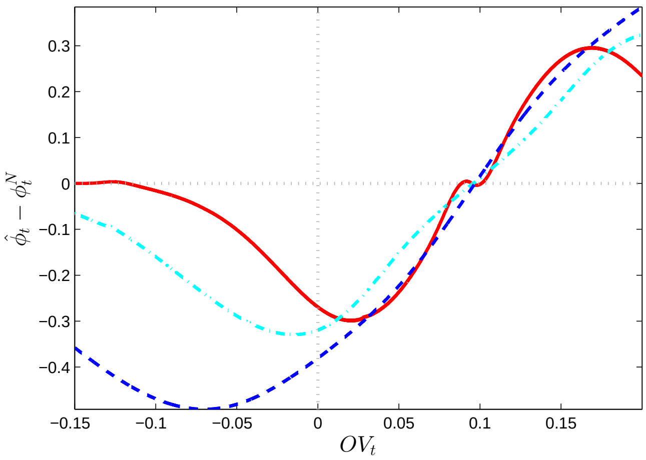


Figure 5: **Hedge Fund Manager's Average Over-Investment in the Mispriced Stock**
Average time-t optimal weight in the stock of the hedge fund manager in excess of the normal portfolio, $\hat{\phi}_t - \phi_t^N$, as of mid-year ($t = 0.5$). For each level of overvaluation OV_t, the average is computed over all the paths of the inferred dividend growth rate $\tilde{\rho}_t$ that result in that particular value of OV_t. The red solid, cyan dash-and-dot and blue dashed lines correspond, respectively, to levels of U-investors' initial uncertainty $\sqrt{v_0}$ equal to 3.7%, 7.4% and 11.1%. We assume: $\alpha_1 = 0, \alpha_2 = 5, \bar{\zeta} = 1.05, \phi^Y = 0, \tilde{\rho}_0 = 0.0238, T = T' = 1, r = 1.5\%, \delta = .0129, \gamma = 5$.

against mispricing and, as we show in the next subsection, worsens precisely when the manager's potential profits from trading against the mispricing are the greatest.

4.2 Trading on Superior Information

An important question we can answer in our setup is whether an increase in expected mispricing leads an informed investor to trade more aggressively against it. Similar to our analysis above, we are interested in assessing first whether the normal policy conforms with the common wisdom as expected mispricing exacerbates, and then how an informed manager subject to convex incentives trades relative to this normal policy in the same circumstances.

We examine this problem by carrying out a comparative statics analysis based on the information advantage parameter v_t that measures U-investors' degree of uncertainty about ρ. Without loss of generality, we focus on the case $t = 0$, making the analysis ex-ante in nature: we assess the informed manager's expected trading by averaging the model results corresponding to different realizations of ρ over its distribution at $t = 0$, $\mathbf{N}(\rho_0, v_0)$. Since we assume that uninformed investors' prior distribution for ρ is correct ($\tilde{\rho}_0 = \rho_0$), ρ_0 can be set to equal any arbitrary value. The only parameter characterizing the manager's ex-ante information advantage is thus v_0.

Let $E_t^\rho(.)$ denote the expectation over the distribution $\mathbf{N}(\tilde{\rho}_t, v_t)$ of ρ, for $t \in [0, T']$. We introduce the following terminology:

1. **Expected Mispricing (EMP)**: refers to a situation in which $E_0^\rho(OV_0) \neq 0$, regardless of whether in expectation $S_0 > S_0^{CI}$ or $S_0 < S_0^{CI}$ (i.e. the stock is expected to be over- or undervalued). According to Section 3.1, this corresponds to all realizations of ρ such that

$\rho \neq \tilde{\rho}_0 - \left(\gamma - \frac{1}{2}\right) v_0 T'$.

2. **Expected Overvaluation (EOV)**: refers to a situation in which $E_0^{\rho}(OV_0) > 0$. According to Section 3.1, this corresponds to all realizations of ρ such that $\rho < \tilde{\rho}_0 - \left(\gamma - \frac{1}{2}\right) v_0 T'$.

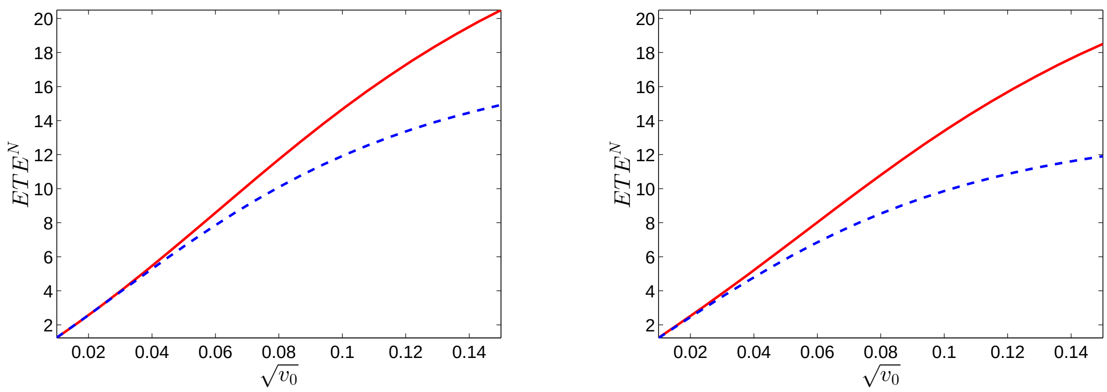
The following lemma characterizes the relation between U-investors' parameter uncertainty v and expected stock mispricing/overvaluation:

Lemma 1. *For $t \in [0, T']$, the time-t expected mispricing $EMP(t) \equiv \left| E_t^{\rho}\left[(1 + OV_t)^{\tau'}\right] - 1 \right|$ and overvaluation $EOV(t) \equiv E_t^{\rho}\left[(1 + OV_t)^{\tau'} \mid OV_t \geq 0\right] - 1$ are given by:*

$$EMP(t) = \left| e^{-(\gamma-1)v_t(\tau')^2} - 1 \right|, \tag{41}$$

$$EOV(t) = e^{-(\gamma-1)v_t(\tau')^2} \frac{\mathcal{N}\left(-(\gamma - \frac{3}{2})\sqrt{v_t}\tau'\right)}{\mathcal{N}\left(-(\gamma - \frac{1}{2})\sqrt{v_t}\tau'\right)} - 1 > 0. \tag{42}$$

From Lemma 1 it is easy to see that $EMP(t)$ is monotonically increasing in v_t. Moreover, for all the model parameterizations that we use in our analysis throughout the paper we verify that $EOV(t)$ is also monotonically increasing in v_t. Hence, there is a *one-to-one positive relationship* between the manager's information advantage (U-investors' parameter uncertainty) and both expected mispricing and expected overvaluation.



(a) Expected Mispricing

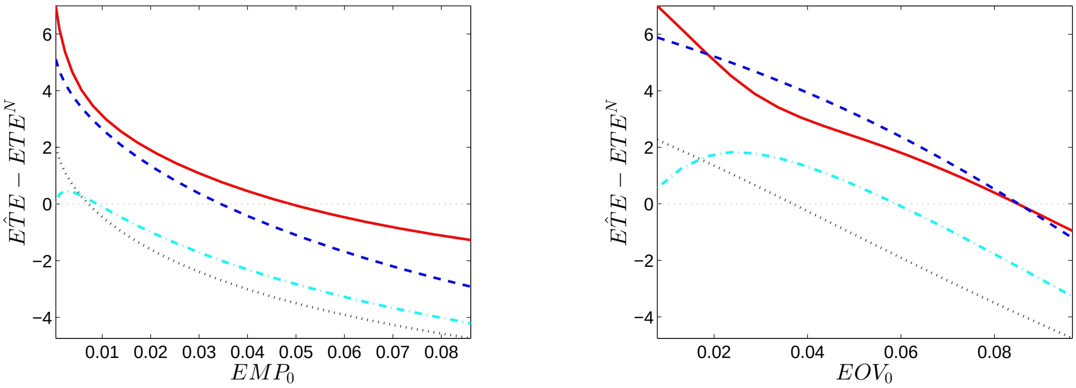
(b) Expected Overvaluation

Figure 6: **Expected Normal Tracking Error**

The solid blue and dashed red lines represent, for the cases $T' = 1$ and $T' = 2$, respectively, the manager's expected normal (i.e., absent convex incentives) tracking error ETE^N (in %) in response to expected mispricing (Panel 6(a)) and to expected overvaluation (Panel 6(b)). Results correspond to the following parameter values: $\tilde{\rho}_0 = 0.0238, T = 1, r = 1.5\%, \delta = .0129, \gamma = 5$.

