How to Squander Your Endowment: Pitfalls and Remedies*

Philip H. Dybvig

Olin Business School, Washington University, St. Louis.

Zhenjiang Qin

Institute of Financial Studies, SWUFE.

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Abstract

University donors choose to contribute to endowment if they want to make a permanent contribution to the university. It is consequently viewed as a responsibility of the university to preserve capital when choosing the investments and spending rule of endowments. Practitioners commonly measure preservation of capital by looking at the excess of expected return over the spending rate, but this criterion involves an incorrect application of the law of large numbers based on products instead of sums. The measure can be corrected by looking expected log return net of spending, which is less by approximately half the variance of returns if period returns are not too volatile. Even if the correct target spending rule is applied, the common practice of smoothing spending using a partial adjustment model for spending tends to makes spending unstable in bad times and in fact the probability of eventual ruin is one. However, we show that a simple modification to the traditional smoothing rule does preserve capital. We look at optimal spending rules that preserve capital and retain some benefits of smoothing.

Preliminary and incomplete

*Phil Dybvig is at Olin Business School, Washington University, St. Louis, E-mail: dybvig@wustl.edu. Zhenjiang Qin is at Institute of Financial Studies, Southwestern University of Finance and Economics. E-mail: zqin@swufe.edu.cn.

1 Introduction

Donors who wish to contribute to universities have a number of options depending on when they want their giving to have an impact. Donors wanting to have an immediate impact can contribute through annual giving, donors who want to have an impact on an intermediate time frame can give funds for a building, and donors who want to have a permanent impact can contribute to endowment. Since contributions to endowment are supposed to have a permanent impact, the university has a responsibility to make sure that the spending rule and investment strategy for endowment, taken together, preserve capital in the long-run. This paper takes a look at the long-term preservation of capital with a focus on the properties of existing practice. We find that the usual criterion (spending less than expected return) for preservation of capital is incorrect because it is based on an incorrect application of the law of large numbers. We provide a corrected formula based on logged return net of spending. We also show that a stylized version of the practice of smoothing spending implies that the endowment never preserves capital, and we suggest an alternative based on an optimization model that preserves capital for appropriately chosen parameter values.

A spending rate less than the expected return on assets, calculated in real terms, has long been used as a criterion for whether an endowment preserves capital. This criterion is based on the intuition of the law of large numbers, since it means that on average the expected return on the portfolio should cover spending. However, this intuition represents a mis-application of the law of large numbers (or central limit theorem), since the law of large numbers applies to sums but the portfolio problem involves products. In particular, the proportional change in value in a period is one plus the return less spending, and these returns multiply over time. We should not include new contributions in this calculation, since we are asking whether the initial contribution has a permanent impact, not whether the initial contribution plus others' later contributions have a permanent impact. However, the law of large numbers does not apply to products, and it is easy to construct examples to illustrate that spending at a rate less than the expected rate of return on assets does not preserve capital.

As a very simple example, suppose the rate of return is 200% half the time (value triples) and -100% half the time (value disappears), so that the expected return is $0.5 \times 200 + 0.5 \times (-100) = 50\%$. Assume further that the spending rate is zero, which is certainly less than 50%. Then, in each year there is a 50% probability the endowment will be wiped out and the probability of surviving for T years is 2^{-T} which approaches 0 rapidly. Having no endowment at all with probablity close to one certainly does not preserve capital but it satisfies the traditional rule.

More generally, we do not need such a stark example (with a probability the portfolio is actually wiped out) to show the problem with the traditional rule. Provided the portfolio return is random,[1] there is always a spending rate less than the expected return for which the value of the endowment converges to zero over time with probability 1. The error in the rule can be large; in a realistic example, capital may not be preserved even if spending exceed the expected return on assets by a couple of percent.

Although the traditional rule does not work, it is easy to provide an alternative rule that does. Taking logarithms converts products into sums, and there is a natural criterion using logarithms that is the natural fix for the traditional rule. The modified rule is that the expected log net return defined as the log of one plus the proportional change in value of the portfolio, inclusive of investment return and net of spending, should be positive. This rule preserves capital in the sense that the value of the endowment arising from an initial investment grows without limit over time if this assumption is true.[2]

Besides looking at the basic spending rule, we also look at the common practice of overlaying smoothing on the basic spending rule. Smoothing of spending is supposed to prevent the damage done by large fluctuations in spending. This is a reasonable idea: sudden decreases in spending can be disruptive, and sudden increases may be used carelessly. Unfortunately, the usual partial adjustment rule of moving only a fixed fraction of the way toward the target spending level *never* preserves capital in the endowment if the target spending rate is positive (even if very small). This result is based on an assumption that portfolio returns are drawn from the same distribution and are independent over time. Intuitively, random fluctuations imply that the spending rate will eventually be very large, and when the spending rate is large, the high spending depletes capital relatively more quickly than the spending is reduced by the smoothing rule, and as a result the portfolio ends up in a "death spiral" plunging towards zero.

Since smoothing is a good idea and the traditional rule does not preserve capital, we have proposed two possible solutions. One solution is a simple modified smoothing rule that adds a new term that changes spending to compensate for the expected change in spending rate given the excess of current spending over the expected return of assets. For this rule, we have a characterization of the parameter values for which capital is preserved. Our second alternative to the traditional smoothing rule is a theoretical optimization model that

[1] We also need minor technical assumptions, that the mean exists and that we have sufficient independence over time and the degree of randomness not going to zero over time. For example, it suffices to have iid return with a finite mean.

[2] As above, we also need minor technical assumptions, that the mean exists and that we have sufficient independence over time and the expected log return less the spending rate does not got to zero too quickly. For example, it suffices to have idd trturns and constant spending.

penalizes big changes in spending and preserves capital for some parameter values. At this point, we have numerical results showing that the strategy does preserve capital for some parameter values, and it is work in progress trying to characterize for which parameter values the strategy does and does not preserve capital.

The rest of the paper is arranged as follows. Section 2 documents the problem with the traditional rule for preserving capital and provides the new correct rule. Section 3 shows that traditional smoothing implies capital is not preserved. We provide a modified smooth spending rule that preserves capital. Section 4 comes up with the condition for preserving capital with temporarily negative risk-free rate. (To be finished.) Section 5 gives an optimization model of spending that does preserve capital. Section 6 closes the paper.

2 Preserving Capital by Traditional Rule

2.1 Preserving Capital in Discrete Time

One traditional rule says that a spending rate of no more than the average return on the endowment will preserve its value. This traditional rule is widely adopted and clearly stated in the spending policy statements of many university endowments. For example, the spending policy statement of UCSD Foundation (2014) states that its objective is to "achieve an average total annual net return equivalent to the endowment spending rate adjusted for inflation". Moreover, the endowment of Henderson State University (2014) even employs a concrete example to illustrate its objective of achieving that the inflation-adjusted average return equals to the spending rate: "Total return objective 7.00%, spending rate 4.00%, administration fee 1.50%, and inflation rate 1.50%". Furthermore, this rule is also mentioned in some manual books. For instance, the book "Nonprofit Asset Management: Effective Investment Strategies and Oversight" (Rice, DiMeo, and Porter (2012)) states that "the primary objective of the Great State University Endowment fund is to preserve the purchasing power of the endowment after spending. This means that the Great State University Endowment must achieve, on average, an annual total rate of return equal to inflation plus actual spending". Despite it is widely used, does the traditional rule necessarily preserve capital?

Absent risk, this rule makes perfect sense. Let W_t be the amount of wealth in the endowment at time t, with spending C_t and assume it earns a return r, which is assumed to be riskless for the moment and the same each period. Any additions to the portfolio from new contributions are not included, since new contributions are supposed to increase the possible future spending and stand on their own, not replenish the spending power of previous endowment that has been depleted (this is one thing the traditional rule gets right). The traditional rule says that the spending is no more than the return on the portfolio, that is, $C_t/W_t \leq r$, or equivalently $C_t \leq W_t r$. Then we have that

$$W_{t+1} = W_t + rW_t - C_t \geq W_t. \tag{1}$$

In this case, spending no more than the return on the endowment implies the endowment never falls, so we have preservation of capital. So far so good. In the traditional rule, the next step says we can use the same analysis if spending is no larger than the expected return on endowment in an uncertain world, you know, because of the law of averages. Well, we do know the math, and unfortunately this argument is wrong because the law of averages applies to sums, not products. Now that the return is random, we write r_{t+1} for the return from time t to $t+1$, and (1) becomes

$$W_{t+1} = W_t + r_{t+1}W_t - C_t, \tag{2}$$

which implies that

$$W_t = W_1(1 + r_2 - C_1/W_1)(1 + r_3 - C_2/W_2)...(1 + r_{t+1} - C_t/W_t),$$

which is a product not a sum. We can convert this to a sum by taking logarithms:

$$\log(W_t) = \log(W_1) + \log(1+r_2 - C_1/W_1) + \log(1+r_3 - C_2/W_2) + ... + \log(1+r_{t+1} - C_t/W_t),$$

and now we can use the law of averages (i.e., the law of large numbers and the central limit theorem). Assume that r_t is independent over time and has the same distribution in each period (the usual assumption[3] for these calculations), and that $\log(1 + r_{t+1} - C_t/W_t)$ has a finite mean. Then we have:

Theorem 1 *There exsits some constant $c^* < E[r_t]$, which is the average return on the endowment, if the spending rate c is larger than c^*, and the return on the endowment is random, the endowment is never preserved. More formally, assume $\{r_t\}_{t=1,\infty}$ are i.i.d. non-constant random variables, $\exists c^* < E[r_t]$, if $\forall c > c^*$, then $\lim_{t\to\infty} W_t = 0$, that is, for any fixed positive wealth level $\underline{W}$, no matter how small, then in every state of nature there is a critical time T such that $W_t < \underline{W}$ for all $t > T$.*

Proof: Assume without loss of generality that the expected log wealth is finite. We look at

[3] There are many generalizations of the law of large numbers that can be used more generally, but we needn't go into that here.