Based on our discussion in Section 3.2, we next study the optimal trading under mispricing by inspecting the expected tracking error over the total investment period: $ETE \equiv E_0^{\rho}[TE]$. Figure 6 plots the expected normal (i.e., absent convex incentives) tracking error ETE^N for different levels of information advantage of the informed investor over the uninformed traders. We see that the trading pattern implied by the normal policy agrees with the expected behavior of informed investors under the standard paradigm, as well as with our results in the previous section. Specifically, trading against both mispricing (Panel 6(a)) and overvaluation (Panel 6(a)) when either $T = T' = 1$ (blue solid line) or $T < T' = 2$ (red dashed line) intensifies with the information advantage of the informed normal investor or, equivalently, with the degree of expected stock mispricing or overvaluation.

For an informed manager subject to convex incentives, we analyze the trading against mispricing and against overvaluation for different levels of initial information advantage v_0 by computing the *excess* expected tracking error with respect to the normal expected tracking error: $E\hat{T}E - ETE^N$. For exposition brevity, we present the results corresponding only to the one-period investment ($T' = 1$) in Figure 7, as results for the two-period problem ($T' = 2$) are qualitatively similar.



(a) Expected Mispricing (b) Expected Overvaluation

Figure 7: **The Manager's Excess Expected Tracking Error**

The manager's expected excess tracking error $E\hat{T}E - ETE^N$ (in %) over $[0, T]$ for different initial levels of expected mispricing (Panel 7(a)) and expected overvaluation (Panel 7(b)). The solid, dashed, dash-and-dot and dotted lines correspond, respectively, to the policies of a hedge fund manager and of mutual fund managers subject to the flow relationships (i) to (iii) in Section 4. All results correspond to the following parameterization: $\tilde{\rho}_0 = 0.0238, T = 1, r = 1.5\%, \delta = .0129, \gamma = 5$.

We draw the following observations from Figure 7. First, the intensity of the trading against expected mispricing and overvaluation of the informed manager generally falls with the extent of the mispricing.[28] This result generalizes result (ii) at the end of the previous subsection, and follows the same economic intuition. Second, as U-investors uncertainty peaks, resulting in high levels of expected mispricing and overvaluation, all types of managers eventually trade less aggressively against the mispriced stock than required by the normal policy ($\hat{TE} - TE^N < 0$).

We conclude that when managers face convex incentives, their optimal investment strategy can in general be inconsistent with the common-wisdom notion that trading against mispricing should intensify with the extent of mispricing. Furthermore, and contrary to the hypothesis that option-like incentives can induce managers to bet against overpriced securities, we have shown

[28] The only exception is the excess expected tracking error by the informed manager of a mutual fund with no flow sensitivity to bottom relative performance but high flow sensitivity to mid-range performance (case (ii) in Section 3.2), for low levels of overpricing.

that this type of incentives can exacerbate, rather than lessen, informed money managers' over-investment in overpriced assets. This incentive-induced failure to trade against mispricing can worsen as expected mispricing heightens. Following Pastor and Veronesi (2009)'s interpretation of a bubble in stock prices as a period of high uncertainty about the productivity of a technology, we conclude that convex incentives can induce bubble-riding behavior by informed money managers even in the absence of career concerns or other market frictions.

5 Conclusions

In this paper we consider the effects of convex incentives on the trading against mispricing of a money manager with superior information. According to the standard paradigm, the trading of an agent with superior information should vary depending on the level of mispricing or deviation of the security price from the fundamentals; in particular, the investor should underweight an overpriced security and overweight an underpriced security. Even in the presence of career concerns, a recent line of research suggests that short-term incentive contracts should induce trading against overpricing and offset the bubble-riding behavior resulting from these concerns.

We find that convex incentives alter these conclusions. In particular, it can be optimal for an informed money manager to over-invest—relative to the standard level or even to the market portfolio—in overpriced securities, so as if "riding the bubble." We further show that this behavior worsens as expected overpricing increases. Our model is able to reconcile some puzzling empirical findings without recurring to behavioral arguments, and only using incentives documented in the literature—although not standard in financial models. We do not solve a full equilibrium model, but such behavior would arguably "fuel" the bubble.

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Appendix

A Proofs and Auxiliary Results

We start by stating two auxiliary lemmas that are used throughout the remaining proofs.

Lemma A1. *Let $\tau' = T' - t$. For $0 \leq t \leq T'$, $\alpha \in \mathbb{R}$:*

$$\tilde{E}_t\left[D^{\alpha}_{T'}\right] = D^{\alpha}_t \exp\left\{\left(\alpha\tilde{\rho}_t - \frac{1-\alpha}{2}\alpha\delta^2 + \frac{\alpha^2}{2}v_t\tau'\right)\tau'\right\} \tag{43}$$

Proof. The dynamics of D_t under the filtered probability are $dD_t = D_t(\tilde{\rho}_t dt + \delta d\tilde{B}_t)$, or, for $0 \leq t \leq t' \leq T'$:

$$\begin{aligned} D_{t'} &= D_t e^{\int_t^{t'}\left(\tilde{\rho}_s - \frac{\delta^2}{2}\right)ds + \delta(\tilde{B}_{t'} - \tilde{B}_t)} \\ &= D_t e^{-\frac{\delta^2}{2}(t'-t) + \int_t^{t'}\tilde{\rho}_s ds + \delta(\tilde{B}_{t'} - \tilde{B}_t)}. \end{aligned} \tag{44}$$

From (12), and using the solution to v as: $v_t = \frac{\delta^2 v_0}{\delta^2 + v_0 t}$,

$$\begin{aligned} \frac{d\tilde{\rho}_t}{v_t} = \delta d\tilde{B}_t &\Rightarrow \int_t^{t'}\frac{d\tilde{\rho}_t}{v_t} = \delta(\tilde{B}_{t'} - \tilde{B}_t) \\ &\Rightarrow \frac{\tilde{\rho}_{t'}}{v_{t'}} - \frac{\tilde{\rho}_t}{v_t} - \int_t^{t'}\tilde{\rho}_s ds = \delta(\tilde{B}_{t'} - \tilde{B}_t) \\ &\Rightarrow \frac{\tilde{\rho}_{t'}}{v_{t'}} - \frac{\tilde{\rho}_t}{v_t} = \int_t^{t'}\tilde{\rho}_s ds + \delta(\tilde{B}_{t'} - \tilde{B}_t), \end{aligned} \tag{45}$$

which allows us to re-express (44) as:

$$D_t e^{-\frac{\delta^2}{2}(t'-t) + \int_t^{t'}\tilde{\rho}_s ds + \delta(\tilde{B}_{t'} - \tilde{B}_t)} = D_t e^{-\frac{\delta^2}{2}(t'-t) + \frac{\tilde{\rho}_{t'}}{v_{t'}} - \frac{\tilde{\rho}_t}{v_t}}. \tag{46}$$

Note that, conditioning on $\mathcal{F}^D_t$, the only random variable in the former expression is $\tilde{\rho}_{t'}$. Moreover, from (12) we know that $\tilde{\rho}_t$ is a linear diffusion with deterministic volatility, so:

$$\tilde{\rho}_{t'}|\tilde{\rho}_t = \tilde{\rho}_t + \frac{1}{\delta}\int_t^{t'} v_s d\tilde{B}_s|\tilde{\rho}_t \sim \mathbf{N}\left(\tilde{\rho}_t, \sigma^2_{\tilde{\rho},t,t'}\right), \tag{47}$$

with $\sigma^2_{\tilde{\rho},t,t'} = \frac{1}{\delta^2}\int_t^{t'}(v_s)^2 ds = v_t - v_{t'}$. This implies that $D_{T'}|D_t$ is log-normally distributed with deterministic mean and variance, so:

$$\begin{aligned} \tilde{E}_t\left[D^{\alpha}_{T'}\right] =& D^{\alpha}_t e^{-\alpha\frac{\delta^2}{2}\tau' - \alpha\frac{\delta^2}{v_t}\tilde{\rho}_t} E_t\left[e^{\alpha\frac{\delta^2}{v_{T'}}\tilde{\rho}_{T'}}\right] \\ =& D^{\alpha}_t \exp\left\{-\alpha\frac{\delta^2}{2}\tau' - \alpha\delta^2\left(\frac{1}{v_t} - \frac{1}{v_{T'}}\right)\tilde{\rho}_t + \frac{\alpha^2\delta^4}{2v^2_{T'}}(v_t - v_{T'})\right\}, \end{aligned} \tag{48}$$

which results in (43) after some algebraic manipulations. $\square$

Lemma A2. *Let $z \sim \mathbf{N}(0, \sigma_z^2)$, and let $\breve{\rho}, c, \bar{z} \in \mathbb{R}$. We have:*

$$E\left[e^{-\breve{\rho}(z-c)^2}\mathbb{1}_{\{z\leq\bar{z}\}}\right] = \frac{e^{-\frac{\breve{\rho}c^2}{1+2\breve{\rho}\sigma_z^2}}}{\sqrt{1+2\breve{\rho}\sigma_z^2}}\mathcal{N}\left(\frac{\bar{z}-\frac{2\breve{\rho}\sigma_z^2}{1+2\breve{\rho}\sigma_z^2}c}{\sigma_z/\sqrt{1+2\breve{\rho}\sigma_z^2}}\right), \tag{49}$$

where $\mathcal{N}(.)$ is the standard normal cumulative distribution function.

Proof. Follows from direct integration against the normal density, using the change of variables $\check{z} = \frac{z-\frac{2\breve{\rho}\sigma_z^2}{1+2\breve{\rho}\sigma_z^2}}{\sigma_z/\sqrt{1+2\breve{\rho}\sigma_z^2}}$, $\check{z}_l = \frac{\bar{z}-\frac{2\breve{\rho}\sigma_z^2}{1+2\breve{\rho}\sigma_z^2}}{\sigma_z/\sqrt{1+2\breve{\rho}\sigma_z^2}}$. □

Proof of Proposition 1. The standard solution to uninformed investors' optimization problem is:

$$\hat{W}_{T'}^U = (\lambda\tilde{\pi}_{T'})^{-\frac{1}{\gamma}} \Rightarrow \tilde{\pi}_{T'} = \frac{1}{\lambda}(\hat{W}_{T'}^U)^{-\gamma}. \tag{50}$$

By market clearing condition (11):

$$\hat{W}_{T'}^U = D_{T'} \Rightarrow \tilde{\pi}_{T'} = \frac{1}{\lambda}D_{T'}^{-\gamma}. \tag{51}$$

Uninformed investors' equilibrium SPD is:

$$\tilde{\pi}_t = e^{r(T'-t)}\tilde{E}_t[\tilde{\pi}_{T'}] = e^{r(T'-t)}\tilde{E}_t\left[D_{T'}^{-\gamma}\right]. \tag{52}$$

Applying Lemma A1 for $\alpha = -\gamma$, uninformed investors' equilibrium SPD is then:

$$\tilde{\pi}_t = \lambda^{-1}D_t^{-\gamma}\exp\left\{\left(r - \gamma\tilde{\rho}_t + \frac{1+\gamma}{2}\gamma\delta^2 + \frac{\gamma^2}{2}v_t\tau'\right)\tau'\right\}. \tag{53}$$

Using (53) to solve for λ in the equation $\tilde{\pi}_0 = 1$:

$$\lambda = D_0^{-\gamma}\exp\left\{\left(r - \gamma\tilde{\rho}_0 + \frac{1+\gamma}{2}\gamma\delta^2 + \frac{\gamma^2}{2}v_0T'\right)T'\right\}. \tag{54}$$

By no-arbitrage, equilibrium stock prices are:

$$S_t = \tilde{\pi}_t^{-1}\tilde{E}_t[\tilde{\pi}_{T'}D_{T'}] = (\lambda\tilde{\pi}_t)^{-1}\tilde{E}_t\left[D_{T'}^{1-\gamma}\right] \tag{55}$$

Using Lemma A1 for $\alpha = 1-\gamma$ and equation (53):

$$S_t = D_t\exp\left\{\left(\tilde{\rho}_t - r - \gamma\delta^2 - \left(\gamma - \frac{1}{2}\right)v_t\tau'\right)\right\}. \tag{56}$$

Applying Itô's Lemma to (56)

$$dS_t = \left(r + \gamma\left(\delta + \frac{v_t}{\delta}\tau'\right)^2\right)S_tdt + \left(\delta + \frac{v_t}{\delta}\tau'\right)S_td\tilde{B}_t. \tag{57}$$

Under the $\tilde{P}$-probability, the stock dynamics (2) can be rewritten as:

$$dS_t = \tilde{\mu}_tS_tdt + \sigma_tS_td\tilde{B}_t. \tag{58}$$

Comparing the drift and diffusion terms of (57) and (58) we get equations (18). □

Proof of Corollary 1. Equations (19)-(20) follow by letting $\rho_0 \to \rho$ and $v_0 \to 0$ in Proposition 1. To obtain (21), we divide (16) by (19) to get:

$$\frac{S_t}{S_t^{CI}} = \exp\left\{\left(\tilde{\rho}_t - \rho - \left(\gamma - \frac{1}{2}\right) v_t \tau'\right) \tau'\right\}. \tag{59}$$

The result then follows from the definition of OV_t. □

Proof of Proposition 2. For an informed direct investor with RRA coefficient $\tilde{\gamma}$, the normal optimization problem is:

$$\max_{W_{\tilde{\gamma},T'}} E_0\left[\frac{(W_{\tilde{\gamma},T'})^{1-\tilde{\gamma}}}{1-\tilde{\gamma}}\right], \tag{60}$$

subject to:

$$E_0\left[\pi_{T'} W_{\tilde{\gamma},T'}\right] = w_0. \tag{61}$$

Attaching Lagrange multiplier λ_N to the budget constraint (61),the normal time-T' optimal wealth profile is given by the first order condition:

$$W^N_{\tilde{\gamma},T'} = \left(\lambda_N \pi_{T'}\right)^{-\frac{1}{\tilde{\gamma}}}, \tag{62}$$

where the Lagrange multiplier λ_N is given by:

$$\lambda_N = w_0^{-\tilde{\gamma}}\left(E_0\left[(\pi_{T'})^{1-\frac{1}{\tilde{\gamma}}}\right]\right)^{\tilde{\gamma}} = \left(\frac{Z_{1-\frac{1}{\tilde{\gamma}},0,T'}}{w_0}\right)^{\tilde{\gamma}}. \tag{63}$$

The normal time-t ($0 \le t \le T'$) portfolio value $W^N_{\tilde{\gamma},t}$ is given by the no-arbitrage condition:

$$\begin{aligned} \pi_t W^N_{\tilde{\gamma},t} &= E_t\left[\pi_{T'} W^N_{\tilde{\gamma},T'}\right] \\ \Rightarrow W^N_{\tilde{\gamma},t} &= \left(\lambda_N \pi_t\right)^{-\frac{1}{\tilde{\gamma}}} E_t\left[\left(\frac{\pi_{T'}}{\pi_t}\right)^{1-\frac{1}{\tilde{\gamma}}}\right] = \left(\lambda_N \pi_t\right)^{-\frac{1}{\tilde{\gamma}}} Z_{1-\frac{1}{\tilde{\gamma}},t,T'}, \end{aligned} \tag{64}$$

with $Z_{1-\frac{1}{\tilde{\gamma}},t,T'} = E_t\left[\left(\frac{\pi_{T'}}{\pi_t}\right)^{1-\frac{1}{\tilde{\gamma}}}\right]$. The following lemma provides a closed-form expression for $Z_{1-\frac{1}{\tilde{\gamma}},t,T'}$:

Lemma A3. *Let $\psi \in \mathbb{R}$. For $0 \le t \le t' \le T'$:*

$$\begin{aligned} Z_{\psi,t,t'} \equiv& E_t\left[\left(\frac{\pi_{t'}}{\pi_t}\right)^{\psi}\right] \\ =& \delta^{\psi}\sqrt{\frac{(\delta^2 + v_t(t'-t))^{1-\psi}}{\delta^2 + (1-\psi)v_t(t'-t)}} \exp\left\{-\psi r(t'-t) - \frac{\psi(1-\psi)\delta^2(t'-t)}{\delta^2 + (1-\psi)v_t(t'-t)}\frac{\eta_t^2}{2}\right\} \end{aligned} \tag{65}$$

Proof. We first simplify the expression of the likelihood process ξ_t. Defining $\kappa_t \equiv \frac{\rho - \tilde{\rho}_t}{\delta}$, the likelihood process can be rewritten as:

$$\xi_t = e^{-\frac{1}{2}\int_0^t \kappa_s^2 ds - \int_0^t \kappa_s dB_s}. \tag{66}$$

The manager sees the dynamics of $\tilde{\rho}$ in (12) as:

$$d\tilde{\rho}_t = \frac{v_t}{\delta}\left(\frac{\rho - \tilde{\rho}_t}{\delta} + dB_t\right) = \frac{v_t}{\delta^2}(\rho - \tilde{\rho}_t)dt + \frac{v_t}{\delta}dB_t. \tag{67}$$

An application of Itô's Lemma gives the dynamics of $\frac{\tilde{\rho}_t}{v_t}$ as:

$$d\left(\frac{\tilde{\rho}_t}{v_t}\right) = \frac{\rho}{\delta^2}dt + \frac{1}{\delta}dB_t \tag{68}$$

A further application of Itô's Lemma to the product $\kappa_t \frac{\tilde{\rho}_t}{v_t}$ gives:

$$\begin{aligned}
d\left(\kappa_t \frac{\tilde{\rho}_t}{v_t}\right) =& \frac{\tilde{\rho}_t}{v_t} d\kappa_t + \kappa_t d\left(\frac{\tilde{\rho}_t}{v_t}\right) + d\left\langle \kappa, \frac{\tilde{\rho}}{v}\right\rangle_t \\
=& -\frac{\tilde{\rho}_t}{\delta^2}\left(\kappa_t dt + dB_t\right) + \kappa_t\left(\frac{\rho}{\delta^2}dt + \frac{dB_t}{\delta}\right) - \frac{v_t}{\delta^3}dt \\
=& \frac{2}{\delta}\left(\frac{\kappa^2}{2}dt + \kappa_t dB_t - \frac{v_t}{2\delta^2}dt - \frac{\rho}{2\delta}dB_t\right) \\
\Rightarrow& \frac{\kappa^2}{2}dt + \kappa_t dB_t = \frac{\delta}{2} d\left(\kappa_t \frac{\tilde{\rho}_t}{v_t}\right) + \frac{1}{2}\left(\frac{v_t}{\delta^2}dt + \frac{\rho}{\delta}dB_t\right).
\end{aligned} \tag{69}$$

Integrating both side from 0 to t:

$$\frac{1}{2}\int_0^t \kappa_s^2 ds + \int_0^t \kappa_s dB_s = \frac{\delta}{2}\int_0^t d\left(\kappa_s \frac{\tilde{\rho}_s}{v_s}\right) + \frac{1}{2\delta^2}\int_0^t v_s ds + \frac{\rho}{2\delta}\int_0^t dB_s. \tag{70}$$

Direct integration gives the second integral on the RHS as: $\int_0^t v_s ds = -\delta^2 \ln(v_t/v_0)$, which allows us to re-express the likelihood process as:

$$\xi_t = e^{-\frac{1}{2}\int_0^t \kappa_s^2 ds - \int_0^t \kappa_s dB_s} = \sqrt{\frac{v_t}{v_0}} e^{-\frac{\delta}{2}\left(\frac{\rho-\tilde{\rho}_t}{\delta}\frac{\tilde{\rho}_t}{v_t} - \frac{\rho-\tilde{\rho}_0}{\delta}\frac{\tilde{\rho}_0}{v_0}\right) - \frac{\rho}{2\delta}B_t}, \tag{71}$$

or, for $t' \geq t$:

$$\frac{\xi_{t'}}{\xi_t} = \frac{\delta}{\sqrt{\delta^2 + v_t(t'-t)}} e^{-\frac{1}{2}\left((\rho-\tilde{\rho}_{t'})\frac{\tilde{\rho}_{t'}}{v_{t'}} - (\rho-\tilde{\rho}_t)\frac{\tilde{\rho}_t}{v_t} - \frac{\rho}{\delta}(B_{t'}-B_t)\right)}. \tag{72}$$

Integrating both sides of (68), from t to t', we can solve for $B_{t'} - B_t$ as:

$$B_{t'} - B_t = -\frac{\rho}{\delta}(t'-t) + \delta\left(\frac{\tilde{\rho}_{t'}}{v_{t'}} - \frac{\tilde{\rho}_t}{v_t}\right), \tag{73}$$

so we can re-express (72) as:

$$\frac{\xi_{t'}}{\xi_t} = \frac{\delta}{\sqrt{\delta^2 + v_t(t'-t)}} e^{\frac{1}{2v_{t'}}\left(\tilde{\rho}_{t'}^2 - 2\rho\tilde{\rho}_{t'}\right) - \frac{1}{2v_t}\left(\tilde{\rho}_t^2 - 2\rho\tilde{\rho}_t\right) + \frac{\rho^2}{2\delta^2}(t'-t)}. \tag{74}$$

Given expressions (24) and (17) for the manager's and uninformed investors' state-price deflators, and equation (74) for the likelihood process, we can write:

$$\begin{aligned}
E_t\left[\left(\frac{\pi_{t'}}{\pi_t}\right)^{\psi}\right] =& \delta^{\psi}\sqrt{\delta^2 + (1-\psi)v_t(t'-t)} \exp\left\{-\psi\left(r + \gamma(\rho+\gamma\delta^2) + \frac{\gamma^2}{2}v_t(t'-t) - \gamma\tilde{\rho}_t\right)\right. \\
&\left.\times (t'-t) - \frac{\psi}{2v_t}\left(\tilde{\rho}_t - (\rho+\gamma\delta^2)\right)^2\right\} \\
&\times E_t\left[\exp\left\{\frac{\psi}{2v_{t'}}\left(\tilde{\rho}_{t'} - (\rho+\gamma\delta^2)\right)^2\right\}\right].
\end{aligned} \tag{75}$$

We know from (73) that:

$$\tilde{\rho}_{t'} = \tilde{\rho}_t + (\rho - \tilde{\rho}_t)\frac{v_{t'}}{\delta^2}(t'-t) + \frac{v_{t'}}{\delta}(B_{t'} - B_t), \tag{76}$$

so the under P $\tilde{\rho}_{t'}$ is normally distributed with conditional mean and variance:

$$\begin{cases} E_t[\tilde{\rho}_{t'}] = \frac{\delta^2}{\delta^2+v_t(t'-t)}\tilde{\rho}_t + \frac{v_t(t'-t)}{\delta^2+v_t(t'-t)}\rho, \\ Var_t[\tilde{\rho}_{t'}] = \frac{\delta^2 v_t^2}{(\delta^2+v_t(t'-t))^2}(t'-t). \end{cases} \tag{77}$$