$\log(W_T)$ which is $\log(W_1)+\sum_{t=1}^{t=T}\log(1+r_{t+1}-C_t/W_t)$, and the typical term in the sum has negative expectation because of Jensen's inequality and the assumption $C_t/W_t=\mathrm{E}[r_t]$. ■

In Theorem 1, we have assumed that returns are iid and that $C_t/W_t=\mathrm{E}[r_t]$, but by continuity the result still holds for $C_t/W_t<\mathrm{E}[r_t]$ but sufficiently close to $\mathrm{E}[r_t]$. It also obviously holds if instead of i.i.d. we have the sort of mixing property that implies the law of large numbers.

Example 1 (Implausible implication of traditional rule): To get more intuition about why the traditional rule does not make sense, consider investing in a riskless asset with mean return r and a risky asset with a mean return $\mu>r$. Then if we put a proportion θ in the stock (θ could be larger than one for a levered position), the traditional rule says we preserve capital if $r+\theta(\mu-r)>c$, where c is spending ratio C_t/W_t. However, this implies that we can spend at as high a rate c as we want, so long as we take on enough risk by taking θ to be high enough!

Knowing that we have to deal with a sum in logs, we make a statement about log returns. We provide the following Theorem describing the condition needed for preserving capital.

Proposition 1 *If the random return is $\{r_t\}_{t=1,\infty}$ and the spending at the end of the period is a fraction of wealth $c_t=C_t/W_t$ at the beginning of the period, $E[\log(1+r_t-c_t)]>0$ suffices for preserving capital. (If spending is done at the start of the period, $E[\log((1-c)(1+r_t))]>0$ suffices.)*

Remark 1 ***(Jensen's Inequality Argument)*** *Mathematically we can view the problem in terms of concavity of the logarithm. By Jensen's inequality and concavity of the logarithm,*

$$\mathrm{E}[\log(1+r_t-c_t)]<\log(\mathrm{E}[1+r_t-c_t]). \tag{3}$$

Positivity of the right-hand side is the traditional rule $E[r_t-c_t]>0$, and positivity of the left-hand side is the correct rule, which require a low level of spending to preserve capital. For example, assume $\log(1+r_t-c_t)\sim N(\mu-\sigma^2/2,\sigma^2)$, then we have

$$\log(\mathrm{E}[1+r_t-c_t])=\mu \quad \textit{and} \quad \mathrm{E}[\log(1+r_t-c_t)]=\mu-\sigma^2/2.$$

However, the traditioanl rule fails if $\sigma>0$, $\mu\simeq 0$ but still positive, since the right-hand side of (3) is positive, but capital is not preserved ($\mu-\sigma^2/2<0$). To obtain a positive left-hand side of (3) with the same investment opportunity $1+r_t$, one needs to decrease the spending rate c_t to $\hat{c}_t$ to obtain $\mathrm{E}[\log(1+r_t-\hat{c}_t)]=\hat{\mu}>0$.

Returning to the example in the introdution which illustrating traditional rule does not preserve capital, the following example 2 shows that if we replcace the zero payoff at the bad state by a small positive number, the capital is still not preserved.

Example 2 (Unsuccessful preservation of capital by setting expected return higher than spending rate): Assume an endowment has a spending rate of 0% and an investment of 1 which has half probability of tripling and going to zero respectively:

$$1 < \begin{array}{ll} 3 & \text{Prob } 0.5, \\ 1/9 & \text{Prob } 0.5. \end{array}$$

Hence, the expected return is $5/9 > 0\%$, but the whole endowment is exhausted in finite time with probability 1.

A couple of qualifications are in order for the positive result for the riskless case and are also relevant for the risky case. First, we should work with real returns, that is, returns in excess of inflation. This adjustment is normally done correctly in practice when using the traditional rule: we are not really preserving capital if we are keeping the same dollar amount in an inflationary environment. The second qualification is a little trickier but probably not too big a deal. The assumption in (1) is that spending takes place at the end of the period, so the wealth relative $W_t/W_{t-1} = 1+r_t-c_t$. However, the actual timing depends on the local convention. For example, if budgeted spending for the year is taken out of the endowment and placed in a separate account at the beginning of the year, the wealth relative would be $(1-c_t)(1+r_t)$. Calculation given other convention are straightforward but can be messy. For example, if the spending $C_t = c_tW_{t-1}$, is computed at the beginning of the year but taken out in two parts, half at the start of the year and half in the middle, the wealth relative is $(1-c_t/2)(1+r_t^{H^1} - c_t/2)(1+r_t^{H^2})$, where $r_t^{H^1}$ is the return on the assets in the first half of the year and $r_t^{H^2}$ is the return in the second half. In general, we want to compute W_{t+1}/W_t.

2.2 Preserving Capital in Continuous Time

Assume the process for the reinvested stock price is governed by the usual lognormal process solving,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t, \tag{4}$$

subject to some given initial stock price S_0, where μ and $\sigma > 0$ are constants, and Z is a standard Wiener process. Then similar results can be obtained in continuous time model.

Here the wealth dynamic follows

$$dW_t/W_t = \mu dt + \sigma dZ_t - c,$$

with spending rate c. Consequently,

$$W_t = W_0 e^{(\mu-\sigma^2/2-c)t+\sigma Z_t}. \tag{5}$$

If the log return $\mu - \sigma^2/2$ is larger than c, $p\lim_{t\to\infty} W_t = \infty$, capital is preserved, while if it is smaller than c, $p\lim_{t\to\infty} W_t = 0$, and capital is not preserved. The traditional result fails if $\mu - \sigma^2/2 < c < \mu$. With equality, probably we would say capital is not preserved, but that depends on what definition we use.

We can also look at this in terms of Itô's lemma (and concavity of the logarithm because of the second derivative in the Itô term)). We have that

$$d\log(W_t) = (\mu - \frac{\sigma^2}{2} - c)dt + \sigma dZ_t, \tag{6}$$

so the drift is positive if the coefficient of dt is positive. Only with a positive drift can the capital be preserved. We summarize the results in the following Theorem.

Theorem 2 *Given the endowment can invest in a risky asset continuously, the capital is preserved if the expected log growth rate is larger than the spending rate, i.e.,* $\mu - \sigma^2/2 > c$.

3 Preserving Capital with Smooth Spending

Instead of making spending strictly proportional to the size of the endowment, it is common to smooth spending using a moving-average (partial adjustment) rule to move from current spending towards a spending target. Probably there is some economic sense to smoothing, since a sudden decrease in a budget can cause distress, while a sudden increase can invite waste. As a result, there are many endowments which are empolying some kinds of smooth spending formulas. For instance, several universities in the UC system use smooth spending policy: UC Berkeley, UC Irvine, and UC Santa Cruz plan to spend 4.5% of a twelve-quarter (three year) moving average market value of the endowment pool. Another example: Grinnell College Endowment states that endowment distribution is calculated as 4.0% of the 12-quarter moving average endowment market value determined annually as of the December 31 immediately prior to the beginning of the fiscal year. Actually, according to the Commonfund Benchmarks Study: Educational Endowment Report, 2005, 63 per cent of institutions in the

US report ‘they employ either a three-year or 12-quarter moving average of market value as a smoothing mechanism in their spending formula; 38 per cent use the three-year and 25 per cent use a 12-quarter moving average (Also see page 112, Chapter 4, Acharya and Dimson (2007)).

However, the moving average rule tends to destabilize the endowment. We illustrate this with a riskless example for which an initial high spending rate sends the fund into a "death spiral" with the wealth going to zero for sure at a known finite time. Then we give a result for risky iid returns. When stock returns are bad, wealth goes down but spending is slow to adjust so the spending rate goes up. This pushes wealth down and at some point the fall in wealth becomes unstable because the adjustment is not fast enough to keep the spending rate from getting large as wealth (in the denominator) falls. Over time, this scenario will play out sooner or later, with probability one the fund's wealth will reach zero at some (random) future time.

3.1 Benchmark: Traditional Moving Average Spending Rule with Only A Riskless Bond

A traditional moving average spending rule assumes the dynamic of spending to be[4]

$$dC_t = \kappa\left(\tau W_t - C_t\right)dt. \tag{7}$$

We will assume $\tau < r$, which implies that the target spending rate would preserve capital, so our policy has a fighting chance. If the endowment only invests in a riskless bond with constant risk-free rate r, then the wealth process is given as

$$dW_t = rW_t dt - C_t dt. \tag{8}$$

We have the following result.

Proposition 2 *When the endowment invests in only the riskless asset, the moving average spending rule does not preserve capital when the initial spending rate C_0/W_0 is sufficiently high. Specifically, given the dynamic (7) and (8), wealth W_t reaches 0 at some (random) finite time if C_0/W_0 is larger enough.*

If the endowment starts with high spending under the moving average rule, capital will be wiped out quickly. The intuition is that with a high initial spending, the wealth declines

[4]Often practitioners use a moving average rule, e.g., a 10-year average, in place of this autoregressive rule, the distinction is not important for us.

dramatically. Decline in wealth is faster than decline in spending under moving average rule, hence, the spend rate becomes increasingly higher and much higher then return of the riskless investment. As a results, wealth converges to zero in a "death spiral".

3.2 Traditional Moving Average Spending Rule with Only Risky Asset

In practice, the portfolio choice is not often link to the current spending rate. Usually, the portfolios in different asset classes look fixed or around fixed. As a result, it is reasonable to model that the endowment invest in a single risky asset with constant mean and variance. Given the moving average spending rule still follows (7), if the endowment only invests in a risky asset with price process following (4), then the wealth process is given as

$$dW_t = W_t\left(\mu dt + \sigma dZ\right) - C_t dt = \left(W_t \mu - C_t\right) dt + W_t \sigma dZ, \tag{9}$$

so long as wealth as positive. Also assume that if W_t reaches zero, then the endowment is shut down and W_t and C_t are both zero forever afterwords if wealth reaches zero. We have the following result.

Proposition 3 *When the endowment invests in only a risky asset (9), the moving average spending rule (7) cannot preserve capital and survival forever has zero probability, i.e., for any initial positive wealth* W_0 *and spending* C_0, $\lim_{t\to\infty} prob(W_t = 0) = 1$. *In other words, always reaches zero in finite time.*

Sketch of proof: Given the dynamic of wealth and spending, we can write the dynamics of wealth over spending (which is Markov). Then find a function F of the variable W_t/C_t that is a local martingale (by deriving the dynamics of F using Itô's Lemma, and set the drift term of F equal to zero). Note that $F(0)$ is finite and $F(\infty) = \infty$. Considered $F(W_t/C_t)$ stopped at the first time it reaches $F(0)$ or K (where K is chosen larger than $F(W_0/C_0)$). This is a bounded martingale, so it must converge over time, and since the volatility is positive on the interior, it must converge to either $F(0)$ or K. The martingale condition gives the probability that $F(W_t/C_t)$ converges to the two boundaries. Computing the probability that $F(W_t/C_t) \to F(0)$, and taking the limit as $K \to \infty$ gives us the results.

See the Appendix for the proof.

When wealth delines a lot over a short time, smoothed spending does not change much but the wealth changes quickly so that the spending rate now exceeds the return on investments. This causes further decline in wealth. For a sufficiently large initial decline, the mean reversion towards the target spending rate is too slow to overcome the current loss due to spending too much. Luck might increase wealth enough to save the endowment from falling to zero, but sooner or later we will encounter a large enough shock, without subsequent offsetting good luck, that will pull wealth down to zero in finite time.

3.3 A Smooth Spending Rule that Preserves Capital

The problem with the traditional mean reverting spending rule is that the endowment can spending too much when wealth is low and, thus, target spending moves away more quickly than spending can adjust and capital is not preserved. Hence, to keep the target within a reasonable distance, we need to change the smoothing rule. To give the rule a fighting chance, we assume the traget spending rate would preserve capital, i.e., $\mu-\sigma^2/2$. We propose the smooth spending rule which has potential to preserve capital as

$$dC_t = C_t\left(\kappa\left(\tau - \log\left(\frac{C_t}{W_t}\right)\right) + \mu - \frac{\sigma^2}{2} - \frac{C_t}{W_t}\right)dt, \tag{10}$$

where the wealth process follows

$$dW_t = W_t\left(\mu dt + \sigma dZ\right) - C_t dt = \left(W_t\mu - C_t\right)dt + W_t\sigma dZ. \tag{11}$$

We have the following theorem.

Theorem 3 *The smooth spending rule given by (10) preserves capital in the sense that*[5]

$$\lim_{t\to\infty} \Pr\left(W_t < W_0\right) = 0,$$

provided the parameters satisfy the following condition[6]

$$\mu - \frac{\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right] = Q > 0. \tag{12}$$

However, if

$$\mu - \frac{\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right] < 0,$$

[5] Definition (1) is not the only possible definition of preserving capital. A stronger definition can be that at every state of nature, the wealth converges to positive infinity as time evolves. In other words, if we measure once a year, e.g., in the end of every year, the wealth converges to positive infinity almost surely. Although with a stronger definition, the proof is similar to the weaker version, but needs to apply a functional limit theorem.

[6] The proof can be generalized to the case that given $\forall K > 0$, $\lim_{t\to\infty} \Pr\left(W_t < K\right) = 0$.

then the smooth spending rule does not preserves capital, i.e.,

$$\lim_{t\to\infty} \Pr\left(W_t < W_0\right) = 1.$$

Sketch of proof: Given the proposed dynamics of spending and the wealth dynamics, we can derive the dynamic of $\log\left(C_t/W_t\right)$ by Itô's Lemma, which turns out to be a Gaussian and stationary process if assuming the starting point follow a specific distribution. Furthermore, we can prove that C_t/W_t is a stationary and mean-square ergodic process by mean-square ergodic theorem (Finite Autocovariance Time). Then turn to the expression of the wealth. To prove the capital is preserved, we only need to prove $\text{plim}_{t\to\infty}\left(\log W_t/W_0\right) = \infty$, which can be proved if $\text{plim}_{t\to\infty}\left(\log W_t/W_0\right)/t$ equals to a positive number. Note by the expression of wealth, $\left(\log W_t/W_0\right)/t$ can be written as the sum of the log growth rate of stock $\mu - \sigma^2/2$, the time average of spending ratio C_t/W_t, and the time scaled Brownian motion. Given the ergodic properties of spending ratio, the time average of spending ratio converges to the mean of C_t/W_t with L^2. Finally, by Chebyshev's inequality, we can prove that $\text{plim}_{t\to\infty}\left(\log W_t/W_0\right)/t = Q$.

See the Appendix for the proof.

The condition (12) means that the log growth rate of the risk asset have to be larger than the expected spending rate $\text{E}[C_t/W_t] = \exp\left(\tau + \sigma^2/\left(4\kappa\right)\right)$. Note C_t/W_t is lognormally distributed. This results carry quite similar intuitions as that in subsection 2.2.

Note the term $\mu - \sigma^2/2$ in the drift of spending is the log growth rate of wealth from investment, and term $-C_t/W_t$ is the reduction in wealth from consumption. Recall the spending rule preserving capital in the previous subsection requires that $C_t/W_t \le \mu - \sigma^2/2$. Hence, the smooth spending rule in (10) demonstrate that if the spending is too high, i.e., $C_t/W_t - \left(\mu - \sigma^2/2\right) \le 0$, then reduction in spending at expected rate of decline in wealth is needed to preserve capital. Moreover, the term $\kappa\left(\tau - \log\left(C_t/W_t\right)\right)$ means that on top of preservation of capital, the spending mean reverts to the contant target level.

Remark 2 ***(Knife-edge Case):*** *It is a knife-edge case when the expected log return equals to the expected spending rate. Since we never know when the knife-edge case is exactly true, it is not an case of great interest. It is easy to find out that just like standard winner process returns to initial value infinitely many times over an infinite horizon, the wealth reaches the initial wealth infinitely many times in the knife-edge case. If defining preservation of capital by returning to original value of wealth infinitely many times, this knife-edge case can preserve capital. However, this is not a quite reasonable definition of preservation of capital. Because as time evolves to infinity, most of the probability mass concentrates in the* *tails. As a result, as time t evolves, the probability that wealth is in an interval from $1/t$ times initial wealth to t times initial wealth converges to zero. The wealth can be either above the interval or below the interval. Conventionally, this is not consistent with the sprite of preserving capital. Therefore, in most senses of preservation of capital, the knife-edge case does not preserve capital over time.*

Remark 3 ***(Definition of Preservation of Capital):*** *In this paper, we do not have a formal definition of preservation of capital. Basically, a formal definition does not matter, since our results are consistent with any reasonable definitions of preservation of capital. For example, one can define preservation of capital in a sense that for a sufficient large time, the wealth is larger than the initial wealth, or wealth explodes to infinity as time evolves, or wealth comes across the initial wealth many times. Our results remain established for all of these definitions. On the other hand, it is also sufficient to say that capital is not persevered, if wealth converges to zero over time.*

4 General Condition for Preservation of Capital

The moving average spending rule in the previous section assumes continuity of underlying parameters, e.g., constant volatility of stock return and constant return growth rate. In this section, we provide a general condition of preservation of capital which allows stock return growth, volatility, and the spending rate to follow numerous general type of processes.