We can then rewrite the expectation on the RHS of (75) as:

$$\begin{aligned} E_t\left[\exp\left\{\frac{\psi}{2v_{t'}}\left(\tilde{\rho}_{t'}-(\rho+\gamma\delta^2)\right)^2\right\}\right] =& E_t\Bigg[\exp\Bigg\{\frac{\psi}{2v_{t'}}\Bigg(\tilde{\rho}_{t'}-E_t[\tilde{\rho}_{t'}] \\ & -\frac{\delta^2}{\delta^2+v_t(t'-t)}(\rho-\tilde{\rho}_t)-\gamma\delta^2\Bigg)^2\Bigg\}\Bigg]. \end{aligned} \tag{78}$$

Using Lemma A2 for $z=\tilde{\rho}_{t'}-E_t[\tilde{\rho}_{t'}]$, $\sigma_z^2 = Var_t[\tilde{\rho}_{t'}]$, $\check{\rho}=\frac{\psi}{2v_{t'}}$, $c=\frac{\delta^2}{\delta^2+v_t(t'-t)}(\rho-\tilde{\rho}_t)+\gamma\delta^2$,$\bar{z}=+\infty$, we can compute this expectation as:

$$\begin{aligned} E_t\left[\exp\left\{\frac{\psi}{2v_{t'}}\left(\tilde{\rho}_{t'}-(\rho+\gamma\delta^2)\right)^2\right\}\right] =& \sqrt{\frac{\delta^2+v_t(t'-t)}{\delta^2+(1-\psi)v_t(t'-t)}} \\ & \times\exp\left\{\frac{\psi\frac{\delta^2+v_t(t'-t)}{2v_{t'}}\left(\frac{\delta^2}{\delta^2+v_t(t'-t)}(\rho-\tilde{\rho}_t)+\gamma\delta^2\right)^2}{\delta^2+(1-\psi)v_t(t'-t)}\right\}. \end{aligned} \tag{79}$$

Plugging (79) in (75) we get, after some algebraic manipulation, equation (65). □

In order to derive the investment policy (25) replicating the optimal portfolio value (26), note that this can be rewritten as $W^N_{\tilde{\gamma},t} = f(t,\eta_t)$, where the diffusion term σ_η of η can be computed as $\sigma_\eta = -v_t/\delta^2$ and $f \in C^{1,2}$. Applying Itô's Lemma the diffusion term of $dW^N_{\tilde{\gamma},t}$ is:

$$-\frac{v_t}{\delta^2}\frac{\partial W^N_{\tilde{\gamma},t}}{\partial \eta_t} = W^N_{\tilde{\gamma},t}\frac{\delta^2+v_t\tau'}{\delta^2+\frac{v_t}{\tilde{\gamma}}\tau'}\frac{\eta_t}{\tilde{\gamma}} \tag{80}$$

Equating (80) to the diffusion term of W_t in (3) gives the optimal portfolio (25).

□

Proof of Corollary 2. Equation (28) follows from plugging in the equilibrium values $\eta_t = \tilde{\eta}_t + \frac{\rho-\tilde{\rho}_t}{\delta}$ and $\tilde{\eta}_t = \gamma\sigma_t$ in equation (25), letting $\tilde{\gamma}=\gamma$, subtracting 1 from $\phi^N_{\gamma,t}$ and rearranging. Since $\delta^2+v_t/\gamma\tau'$ and γ are positive, the LHS of (28) is negative iff the numerator on the RHS is negative, i.e.:

$$\tilde{\rho}_t > \rho + (\gamma-1)v_t\tau'. \tag{81}$$

To obtain condition (29), we apply the natural logarithm on both sides of equation (59) to get:

$$\begin{aligned} \ln\left(\frac{S_t}{S_t^{CI}}\right) &= -\left((\rho-\tilde{\rho}_t)+\left(\gamma-\frac{1}{2}\right)v_t\tau'\right)\tau' \\ \Leftrightarrow \frac{1}{\tau'}\ln\left(\frac{S_t}{S_t^{CI}}\right) &= -\left((\rho-\tilde{\rho}_t)+\left(\gamma-\frac{1}{2}\right)v_t\tau'\right) \\ \Leftrightarrow \frac{1}{\tau'}\ln\left(\frac{S_t}{S_t^{CI}}\right)-\frac{1}{2}v_t\tau' &= (\rho-\tilde{\rho}_t)+(\gamma-1)\,v_t\tau'. \end{aligned} \tag{82}$$

Therefore, condition (81) holds iff condition (29) holds. □

Proof of Proposition 3. Given in the second part of the proof of Proposition 4. □

Proof of Corollary 4. During $[T, T']$, the manager's optimization problem is:

$$\max_{W_{T'}} E_T\left[\frac{(W_{T'})^{1-\gamma}}{1-\gamma}\right], \tag{83}$$

subject to:

$$E_T\left[\pi_{T'}W_{T'}\right] = \pi_T f_T W_T \equiv w_T. \tag{84}$$

Attaching Lagrange multiplier $\lambda_M^{'}$ to the budget constraint (61), the manager's time-T' optimal wealth profile is given by the first order condition:

$$\hat{W}_{T'} = \left(\lambda_M^{'}\pi_{T'}\right)^{-\frac{1}{\gamma}}, \tag{85}$$

where the Lagrange multiplier $\lambda_M^{'}$ is given by:

$$\lambda_M^{'} = \pi_T^{\gamma-1}w_T^{-\gamma}\left(E_T\left[\left(\frac{\pi_{T'}}{\pi_T}\right)^{1-\frac{1}{\gamma}}\right]\right)^{\gamma} = \pi_T^{\gamma-1}\left(\frac{Z_{1-\frac{1}{\gamma},T,T'}}{w_T}\right)^{\gamma}. \tag{86}$$

The manager's time-t ($T \leq t \leq T'$) optimal AUM $\hat{W}_t$ are given by the no-arbitrage condition:

$$\begin{aligned}\pi_t\hat{W}_t &= E_t\left[\pi_{T'}\hat{W}_{T'}\right]\\ \Rightarrow\hat{W}_t &= \left(\lambda_M^{'}\pi_t\right)^{-\frac{1}{\gamma}}E_t\left[\left(\frac{\pi_{T'}}{\pi_t}\right)^{1-\frac{1}{\gamma}}\right] = \left(\lambda_M^{'}\pi_t\right)^{-\frac{1}{\gamma}}Z_{1-\frac{1}{\gamma},t,T'},\end{aligned} \tag{87}$$

with $Z_{1-\frac{1}{\gamma},t,T'} = E_t\left[\left(\frac{\pi_{T'}}{\pi_t}\right)^{1-\frac{1}{\gamma}}\right]$. The equality (37) follows from recognizing that $\hat{W}_t$ in (87) and $W^N_{\tilde{\gamma},t}$ in (64) differ only by a constant, in which case the diffusion terms in the SDEs characterizing their respective dynamics are identical.

□

Proof of Proposition 4. In order to characterize the optimal Policy during the first period $[0, T]$, we first need to solve for the indirect utility of the informed manager at $t = T$. Let $\zeta \equiv \bar{\zeta}W_0/Y_0$ be the normalized performance fee threshold. We use the following

Lemma A4. *As of $t = T$, the manager's indirect utility $U_T(W_T) \equiv E_T\left[u\left(\hat{W}_{T'}\right)\right]$ (where $\hat{W}_{T'}$ is the optimal terminal AUM as given by (26) when $w_T = \pi_T f_T W_T$), is given by:*

$$U_T(W_T) = \frac{Z^{\gamma}_{1-\frac{1}{\gamma},T,T'}}{1-\gamma} \times \begin{cases} W_T^{1-\gamma_1}\left(\zeta Y_T\right)^{\gamma_1-\gamma}, & \textit{if } W_T < \zeta Y_T,\\ W_T^{1-\gamma_2}\left(\zeta Y_T\right)^{\gamma_2-\gamma}, & \textit{if } W_T \geq \zeta Y_T.\end{cases} \tag{88}$$

Proof. Using (85) we can write:

$$\left(\hat{W}_{T'}\right)^{1-\gamma} = \left(\lambda_M^{'}\right)^{1-\frac{1}{\gamma}}\pi_{T'}^{1-\frac{1}{\gamma}}, \tag{89}$$

so:

$$U_T(W_T) = E_T\left[u\left(\hat{W}_{T'}\right)\right] = \frac{\left(\lambda_M^{'}\right)^{1-\frac{1}{\gamma}}E_T\left[\pi_{T'}^{1-\frac{1}{\gamma}}\right]}{1-\gamma}. \tag{90}$$