4.1 General Condition

Assume the process for the reinvested stock price is governed by a lognormal process with time-varying parameters solving,

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dZ_t, \tag{13}$$

subject to some given initial stock price S_0, where μ_t and σ_t are some general processes, and Z is a standard Wiener process. Then the wealth dynamic follows

$$dW_t = W_t \left(\mu_t dt + \sigma_t dZ\right) - C_t dt = \left(W_t \mu_t - C_t\right) dt + W_t \sigma_t dZ.$$

which implies that

$$W_t = W_0 \exp\left[\int_{s=0}^{t} \left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right) ds - \int_{s=0}^{t} \sigma_s dZ_s\right].$$

Then the Theorem (3) can be easily generalized to a more general case as the following theorem.

Theorem 4 *Given some general stochastic processes of μ_s, σ_s^2, and c_s, and for $\forall s > 0$, $\sigma_s > 0$ and $c_s > 0$, and the following limit exist*

$$\lim_{t\to\infty} \frac{1}{t} \mathrm{E}\left[\int_{s=0}^{t} \left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right) ds - \int_{s=0}^{t} \sigma_s dZ_s\right] = B,$$
$$\lim_{t\to\infty} \frac{1}{t^2} \mathrm{Var}\left[\int_{s=0}^{t} \left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right) ds - \int_{s=0}^{t} \sigma_s dZ_s\right] = 0.$$

then the spending process preserves capital in the sense that

$$\lim_{t\to\infty} \Pr\left(W_t < W_0\right) = 0,$$

if $B > 0$. However, if $B < 0$, then the spending rule does not preserves capital, i.e.,

$$\lim_{t\to\infty} \Pr\left(W_t < W_0\right) = 1.$$

Note the process of μ_s, σ_s^2, and c_s do not need to be each stationary and ergodic, as long as the conditions are satisfied. However, these conditions might not be easily utilized by practitioners, since they are not explicit and simple enough. Hence, we provide some simple conditions which are the special cases of the general condition and capture the basic properties growth rate, volatility, and spending rate in the real world, and obtian the following corollary.

Corollary 1 *Assume μ_s and σ_s^2 is covariance-stationary, and c_s is ergodic, then*

$$\mathrm{E}\left[\mu_s - \sigma_s^2/2\right] > \mathrm{E}\left[c_s\right]$$

ensures preservation of capital.

Moreover, by the general condition, we can study some interesting cases: spending with temporarily negative real risk-free rate and spending with stochastic volatility.

4.2 Preserving Capital with Temporarily Negative Real Risk-Free Rate

These calculations by practitioners are done in real terms (as they should be). An interest rate environment like the current one where inflation exceeds the nominal rate is a special challenge. The endowment never preserve capital if the real risk-free rate is always negative. For example, if investments in real riskless bonds are available but the local expectations hypotheses holds, then given a little regularity, no strategy with non-negative spending will preserve capital if the long-term expected short real interest rate is negative. However, under some condition, capital can be persevered even the risk-free rate is temporarily negative. This subsection models temporarily negative real rate and provides the conditions needed for preserving capital by employing the results of Theorem (4).

Let the nominal interest rate r_t modeled by some Ornstein–Uhlenbeck processes. Hence, the stock price follows

$$\frac{dS_t}{S_t} = (r_t - \iota + \pi)\, dt + \sigma dZ_t,$$

where ι is a constant inflation rate and π is a constant risk premium. With a fixed portfolio θ, the wealth process follows,

$$\begin{aligned} dW_t &= (r_t - \iota)\, W_t dt + W_t \theta\, (\sigma dZ_t + \pi dt) - C_t dt, \\ &= W_t\, ((r_t - \iota + \theta\pi)\, dt + \theta\sigma dZ_t) - C_t dt. \end{aligned}$$

Employing the results in Theorem (4), we can obtain the following theorem:

Theorem 5 *With a constant portfolio in stock, the endowment can preserve capital if*

$$\mathrm{E}\left[r_t - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2}\right] > \mathrm{E}\left[c_t\right], \tag{14}$$

where the spending rate c is covariance-stationary process. If

$$\mathrm{E}\left[r_t - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2}\right] < \mathrm{E}\left[c_t\right],$$

then capital is not preserved.

We can provide examples of spending rule with negative real interest rate, both rules preserving capital and rules not preserving capital.

Example 3 (Successful preservation of capital with negative real rate): Let the nominal interest rate follows a CIR model, i.e.,

$$dr_t = a_0\, (b - r_t)\, dt + \sigma\sqrt{r_t} dZ_t, \tag{15}$$

where a_0 is a constant and b is the long-term mean. Let the spending rule be a moving average rule as

$$dC_t = C_t \left(\kappa \left(\tau - \log \left(\frac{C_t}{W_t} \right) \right) + r_t - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2} - \frac{C_t}{W_t} \right) dt,$$

which, by the results in Theorem (3), implies that $\mathrm{E}[c_t] = \exp\left[\tau + \sigma^2/(4\kappa)\right]$.

Given $\iota = 4\%$, $b = 4\%$, $\pi = 5\%$, $\sigma = 15\%$, $\theta = 0.8$, $\tau = -3.5$ (with target rate of $c = 0.03$, i.e., $-3.5 = \log(0.03)$), and $\kappa = 1$, then the real interest rate is zero, just quite similar to real rate in the current financial market. However, the spending still can be covered by a high enough risk premium. Consequently, in a long horizon, the capital can be preserved. For instance, suppose at a point of time, the inflation rate is 4% and the real rate is -4%, then given the risk premium is 5% and the endowment cannot cover a positive spending rate with a negative return at this point. However, capital is still preserved since when during a good time, say, real interest rate is 8% and, thus, the expected return of portfolio is 13%. If the endowment still has the target spending rate, then capital is preserved. To sum up, the point is that preservation of capital is not about a point of time, it is about the whole paths of underlying dynamics. Finally, by applying Theorem (5), it is easy to see (14) is satisfied, since

$$b - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right] = 0.0024,$$

hence, capital is preserved.

Example 4 (Unsuccessful preservation of capital with negative real rate): Given $\iota = 6\%$, $\mathrm{E}[r_t] = 0$, $\pi = 5\%$, and $\sigma = 15\%$, then no choice of a fixed portfolio θ can preserve capital locally. Since even the portfolio which maximizes the growth rate of log wealth, i.e., $\theta = \pi/\sigma^2$ maximizing $\theta\pi - \theta^2\sigma^2/2$, can not preserve capital. Note according to (14) in Theorem (5), we can calculate the expected log turn with highest growth rate:

$$\mathrm{E}\left[r_t\right] - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2} = \mathrm{E}\left[r_t\right] - \iota + \frac{\pi^2}{2\sigma^2} = -0.004444,$$

which is a negative number. However, expected spending cannot be negative. Hence, (14) is not satisfied, and capital is not preserved due to a too high expected inflation and a too low expected nominal interest rate. There are also good reason not to take on so much leverage. If $\theta = 0.8$, and $\iota = 3.5\%$, then capital is still not preserved, since $\mathrm{E}[r_t] - \iota + \theta\pi - \theta^2\sigma^2/2 = -0.0022$.

Example 5 (Preservation of capital by spending rule with stochastic volatility):

Assume the spending rate is given as an affine function of nominal interest rate, i.e.,

$$c_t = C_0 + C_1 r_t,$$

where r_t is the nominal interest rate following the CIR model (15), with $C_0 > 0$, and $0 < C_1 < 1$. Therefore, the spending rate is covariance-stationary and always positive, and we have

$$\mathrm{E}\left[c_t\right] = C_0 + C_1 \mathrm{E}\left[r_t\right].$$

Given $\iota = 4\%$, $b = 4\%$, $\pi = 5\%$, $\sigma = 15\%$, $\theta = 0.8$, $C_0 = 3\%$, and $C_1 = 0.6$, we have capital preserved, since according to (14) in Theorem (5), we have

$$b - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2} - \left(C_0 + C_1 \mathrm{E}\left[r_t\right]\right) = \left(1 - C_1\right) b - \iota + \theta\pi - \frac{\theta^2\sigma^2}{2} - C_0 = 0.0088.$$

Note it is possible that at some point, the nominal rate reaches zero, meanwhile, the spending rate is positive. However, even this case happens, the endowment can still preserve capital. Since, again, preservation of capital is not about several points of times, it is about an infinitely long horizon. Hence, even the expected log real rate of the assets can be less than the spending rate when the interest rate is temporarily low, however, the turn of asset can well covers the spending when interest rate is high. Consequently, capital is preserved.

5 Optimal Spending and Portfolio Choice with Smooth Property

5.1 Preserving Capital by Setting Lower Bound of Ratio of Wealth to Spending

By (5), we know that if we set a lower bound for the ratio of wealth to spending equals to $1/\left(\mu - \sigma^2/2\right)$, i.e., when the ratio reaches the bound, the endowment immediately reduces spending to ensure $W/C \geqslant 1/\left(\mu - \sigma^2/2\right)$. Then no matter what kinds of dynamic that spending C_t follows, the spending rule always preserves capital. However, in the real world, the endowment usually has motivation to spend as much as possible conditional on that the capital is preserved. This motivates us to find an optimal spending rule with a bound which preserves capital and the bound of the ratio of wealth to spending can be as low as possible. To address this question, we study the following model of optimal spending.

5.2 Model of Optimal Spending Strategy

Model in the above subsections set exogenous lower bound of the ratio of wealth to spending to ensure the preservation of capital. In this section, we propose a model which endogeneizes two boundaries of the ratio of wealth to spending. Within the boundaries, the endowment keeps the spending constant. As we will see, maximization of expected utility implies smooth spending and the preservation of capital.

Consider the portfolio problem faced by the endowment which can possible invest in a riskless asset and a single risky asset (a stock) whose price process evolves according to (4). The instantaneous riskless rate is r. To simplify interpretation later, we assume without loss of generality that $\mu > r$, so that the risky asset is an attractive investment. Assume an endowment has incentive to smooth spending, the problem of the endowment can be described as follows.

Problem 1 *Given the initial wealth $\bar{W}_0$ and initial spending $\bar{C}_0$, choose adapted portfolio process $\left\{\bar{\theta}_t\right\}_{t=0}^{\infty}$ and adapted spending process $\left\{\bar{C}_t\right\}_{t=0}^{\infty}$ to maximize the expected utility,*

$$\max_{\bar{\theta},\bar{C}'}\mathrm{E}\left[\int_{t=0}^{\infty} e^{-\rho t}\frac{\left(\bar{C}_t - C^*\right)^{1-R}}{1-R}dt - \frac{a}{1-R}\int_{\bar{C}'\neq 0}\left|d\left(\bar{C}_t - C^*\right)^{1-R}\right|\right]$$
$$s.t.\ d\bar{W}_t = r\bar{W}_t dt + \bar{\theta}_t\left(\left(\mu - r\right)dt + \sigma dZ_t\right) - \bar{C}_t dt$$

where ρ is the pure rate of time preference, R is the constant relative risk aversion, and a is the adjustment cost rate. It is assumed that $\mu - r$, ρ, σ, C^* and r are all positive.

Let

$$C_t = \bar{C}_t - C^*, \quad \text{and} \quad W_t = \bar{W}_t - C^*/r,$$

and thus

$$\bar{C}_t = C_t + C^*, \quad \text{and} \quad \bar{W}_t = W_t + C^*/r,$$

hence the problem can be restated as

Problem 2 *Given the initial wealth W_0 and initial spending C_0, choose adapted portfolio* *process* $\{\theta_t\}_{t=0}^{\infty}$ *and adapted spending process* $\{C_t\}_{t=0}^{\infty}$ *to maximize the expected utility,*

$$\max_{\theta, C'} \mathrm{E}\left[\int_0^{\infty} e^{-\rho t}\left(\frac{C_t^{1-R}}{1-R}\right) dt - \frac{a}{1-R}\int_{C'\neq 0}\left|dC_t^{1-R}\right|\right]$$

$$\begin{aligned} s.t.\ dW_t &= r\bar{W}_t dt + \bar{\theta}_t\left((\mu - r)\, dt + \sigma dZ_t\right) - \bar{C}_t dt \\ &= \left(rW_t + C^*\right) dt + \theta_t\left((\mu - r)\, dt + \sigma dZ_t\right) - \left(C_t + C^*\right) dt \\ &= rW_t dt + \theta_t\left((\mu - r)\, dt + \sigma dZ_t\right) - C_t dt. \\ and\ W &\geq 0,\ \ for\ t \geq 0. \end{aligned}$$

Note a positive C^* ensure a reasonable sense of preservation of capital, since the wealth has to be higher than C^*/r. If assuming wealth cannot fall to 70% of the initial wealth, we can set $C^* = 0.7 \times \bar{W}_0 r$ such that $C^*/r = 0.7 \times \bar{W}_0$. Moreover, by

$$\bar{\theta}_t\left((\mu - r)\, dt + \sigma dZ_t\right) = \theta_t\left((\mu - r)\, dt + \sigma dZ_t\right),$$

it is easy to see, the optimal portfolios in the two problems are identical.

Now we focus on the reformed problem. The value function satisfies the Bellman equation

$$\frac{C_t^{1-R}}{1-R} - \frac{a}{1-R}\left|(1-R)\, C_t^{-R} dC\right| - \rho V + V_C dC + V_W\left(rw_t + \theta_t(\mu - r) - C_t\right) + \frac{\sigma^2\theta_t^2}{2} V_{WW} = 0$$

The optimal strategy of consumption in the interior region is $dC = 0$ and optimal portfolio in stock is given as

$$\theta = -\frac{\mu - r}{\sigma^2}\frac{V_W}{V_{WW}} = -\frac{\kappa}{\sigma}\frac{V_W}{V_{WW}}$$

where $\kappa = \frac{\mu - r}{\sigma}$, thus we have

$$\frac{C_t^{1-R}}{1-R} - \rho V + V_W\left(rW_t - C_t\right) - \frac{\kappa^2}{2}\frac{V_W^2}{V_{WW}} = 0$$

we can simplify it by let $\omega \equiv W/C$, and conjecture $V(C, W) = C^{1-R} v(\omega)$. As a result, we have

$$\frac{1}{1-R} - \rho v(\omega) + v_\omega\left(r\omega - 1\right) - \frac{\kappa^2}{2}\frac{v_\omega^2}{v_{\omega\omega}} = 0$$

Define the dual variable as $z = v_\omega$ and, thus, we have

$$\omega = -J_z, \quad v = J - zJ_z, \quad v_{\omega\omega} = \frac{dv_\omega}{d\omega} = \frac{dz}{d\left(-J_z\right)} = -\frac{1}{J_{zz}}. \tag{16}$$

With above transformations, we can rewrite the ODE with dual variable as

$$J_{zz}+\frac{2\left(\rho-r\right)}{\kappa^{2}z}J_{z}-\frac{2\rho}{\kappa^{2}z^{2}}J=\frac{2}{\kappa^{2}z^{2}}\left(z-\frac{1}{1-R}\right)$$

which is a Euler-Cauchy ODE. Solve for the ODE in dual, we achieve the following Proposition.

Proposition 4 *The dual function in (16) can be expressed as*

$$J\left(z\right)=C_{1}z^{\beta_{1}}+C_{2}z^{\beta_{2}}+\frac{1}{\rho\left(1-R\right)}-\frac{z}{r},$$

where C_1 *and* C_2 *are determined by the boundary conditions described below, and*

$$\beta_{1}=\frac{-\left(p-1\right)+\sqrt{\left(p-1\right)^{2}+4q}}{2},\qquad\beta_{2}=\frac{-\left(p-1\right)-\sqrt{\left(p-1\right)^{2}+4q}}{2},$$
$$p=\frac{2\left(\rho-r\right)}{\kappa^{2}},\qquad q=\frac{2\rho}{\kappa^{2}}.$$

5.2.1 Boundaries for Ratio of Wealth to Spending

Assume the lower boundary and the upper boundary for ω are ω_* and ω^* respectively. For $\omega<\omega_*$, the endowment decreases C immediately so that $\omega=\omega_*$, i.e., move C to W/ω_*. Thus, the value function for the lower boundary $\underline{V}(W_t,C_t)$ is given as

$$\underline{V}\left(C_{t},W_{t}\right)=V\left(\frac{W}{\omega_{*}},W_{t}\right)-\frac{a}{1-R}\left(C^{1-R}-\left(\frac{W}{\omega_{*}}\right)^{1-R}\right),$$

therefore, for $\omega<\omega_*$,

$$v\left(\omega\right)=\left(\frac{\omega}{\omega_{*}}\right)^{1-R}\left(v\left(\omega_{*}\right)+\frac{a}{1-R}\right)-\frac{a}{1-R}.$$