From (86):

$$E_T\left[\pi_{T'}^{1-\frac{1}{\gamma}}\right]=\left(\lambda_M^{'}\right)^{\frac{1}{\gamma}}\pi_T f_T W_T, \tag{91}$$

so:

$$U_T(W_T)=\frac{\lambda_M^{'}\pi_T f_T W_T}{1-\gamma}. \tag{92}$$

Plugging in the value of $\lambda_M^{'}$ as given by (86) for $w_T=\pi_T f_T W_T$:

$$U_T(W_T)=\frac{(f_T W_T)^{1-\gamma}\left(E_T\left[\left(\frac{\pi_{T'}}{\pi_T}\right)^{1-\frac{1}{\gamma}}\right]\right)^{\gamma}}{1-\gamma}=\frac{(f_T W_T)^{1-\gamma}Z^{\gamma}_{1-\frac{1}{\gamma},T,T'}}{1-\gamma}. \tag{93}$$

Finally, plugging in the expression for f_T in (5), using the definition of ζ,γ_1 and γ_2 in Section 3.2, and omitting the resulting term in k (since it does not change incentives in the margin and is absorbed by the Lagrange multiplier attached to the manager's budget constraint at $t=0$), we get expression (88). $\square$

At $t=0$, the problem of the informed money manager is then:

$$\max_{W_T} E_0\left[U_T(W_T)\right] \qquad \text{s.t. } E_0[\pi_T W_T]=w_0. \tag{94}$$

The objective function (88) in the manager's problem (94) is locally non-concave in a neighborhood of $\delta W_T=\zeta Y_T$ Standard optimization techniques cannot be applied directly to this problem. Following Basak and Makarov (2014), the first step consists in constructing the concavification $\tilde{U}_T(.)$ of the manager's indirect utility function $U_T(.)$ (i.e. the smallest concave function $\tilde{U}_T(w)$ satisfying $\tilde{U}_T(w)\geq U_T(w)$ for all $w\geq 0$), restate and solve the original problem (94) in terms of $\tilde{U}_T(.)$.

In order to construct the concavified function for a general intermediate horizon $T\leq T$, we look for functions $\underline{W}\left(\zeta Y_T, Z_{1-\frac{1}{\gamma},T,T'}\right)$, $\overline{W}\left(\zeta Y_T, Z_{1-\frac{1}{\gamma},T,T'}\right)$, $a\left(\zeta Y_T, Z_{1-\frac{1}{\gamma},T,T'}\right)$ and $b\left(\zeta Y_T, Z_{1-\frac{1}{\gamma},T,T'}\right)$ so that (omitting the arguments for notational simplicity):

$$\tilde{U}_T(W_T)=\begin{cases} U_T(W_T), & \text{if } W_T<\underline{W}\leq\zeta Y_T,\\ a+bW_T, & \text{if } \underline{W}\leq W_T<\overline{W},\\ U_T(W_T), & \text{if } \zeta Y_T\leq\overline{W}\leq W_T,\end{cases} \tag{95}$$

and

$$\tilde{U}_T^{'}(W_T)=\begin{cases} U_T^{'}(W_T), & \text{if } W_T<\underline{W}\leq\zeta Y_T,\\ b, & \text{if } \underline{W}\leq W_T<\overline{W},\\ U_T^{'}(W_T), & \text{if } \zeta Y_T\leq\overline{W}\leq W_T.\end{cases} \tag{96}$$

where:

$$U_T^{'}(W_T)=Z^{\gamma}_{1-\frac{1}{\gamma},T,T'}\times\begin{cases}(1+\alpha_1)W_T^{-\gamma_1}\left(\zeta Y_T\right)^{\gamma_1-\gamma}, & \text{if } W_T<\zeta Y_T,\\ (1+\alpha_2)W_T^{-\gamma_2}\left(\zeta Y_T\right)^{\gamma_2-\gamma}, & \text{if } W_T>\zeta Y_T.\end{cases} \tag{97}$$

Eqs. (97) and using (88) give us a system of 4 equations in our 4 unknowns $\underline{W}$, $\overline{W}$, a and b:

$$\begin{cases} a+b\underline{W}=\frac{Z^{\gamma}_{1-\frac{1}{\gamma},T,T'}}{1-\gamma}\underline{W}^{1-\gamma_1}\left(\zeta Y_T\right)^{\gamma_1-\gamma}\\ a+b\overline{W}=\frac{Z^{\gamma}_{1-\frac{1}{\gamma},T,T'}}{1-\gamma}\overline{W}^{1-\gamma_2}\left(\zeta Y_T\right)^{\gamma_2-\gamma}\\ b=(1+\alpha_1)Z^{\gamma}_{1-\frac{1}{\gamma},T,T'}\underline{W}^{-\gamma_1}\left(\zeta Y_T\right)^{\gamma_1-\gamma}\\ b=(1+\alpha_2)Z^{\gamma}_{1-\frac{1}{\gamma},T,T'}\overline{W}^{-\gamma_2}\left(\zeta Y_T\right)^{\gamma_2-\gamma}.\end{cases} \tag{98}$$

The solution to this system of equation yields

$$b\left(\zeta Y_T/Z_{1-\frac{1}{\gamma},T,T'}\right) = \left[\frac{(1+\alpha_2)^{\gamma_1(\gamma_2-1)}}{(1+\alpha_1)^{\gamma_2(\gamma_1-1)}}\left(\frac{\gamma_1}{\gamma_2}\right)^{\gamma_1\gamma_2}\right]^{\frac{1}{\gamma_2-\gamma_1}}\left(\frac{\zeta Y_T}{Z_{1-\frac{1}{\gamma},T,T'}}\right)^{-\gamma}, \tag{99}$$

$$\underline{W}(\zeta Y_T) = \left[\left(\frac{1+\alpha_2}{1+\alpha_1}\right)^{\gamma_2-1}\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2}\right]^{\frac{1}{\gamma_2-\gamma_1}}\zeta Y_T, \tag{100}$$

$$\overline{W}(\zeta Y_T) = \left[\left(\frac{1+\alpha_1}{1+\alpha_2}\right)^{\gamma_1-1}\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_1}\right]^{\frac{1}{\gamma_2-\gamma_1}}\zeta Y_T. \tag{101}$$

In order to verify that (99) to (101) are indeed the solutions we are after, it remains to verify that $\underline{W}$ and $\overline{W}$ satisfy the condition:

$$\underline{W} \leq \zeta Y_T \leq \overline{W}, \tag{102}$$

which holds iff:

$$\left[\left(\frac{(1+\alpha_2)}{(1+\alpha_1)}\right)^{\gamma_2-1}\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2}\right]^{\frac{1}{\gamma_2-\gamma_1}} < 1, \tag{103}$$

and

$$\left[\left(\frac{(1+\alpha_1)}{(1+\alpha_2)}\right)^{\gamma_1-1}\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_1}\right]^{\frac{1}{\gamma_2-\gamma_1}} > 1. \tag{104}$$

Since

$$\frac{1+\alpha_2}{1+\alpha_1} < \left(\frac{1+\alpha_2}{1+\alpha_1}\frac{\gamma_1}{\gamma_2}\right)^{\gamma_2}, \tag{105}$$

and

$$\frac{1+\alpha_1}{1+\alpha_2} < \left(\frac{1+\alpha_1}{1+\alpha_2}\frac{\gamma_2}{\gamma_1}\right)^{\gamma_1}, \tag{106}$$

both conditions indeed verify.