For $\omega>\omega_*$, the endowment increases C immediately so that $\omega=\omega^*$, i.e., move C to W/ω^*. Thus, the value function for the upper boundary $\bar{V}\left(W_t,C_t\right)$ is given as

$$\bar{V}\left(C_{t},W_{t}\right)=V\left(\frac{W}{\omega^{*}},W_{t}\right)-\frac{a}{1-R}\left(\left(\frac{W}{\omega^{*}}\right)^{1-R}-C^{1-R}\right),$$

therefore, for $\omega>\omega_*$,

$$v\left(\omega\right)=\left(\frac{\omega}{\omega^{*}}\right)^{1-R}\left(v\left(\omega^{*}\right)-\frac{a}{1-R}\right)+\frac{a}{1-R}.$$

Moreover, outside the lower boundary we have

$$V_C = -aC^{-R},$$

while inside the boundary we have

$$V_C = (1-R)\,C^{-R}v - C^{-R}\frac{W}{C}v_\omega.$$

Note V_C matches at the boundary, yielding

$$-a = (1-R)\,v - \omega_* v_\omega.$$

Outside the upper boundary we have $V_C = aC^{-R}$, matching yields

$$a = (1-R)\,v\left(\omega^*\right) - \omega^* v_\omega\left(\omega^*\right).$$

Moreover, matching V_{CC} give rise to $-\omega v_{\omega\omega} = Rv_\omega$. For the optimal upper boundary and the optimal lower boundary, we need to determine $z^* = v_\omega\left(\omega^*\right)$ and $z_* = v_\omega\left(\omega_*\right)$ respectively. We can obtain analytical expressions for C_1 and C_2 as functions of z^* and z_*. Substitute C_1 and C_2 into the smooth pasting conditions, we can solve for the optimal boundaries z^* and z_* numerically.

5.2.2 Numerical Illustrations

Boundaries of Spending With the following parameters $\mu = 0.1, \sigma = 0.2, R = 3, r = 0.01$, $W_0 = 1$, and $\rho = 0.01$, we plot the lower boundary of spending $C_* = 1/\omega^*\ (\cdot)$ and the higher boundary of spending $C^* = 1/\omega_*\ (+)$ as function of the rate of adjustment cost a.

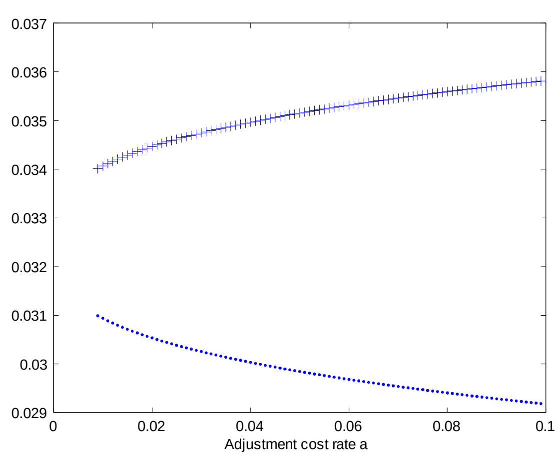

Figure 1: Boundaries of Spending. Lower boundary of spending and the higher boundary of spending are plotted as function of the rate of adjustment cost.

Figure 1 shows that as the adjustment cost rate increases, the interior of the boundaries become wider. Intuitively, the higher cost of adjustment gives rise to a lower frequency of adjusting spending level.

With the following parameters $a = 0.01, \sigma = 0.2, R = 3, r = 0.01$, $W_0 = 1$, and $\rho = 0.01$ we plot the lower boundary of spending $C_* = 1/\omega^*$ $(\cdot)$ and the higher boundary of spending $C^* = 1/\omega_*$ $(+)$ as function of the growth rate of stock μ. The right figure plots the lower boundary of spending $\bar{C}_* = 1/\bar{\omega}^*$ $(\cdot)$ and the higher boundary of spending $C^* = 1/\bar{\omega}_*$ $(+)$ as function of the growth rate of stock μ.

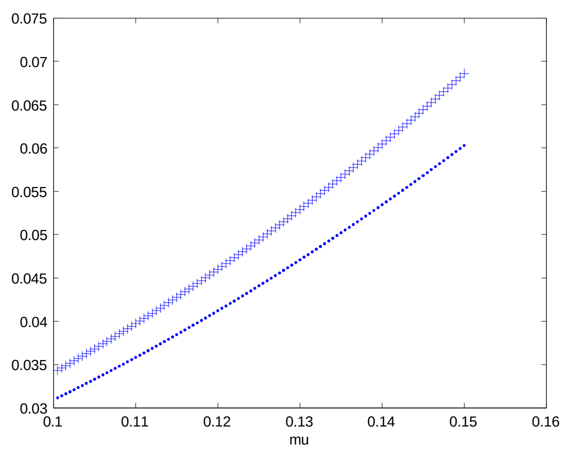

Figure 2: Boundaries of Spending. Lower boundary of spending and the higher boundary of spending are plotted as function of the drift of stock return.

Figure 2 shows that as the drift of stock return increases, the interior of the boundaries become wider. Moreover, both boundaries increases indicating that the endowment tends to increases the spending level. Intuitively, it is optimal to spend more when growth rate of stock return is higher.

Simulation of Wealth Process and Smoothed Spending We simulate the wealth process for many times and find the simulated optimal spending strategy do preserve capital in a sense that $W_t > 0$ in problem 1 and, hence, $\bar{W}_t > C^*/r$ in problem 2. This result confirms that expected utility maximization implies smooth spending and the preservation of capital. See the detailed description of algorithm of simulation in the Appendix.

We simulate the wealth process for 300 times with an initial wealth 1 at time 0, we find all of the simulated terminal wealth at year 100 are positive ($W_t > 0$). Note when simulating, we use following parameters $a = 0.2, \sigma = 0.2, R = 3, r = 0.01$, $W_0 = 1$, $T = 100$, and $\rho = 0.01$. Furthermore, we set the program to ensure that once wealth reaches zero, the simulation is terminated. The complete 300 simulations means wealth is always preserved during the simulation. (Note to ensure the quality of simulation, $dt = T/N$ have to be small enough, e.g., assigning $dt = 100/100000$ in simulations can be acceptable.)

The following figure shows one of the simulated wealth process W_t. The wealth process in Problem 1 can be obtained by swifting the wealth process in Problem 2 upward by C^*/r since $\bar{W}_t = W_t + C^*/r$.

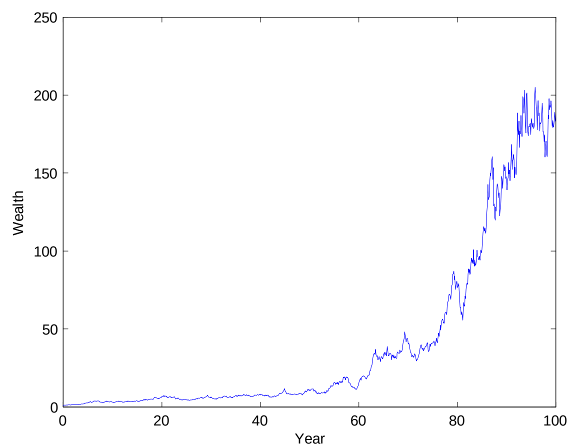

Figure 3: Simulated Wealth Process.

The wealth of the endowment tends increase as time evolves and leads to preservation of capital. The following figure shows one of the simulated spending process. The spending process in Problem 1 can be obtained by swifting the spending process in Problem 2 upward by C^* since $\bar{C}_t = C_t + C^*$.

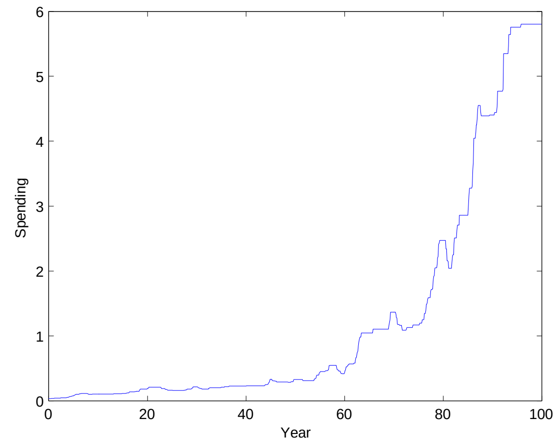

Figure 4: Simulated Spending Process.

The spending also tends to increase as time evolves. Moreover, spending is smoothed, especially compared with the simulated wealth process. The following figure shows one of the simulated process of the ratio of wealth to spending.

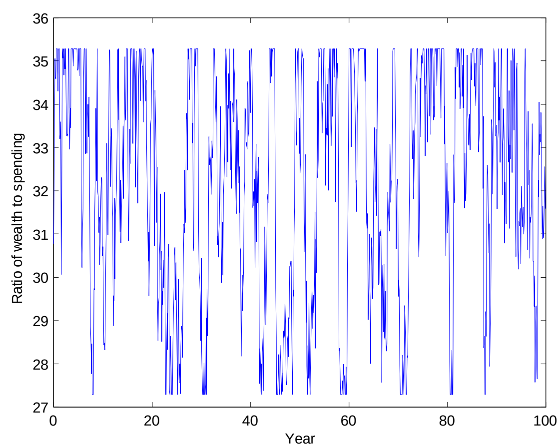

Figure 5: Simulated Ratio of Wealth to Spending.

Note the ratio of wealth to spending is always between $\omega_* = 27.29$ and $\omega^* = 35.27$, which help with smoothing spending.

6 Conclusion

Two commonly used rules of thumb used for managing endowments that are supposed to preserve capital actually do not preserve capital. Having a spending rate less than the expected return on assets is not strong enough and is based on a fallacious application of the law of large numbers. A correct analogous rule would take logs. We can think of an approximate rule (correct for a lognormal world) that the spending rate has to be less than the mean return on the portfolio minus half the variance.

The second rule of thumb that has problems is the use of a moving average rule to smooth spending. This type of rule never preserves capital in a model where returns are random and i.i.d. We provide alternative rules that smooths spending but in a way that preserves capital for appropriate choice of parameter values.

We hope our results will help universities to do a better job managing their endowments.

References

Acharya, S. and E. Dimson (2007). *Endowment asset management : investment strategies in Oxford and Cambridge*. Publisher : Oxford University Press.

Halmos, P. R. (1956). Lectures on ergodic theory. *Math. Soc. Japan*.

Neumann, J. V. (1932). Proof of the quasi-ergodic hypothesis. *Proceedings of the National Academy of Sciences, USA 18*, 70–82.

Rice, M., R. A. DiMeo, and M. Porter (2012). *Nonprofit asset management : effective investment strategies and oversight*. Publisher : John Wiley Sons.

A Appendix: Proofs and Algorithms

A.1 Proof of Proposition 2

Proof. We can rewrite (7) and (8) as

$$d\begin{pmatrix} W_t \\ C_t \end{pmatrix} = A \begin{pmatrix} W_t \\ C_t \end{pmatrix} dt,$$

where

$$A = \begin{pmatrix} r & -1 \\ \kappa\tau & -\kappa \end{pmatrix}.$$

The above ODE can be solved by using an eigenvalue-eigenvector decomposition of A. The solution is given as

$$\begin{pmatrix} W_t \\ C_t \end{pmatrix} = K_1 e^{\lambda_1 t}\phi_1 + K_2 e^{\lambda_2 t}\phi_2,$$

where $\lambda_2 < 0 < \lambda_1$ given by

$$\lambda = \frac{r-\kappa \pm \sqrt{(\kappa - r)^2 - 4\kappa(\tau - r)}}{2} = \frac{r-\kappa \pm \sqrt{(\kappa + r)^2 - 4\kappa\tau}}{2}$$

are the two roots of the eigenvalue equation $\det(A - \lambda I) = 0$, and $\phi_i = (1, r-\lambda_i)^\intercal$. Note that $0 < r - \lambda_1 < r - \lambda_2$, so that if $C_0/W_0 > r - \lambda_2$ (say after an unanticipated negative shock to wealth), then $K_2 > W_0$ and $K_1 = W_0 - K_2 < 0$, so wealth goes to zero in finite time and, thus, capital is not preserved in this case. ■

A.2 Proof of Proposition 3

Assume the dynamic of spending and wealth are given as

$$\begin{cases} dC_t = \kappa(\tau W_t - C_t)\,dt, \\ dW_t = W_t(\mu dt + \sigma dZ) - C_t dt = (W_t\mu - C_t)\,dt + W_t\sigma dZ, \end{cases}$$

i.e.,

$$\begin{cases} \frac{dW_t}{W_t} = \frac{W_t\mu - C_t}{W_t}dt + \sigma dZ, \\ \frac{dC_t}{C_t} = \frac{\kappa(\tau W_t - C_t)}{C_t}dt. \end{cases}$$

Let

$$a = \frac{W_t\mu - C_t}{W_t}, \quad b = \sigma \ \text{ and } \ f = \frac{\kappa(\tau W_t - C_t)}{C_t}, \quad g = 0,$$

let $U = \frac{W}{C}$, then by Itô's lemma

$$\begin{aligned} \frac{dU}{U} &= \left(a - f + g^2\right)dt + bdZ. \\ &= \left(\frac{W_t\mu - C_t}{W_t} - \frac{\kappa(\tau W_t - C_t)}{C_t}\right)dt + \sigma dZ \\ &= \left(\mu + \kappa - \frac{C_t}{W_t} - \kappa\tau\frac{W_t}{C_t}\right)dt + \sigma dZ, \end{aligned}$$

i.e.,

$$d\left(\frac{W_t}{C_t}\right) = \frac{W_t}{C_t}\left(\mu + \kappa - \frac{C_t}{W_t} - \kappa\tau\frac{W_t}{C_t}\right)dt + \frac{W_t}{C_t}\sigma dZ.$$

Let F be a function of the ratio $\frac{W_t}{C_t}$,

$$dF\left(\frac{W_t}{C_t}\right) = F'\left(\frac{W_t}{C_t}\right)\left[\frac{W_t}{C_t}\left(\mu+\kappa-\frac{C_t}{W_t}-\kappa\tau\frac{W_t}{C_t}\right)dt+\frac{W_t}{C_t}\sigma dZ\right]+\frac{1}{2}F''\left(\frac{W_t}{C_t}\right)\left(\sigma\frac{W_t}{C_t}\right)^2 dt.$$

To make the drift F equal to zero, F has to satisfy that

$$F'\left(\frac{W_t}{C_t}\right)\frac{W_t}{C_t}\left(\mu+\kappa-\frac{C_t}{W_t}-\kappa\tau\frac{W_t}{C_t}\right) = -\frac{1}{2}F''\left(\frac{W_t}{C_t}\right)\left(\sigma\frac{W_t}{C_t}\right)^2$$

$$F'\left(\frac{W_t}{C_t}\right)\left((\mu+\kappa)\frac{W_t}{C_t}-1-\kappa\tau\left(\frac{W_t}{C_t}\right)^2\right) = -\frac{1}{2}F''\left(\frac{W_t}{C_t}\right)\left(\sigma\frac{W_t}{C_t}\right)^2$$

$$\frac{F''\left(\frac{W_t}{C_t}\right)}{F'\left(\frac{W_t}{C_t}\right)} = -\frac{2\left((\mu+\kappa)\frac{W_t}{C_t}-1-\kappa\tau\left(\frac{W_t}{C_t}\right)^2\right)}{\left(\sigma\frac{W_t}{C_t}\right)^2}$$

$$\left[\ln F'\left(\frac{W_t}{C_t}\right)\right]' = -\frac{2\left((\mu+\kappa)\frac{W_t}{C_t}-1-\kappa\tau\left(\frac{W_t}{C_t}\right)^2\right)}{\left(\sigma\frac{W_t}{C_t}\right)^2}$$

$$\left[\ln F'\left(\frac{W_t}{C_t}\right)\right]' = -\frac{2}{\sigma^2}\left[(\mu+\kappa)\frac{C_t}{W_t}-\left(\frac{C_t}{W_t}\right)^2-\kappa\tau\right],$$

hence

$$\ln F'\left(\frac{W_t}{C_t}\right) = -\frac{2}{\sigma^2}\left((\mu+\kappa)\ln\frac{W_t}{C_t}+\frac{C_t}{W_t}-\kappa\tau\frac{W_t}{C_t}\right)+C_0$$

$$\begin{aligned}F'\left(\frac{W_t}{C_t}\right) &= \exp\left[-\frac{2}{\sigma^2}\left((\mu+\kappa)\ln\frac{W_t}{C_t}+\frac{C_t}{W_t}-\kappa\tau\frac{W_t}{C_t}\right)+C_0\right]\\ &= \exp C_0\exp\left[-\frac{2}{\sigma^2}\left((\mu+\kappa)\ln\frac{W_t}{C_t}+\frac{C_t}{W_t}-\kappa\tau\frac{W_t}{C_t}\right)\right]\\ &= C_1\exp\left[-\frac{2}{\sigma^2}\left((\mu+\kappa)\ln\frac{W_t}{C_t}+\frac{C_t}{W_t}-\kappa\tau\frac{W_t}{C_t}\right)\right], \text{ where } C_1=\exp C_0,\end{aligned}$$

and

$$F\left(\frac{W_t}{C_t}\right) = C_1\int\exp\left[-\frac{2}{\sigma^2}\left((\mu+\kappa)\ln\frac{W_t}{C_t}+\frac{C_t}{W_t}-\kappa\tau\frac{W_t}{C_t}\right)\right]d\frac{W_t}{C_t}+C_2.$$