We can now restate the manager's optimization problem (94) at $t = 0$ as:

$$\max_{W_T} E_0\left[\tilde{U}_T(W_T)\right] \qquad \text{s.t. } E_0[\pi_T W_T] = w_0. \tag{107}$$

Attaching Lagrange multiplier λ_M to the budget constraint, the solution to the concavified problem (107) is given by the standard (state-by-state) first order condition:

$$\tilde{U}_T'(W_T) = \lambda_M \pi_T. \tag{108}$$

Using (97):

$$\lambda_M \pi_T = Z^{\gamma}_{1-\frac{1}{\gamma},T,T'} \times \begin{cases} (1+\alpha_1)W_T^{-\gamma_1}(\zeta Y_T)^{\gamma_1-\gamma}, & \text{if } W_T < \zeta Y_T, \\ b, & \text{if } \underline{W} \leq W_T < \overline{W}, \\ (1+\alpha_2)W_T^{-\gamma_2}(\zeta Y_T)^{\gamma_2-\gamma}, & \text{if } W_T > \zeta Y_T, \end{cases} \tag{109}$$

which gives the manager's optimal time-T AUM as:

$$\hat{W}_T = \begin{cases} (1+\alpha_1)^{\frac{1}{\gamma_1}} Z^{\frac{\gamma}{\gamma_1}}_{1-\frac{1}{\gamma},T,T'}(\zeta Y_T)^{\frac{\gamma_1-\gamma}{\gamma_1}}(\lambda_M\pi_T)^{-\frac{1}{\gamma_1}}, & \text{if } W_T < \underline{W}, \\ W \in [\underline{W}, \overline{W}], & \text{if } \underline{W} \leq W_T < \overline{W}, \\ (1+\alpha_2)^{\frac{1}{\gamma_2}} Z^{\frac{\gamma}{\gamma_2}}_{1-\frac{1}{\gamma},T,T'}(\zeta Y_T)^{\frac{\gamma_2-\gamma}{\gamma_2}}(\lambda_M\pi_T)^{-\frac{1}{\gamma_2}}, & \text{if } \overline{W} \leq W_T. \end{cases} \tag{110}$$

Using Eqs. (99) through (101), we note that:

$$\hat{W}_T < \underline{W} \Leftrightarrow \lambda_M \pi_T > b, \tag{111}$$

and

$$\hat{W}_T \geq \overline{W} \Leftrightarrow \lambda_M \pi_T \leq b, \tag{112}$$

which allows us to re-express (110) as:

$$\hat{W}_T = \begin{cases} (1+\alpha_1)^{\frac{1}{\gamma_1}} Z_{1-\frac{1}{\gamma},T,T'}^{\frac{\gamma}{\gamma_1}} (\zeta Y_T)^{\frac{\gamma_1-\gamma}{\gamma_1}} (\lambda_M \pi_T)^{-\frac{1}{\gamma_1}}, & \text{if } \lambda_M \pi_T > b, \\ (1+\alpha_2)^{\frac{1}{\gamma_2}} Z_{1-\frac{1}{\gamma},T,T'}^{\frac{\gamma}{\gamma_2}} (\zeta Y_T)^{\frac{\gamma_2-\gamma}{\gamma_2}} (\lambda_M \pi_T)^{-\frac{1}{\gamma_2}}, & \text{if } \lambda_M \pi_T \leq b. \end{cases} \tag{113}$$

This completes our proof of Proposition 4. We now prove Proposition 3. In order to derive Eqs. (31) thorugh (34), we set $T = T'$, so $Z_{1-\frac{1}{\gamma},T',T'}^{\frac{\gamma}{\gamma_1}} = 1$. Time-$T'$ optimal wealth in this case is:

$$\hat{W}_{T'} = \begin{cases} (1+\alpha_1)^{\frac{1}{\gamma_1}} (\zeta Y_{T'})^{\frac{\gamma_1-\gamma}{\gamma_1}} (\lambda_M \pi_{T'})^{-\frac{1}{\gamma_1}}, & \text{if } \lambda_M \pi_{T'} > b(\zeta Y_{T'}), & \mathcal{R}_1, \\ (1+\alpha_2)^{\frac{1}{\gamma_2}} (\zeta Y_{T'})^{\frac{\gamma_2-\gamma}{\gamma_2}} (\lambda_M \pi_{T'})^{-\frac{1}{\gamma_2}}, & \text{if } \lambda_M \pi_{T'} \leq b(\zeta Y_{T'}), & \mathcal{R}_2. \end{cases} \tag{114}$$

In order to define regions $\mathcal{R}_1$ and $\mathcal{R}_2$, we need to obtain an explicit expression for the benchmark $Y_{T'}$. This can be done more easily by first writing the dynamics of Y_t under the uninformed investors' probability $\tilde{P}$:

$$dY_t = Y_t \left(r + \phi^Y \sigma_t \tilde{\eta}_t \right) dt + Y_t \phi^Y \sigma_t d\tilde{B}_t, \tag{115}$$

which implies:

$$\begin{aligned} Y_{T'} &= Y_t \exp\left\{ r\tau' + \phi^Y \left(\gamma - \frac{\phi^Y}{2} \right) \int_t^{T'} \sigma_s^2 ds + \phi^Y \int_t^{T'} \sigma_s d\tilde{B}_s \right\} \\ &= Y_t \exp\left\{ \left(r + \phi^Y \left(\gamma - \frac{\phi^Y}{2} \right) (\delta^2 + v_t \tau') \phi^Y \right) \tau' + \frac{\delta^2 \phi^Y}{v_{T'}} (\tilde{\rho}_{T'} - \tilde{\rho}_t) \right\}, \end{aligned} \tag{116}$$

where we used expressions (12) and (18) to solve for the integrals in (116). Defining:

$$\zeta_0 \equiv \left[\frac{(1+\alpha_2)^{\gamma_1(\gamma_2-1)}}{(1+\alpha_1)^{\gamma_2(\gamma_1-1)}} \left(\frac{\gamma_1}{\gamma_2} \right)^{\gamma_1\gamma_2} \right]^{\frac{1}{\gamma_2-\gamma_1}}, \tag{117}$$

we can express $b(\zeta Y_{T'}) = \zeta_0 (\zeta Y_{T'})^{-\gamma}$. Region $\mathcal{R}_1$ is then given by:

$$\lambda_M \pi_{T'} > b(\zeta Y_{T'}) \Leftrightarrow \lambda_M \pi_{T'} > \zeta_0 (\zeta Y_{T'})^{-\gamma}. \tag{118}$$

Using the above closed-form expressions for $\pi_{T'}$ and $Y_{T'}$ we can express region $\mathcal{R}_1$ as:

$$\left\{ \tilde{\rho}_{T'} < \rho + (1-\phi^Y)\gamma\delta^2 + \Gamma \right\} \cup \left\{ \tilde{\rho}_{T'} > \rho + (1-\phi^Y)\gamma\delta^2 + \Gamma \right\}, \tag{119}$$

where:

$$\Gamma \equiv \sqrt{v_{T'} \Delta(\tilde{\rho}_0, v_0)}, \tag{120}$$

and

$$\Delta(\tilde{\rho}_0, v_0) \equiv \frac{1}{v_0}\left(\tilde{\rho}_0 - \rho - (1-\phi^Y)\gamma\delta^2\right)^2 + 2(1-\phi^Y)\gamma\left\{\ln\left(\frac{D_0}{\beta_0}\right) - \left[r + \left(1-(1-\phi^Y)\gamma\right)\frac{\delta^2}{2} - \rho\right]T'\right\} + 2\ln\left(\frac{\lambda\zeta_0}{\lambda_M\delta\eta^\gamma}\sqrt{\delta^2 + v_0 T'}\right). \tag{121}$$

The existence of a solution to the manager's problem (94) requires $\Delta(\tilde{\rho}_0, v_0) > 0$, implying $\Gamma \geq 0$. Region $\mathcal{R}_2$ is just the relative complement in $\mathbb{R}$ of $\mathcal{R}_1$. We can now derive the interim AUM (32). By no-arbitrage, the deflated wealth process $\pi_t\hat{W}_t$ is a martingale, so using (114) the optimal wealth $\hat{W}_t$ for all $t \in [0, T']$ is given by:

$$\pi_t\hat{W}_t = E_t\left[\pi_{T'}\hat{W}_{T'}\right] \Rightarrow \hat{W}_t = f_{1,t} + f_{2,t}, \tag{122}$$

where :

$$f_{i,t} = \pi_t E_t\left[(1+\alpha_i)^{\frac{1}{\gamma_i}}\left(\zeta Y_{T'}\right)^{\frac{\gamma_i-\gamma}{\gamma_i}}\left(\lambda_M\pi_{T'}\right)^{-\frac{1}{\gamma_i}}\mathbb{1}_{\mathcal{R}_i}\right]. \tag{123}$$

Using the closed-form expressions above for $\pi_{T'}$ and $Y_{T'}$, $\mathcal{R}_1$ and $\mathcal{R}_2$, and applying Lemma A2 to compute the expectation in (123), we get Eq. (33) and thus the manager's optimal wealth process (32). The Lagrange multiplier λ_M is the solution to the equation $\hat{W}_0 = w_0$.

In order to derive the investment policy (31) replicating the optimal portfolio value (32), note that this can be rewritten as $\hat{W}_t = h(t, D_t, \tilde{\pi}_t, \xi_t, X_{1,t}, X_{2,t}, \underline{d}_{1,t}, \underline{d}_{2,t}, \overline{d}_{1,t}, \overline{d}_{2,t})$, where for $i = 1, 2$:

$$X_{i,t} \equiv \exp\left\{\left[\left(1-\frac{\gamma}{\gamma_i}\right)\left(1-\phi^Y\right)r - \left(1-\frac{1}{\gamma_i}\right)\left(k_i\left(\rho - ((\gamma_i-1)k_i+1)\frac{\delta^2}{2}\right) + \frac{1}{2\gamma_i}\frac{\left(\rho - \tilde{\rho}_t - (\gamma_i-1)k_i\delta^2\right)^2}{\delta^2 + \frac{v_t}{\gamma_i}\tau'}\right)\right]\tau'\right\}, \tag{124}$$

for some function $h \in C^{1,2}$. Applying Itô's Lemma the diffusion term $\sigma_{\hat{W}}$ of $d\hat{W}_t$ is:

$$\sigma_{\hat{W}} = \sigma_D h_D + \sigma_{\tilde{\pi}}h_{\tilde{\pi}} + \sigma_\xi h_\xi + \sigma_{X_1}h_{X_1} + \sigma_{X_2}h_{X_2} + \sigma_{\underline{d}_1}h_{\underline{d}_1} + \sigma_{\underline{d}_2}h_{\underline{d}_2} + \sigma_{\overline{d}_1}h_{\overline{d}_1} + \sigma_{\overline{d}_2}h_{\overline{d}_2}, \tag{125}$$

where h_x denotes the partial derivative of h w.r.t. x and σ_X is the diffusion term in the SDE characterizing the dynamics of the process X. Computing the diffusion terms in (125) explicitly and equating the result to the diffusion term of W_t in (3) gives the optimal portfolio (31).

□

Proof of Corollary 3. From equation (34), for $i = 1, 2$ the sign of each risk-shifting component $\Phi_{i,t}$ equals the sign of $\mathcal{N}'(\underline{d}_{i,t}) - \mathcal{N}'(\overline{d}_{i,t})$. By the symmetry of the standard normal density, $\mathcal{N}'(\underline{d}_{i,t}) \geq \mathcal{N}'(\overline{d}_{i,t})$ if and only if $|\underline{d}_{i,t}| \leq |\overline{d}_{i,t}|$. Since $\underline{d}_{i,t} < \overline{d}_{i,t}$,

$$|\underline{d}_{i,t}| \leq |\overline{d}_{i,t}| \Leftrightarrow \underline{d}_{i,t} + \overline{d}_{i,t} \geq 0 \Leftrightarrow 2\frac{\gamma\delta^2\underline{\sigma}_{\gamma,t}}{v_t\sqrt{\tau'}}\sqrt{\frac{\delta^2 + v_t\tau'}{\delta^2 + \frac{v_t}{\gamma_i}\tau'}}\left(\phi^N_{\gamma,t} - \phi^Y\right) \geq 0. \tag{126}$$

Since the factor multiplying the difference $(\phi_{\gamma,t}^N - \phi^Y)$ above is positive, we conclude that $\mathcal{N}'(\underline{d}_{i,t}) \geq \mathcal{N}'(\bar{d}_{i,t})$ if and only if $\phi_{\gamma,t}^N \geq \phi^Y$. Thus, for $i = 1, 2$, $\text{sgn}(\Phi_{i,t}) = \text{sgn}(\phi_{\gamma,t}^N - \phi^Y)$, which leads to equation (36).

$\square$

Proof of Lemma 1. As of time t, $\rho \sim \mathbf{N}(\tilde{\rho}_t, v_t)$ which implies that S_t/S_t^{CI} as given by (82) is log-normally distributed. We then have:

$$
\begin{aligned}
E_t^\rho \left[\frac{S_t}{S_t^{CI}}\right] =& E_t^\rho \left[e^{-\left((\rho-\tilde{\rho}_t)+\left(\gamma-\frac{1}{2}\right)v_t\tau'\right)\tau'}\right] \\
=& e^{-(\gamma-1)v_t(\tau')^2},
\end{aligned} \tag{127}
$$

which, given the definition of EMP, leads to (41).

To compute the expected overvaluation EOV we first note that, by definition, $EOV > 0$. Moreover,

$$
E_t^\rho \left[\frac{S_t}{S_t^{CI}} \mathbb{1}_{\left\{\rho<\tilde{\rho}_t-\left(\gamma-\frac{1}{2}\right)v_t\tau'\right\}}\right] = e^{-\left(\gamma-\frac{1}{2}\right)v_t(\tau')^2} E_t^\rho \left[e^{-(\rho-\tilde{\rho}_t)\tau'} \mathbb{1}_{\left\{\rho-\tilde{\rho}_t<-\left(\gamma-\frac{1}{2}\right)v_t\tau'\right\}}\right]. \tag{128}
$$

Since $\rho - \tilde{\rho}_t \sim \mathbf{N}(0, v_t)$, integrating against the normal density we can compute the expectation on the RHS of (128) as:

$$
E_t^\rho \left[e^{-(\rho-\tilde{\rho}_t)\tau'} \mathbb{1}_{\left\{\rho-\tilde{\rho}_t<-\left(\gamma-\frac{1}{2}\right)v_t\tau'\right\}}\right] = e^{\frac{1}{2}v_t(\tau')^2} \mathcal{N}\left(-\left(\gamma-\frac{3}{2}\right)\sqrt{v_t}\tau'\right). \tag{129}
$$

Moreover,

$$
\begin{aligned}
Prob\left(\rho-\tilde{\rho}_t < -\left(\gamma-\frac{1}{2}\right)v_t\tau'\right) &= Prob\left(\frac{\rho-\tilde{\rho}_t}{\sqrt{v_t}} < -\left(\gamma-\frac{1}{2}\right)\sqrt{v_t}\tau'\right) \\
&= \mathcal{N}\left(-\left(\gamma-\frac{1}{2}\right)\sqrt{v_t}\tau'\right).
\end{aligned} \tag{130}
$$

Then,

$$
\begin{aligned}
EOV(t) &= E_t^\rho \left[S_t/S_t^{CI} \mid \rho < \tilde{\rho}_t - \left(\gamma-\frac{1}{2}\right)v_t\tau'\right] - 1 \\
&= \frac{E_t^\rho \left[\frac{S_t}{S_t^{CI}} \mathbb{1}_{\left\{\rho-\tilde{\rho}_t<-\left(\gamma-\frac{1}{2}\right)v_t\tau'\right\}}\right]}{Prob\left(\rho-\tilde{\rho}_t < -\left(\gamma-\frac{1}{2}\right)v_t\tau'\right)} - 1 \\
&= e^{-(\gamma-1)v_t(\tau')^2} \frac{\mathcal{N}\left(-(\gamma-\frac{3}{2})\sqrt{v_t}\tau'\right)}{\mathcal{N}\left(-(\gamma-\frac{1}{2})\sqrt{v_t}\tau'\right)} - 1.
\end{aligned} \tag{131}
$$

$\square$