Let $x = \frac{W_t}{C_t}$,

$$\begin{aligned}
F(x) &= C_1 \int \exp\left[-\frac{2}{\sigma^2}\left((\mu+\kappa)\ln x + \frac{1}{x} - \kappa\tau x\right)\right] dx + C_2 \\
&= C_1 \int \exp\left(-\frac{2}{\sigma^2}(\mu+\kappa)\ln x - \frac{2}{\sigma^2}\frac{1}{x} + \frac{2}{\sigma^2}\kappa\tau x\right) dx + C_2 \\
&= C_1 \int \exp\left(\ln x^{-\frac{2}{\sigma^2}(\mu+\kappa)} - \frac{2}{\sigma^2}\frac{1}{x} + \frac{2\kappa\tau}{\sigma^2}x\right) dx + C_2 \\
&= C_1 \int \exp\left(\ln x^{-\frac{2}{\sigma^2}(\mu+\kappa)}\right)\exp\left(-\frac{2}{\sigma^2}\frac{1}{x} + \frac{2\kappa\tau}{\sigma^2}x\right) dx + C_2 \\
&= C_1 \int x^{-\frac{2}{\sigma^2}(\mu+\kappa)}\exp\left(-\frac{2}{\sigma^2}\frac{1}{x} + \frac{2\kappa\tau}{\sigma^2}x\right) dx + C_2.
\end{aligned}$$

Note $F(\infty)$ is explosive, i.e., $F(\infty) = +\infty$ is the upper bound of the value of F, $F(0)$ converges to some finite number and it is the lower bound of F. Note F is a martingale and it is a increasing function of x.

As time converges to $+\infty$, let the probability of converging to upper bound of $+\infty$ be P_u, let the probability of converging to lower bound of $F(0)$ be P_l, and we have

$$\begin{cases} P_l F(0) + P_u F(\infty) = F\left(\frac{W_0}{C_0}\right), \\ P_l + P_u + P_0 = 1, \\ P_0 = 0. \end{cases}$$

where P_0 denote the probability converge to any finite number which is larger than $F(0)$, and it is possible to proof that as time evolves to infinity, $P_0 = 0$,, since the volatility of the process is positive. Hence, we have

$$P_l F(0) + F(\infty) - P_l F(\infty) = F\left(\frac{W_0}{C_0}\right) \rightarrow P_l = \frac{F\left(\frac{W_0}{C_0}\right) - F(\infty)}{F(0) - F(\infty)},$$

and $P_u \rightarrow 0$, and $P_l \rightarrow 1$ as $t \rightarrow \infty$. ■

A.3 Proof of Theorem 3

Note the dynamic of spending and wealth are given as

$$\begin{cases} dC_t = h\left(\frac{C_t}{W_t}\right)dt, \text{ where } h\left(\frac{C_t}{W_t}\right) = C_t\left(\kappa\left(\tau - \log\left(\frac{C_t}{W_t}\right)\right) - \frac{C_t}{W_t} + \mu - \frac{\sigma^2}{2}\right), \\ dW_t = W_t(\mu dt + \sigma dZ) - C_t dt = (W_t\mu - C_t)dt + W_t\sigma dZ, \end{cases}$$

i.e.,

$$\begin{cases} \frac{dC_t}{C_t} = \frac{h\left(\frac{C_t}{W_t}\right)}{C_t} dt, \\ \frac{dW_t}{W_t} = \frac{W_t\mu - C_t}{W_t} dt + \sigma dZ. \end{cases}$$

Let

$$a = \frac{h\left(\frac{C_t}{W_t}\right)}{C_t}, \quad b = 0 \text{ and } f = \frac{W_t\mu - C_t}{W_t}, \quad g = \sigma,$$

let $U_t = C_t/W_t$, then by Itô's lemma

$$\frac{dU_t}{U_t} = \left(a - f + g^2\right) dt - g dZ_t = \left(\frac{h\left(\frac{C_t}{W_t}\right)}{C_t} - \frac{W_t\mu - C_t}{W_t} + \sigma^2\right) dt - \sigma dZ_t.$$

Therefore,

$$\begin{aligned} d\log\left(\frac{C_t}{W_t}\right) &= \left(\frac{h\left(\frac{C_t}{W_t}\right)}{C_t} - \frac{W_t\mu - C_t}{W_t} + \sigma^2\right) dt - \sigma dZ_t - \frac{1}{2}\sigma^2 dt \\ &= \left(\frac{h\left(\frac{C_t}{W_t}\right)}{C_t} - \frac{W_t\mu - C_t}{W_t} + \frac{1}{2}\sigma^2\right) dt - \sigma dZ_t. \end{aligned}$$

Simplifying the above equation, we obtain an Ornstein-Uhlenbeck process as

$$d\log\left(\frac{C_t}{W_t}\right) = \kappa\left(\tau - \log\left(\frac{C_t}{W_t}\right)\right) dt - \sigma dZ_t,$$

and

$$\log\left(\frac{C_t}{W_t}\right) = \log\left(\frac{C_0}{W_0}\right) e^{-\kappa t} + \tau\left(1 - e^{-\kappa t}\right) - \sigma e^{-\kappa t}\int_{s=0}^{t} e^{\kappa s} dZ_s.$$

Note $\log\left(C_t/W_t\right)$ is a Gaussian process, in particular, conditional on $\log\left(C_0/W_0\right)$. Assuming $\log\left(C_0/W_0\right)$ has finite variance, then the mean and variance of $\log\left(C_t/W_t\right)$ follow

$$\begin{aligned} \mathrm{E}\left[\log\left(\frac{C_t}{W_t}\right)\right] &= \left(\mathrm{E}\left[\log\left(\frac{C_0}{W_0}\right)\right]\right) e^{-\kappa t} + \tau\left(1 - e^{-\kappa t}\right), \\ \mathrm{Cov}\left[\log\left(\frac{C_s}{W_s}\right), \log\left(\frac{C_t}{W_t}\right)\right] &= \frac{\sigma^2}{2\kappa} e^{-\kappa s}\left(e^{\kappa t} - e^{-\kappa t}\right) + e^{-\kappa(s+t)}\mathrm{Var}\left(\log\left(\frac{C_0}{W_0}\right)\right). \end{aligned}$$

If assume

$$\log\left(\frac{C_0}{W_0}\right) = \tau - \sigma\int_{-\infty}^{0} e^{\kappa t} dZ_s,$$

hence, the process of $\log\left(C_t/W_t\right)$ is stationary with constant mean, variance, and autocovariance as

$$\begin{aligned}
\mathrm{E}\left[\log\left(\frac{C_t}{W_t}\right)\right] &= \tau,\\
\mathrm{Var}\left[\log\left(\frac{C_t}{W_t}\right)\right] &= \frac{\sigma^2}{2\kappa},\\
\mathrm{Cov}\left[\log\left(\frac{C_s}{W_s}\right),\log\left(\frac{C_t}{W_t}\right)\right] &= \frac{\sigma^2}{2\kappa}e^{-\kappa|t-s|}.
\end{aligned}$$

As a result, C_t/W_t is log-normal distributed with mean, variance, and autocorrelation as

$$\begin{aligned}
\mathrm{E}\left[\frac{C_t}{W_t}\right] &= \exp\left(\tau+\frac{\sigma^2}{4\kappa}\right),\\
\mathrm{Var}\left[\frac{C_t}{W_t}\right] &= \left(\exp\left(\frac{\sigma^2}{2\kappa}\right)-1\right)\exp\left(2\tau+\frac{\sigma^2}{2\kappa}\right),\\
\mathrm{Cov}\left[\frac{C_s}{W_s},\frac{C_t}{W_t}\right] &= \left(\exp\left(\frac{\sigma^2}{2\kappa}e^{-\kappa|t-s|}\right)-1\right)\exp\left(2\tau+\frac{\sigma^2}{2\kappa}\right).
\end{aligned}$$

Note that the autocovariance depends only on the lag $|t-s|$ and not on time t. Therefore, C_t/W_t is also stationary.

We now prove it is a mean-square ergodic process. Note the *integral time scale* of the stationary random process C_t/W_t is given as

$$\begin{aligned}
\Upsilon_{int} &= \frac{1}{\left(\exp\left(\frac{\sigma^2}{2\kappa}\right)-1\right)\exp\left(2\tau+\frac{\sigma^2}{2\kappa}\right)}\int_0^\infty\left(\exp\left(\frac{\sigma^2}{2\kappa}e^{-\kappa\varphi}\right)-1\right)\exp\left(2\tau+\frac{\sigma^2}{2\kappa}\right)d\varphi\\
&= \frac{1}{\exp\left(\frac{\sigma^2}{2\kappa}\right)-1}\int_0^\infty\left(\exp\left(\frac{\sigma^2}{2\kappa}e^{-\kappa\varphi}\right)-1\right)d\varphi.
\end{aligned}$$

Let

$$\begin{aligned}
u=\frac{\sigma^2}{2\kappa}e^{-\kappa\varphi} &\Longrightarrow \frac{2\kappa}{\sigma^2}u=e^{-\kappa\varphi}\Longrightarrow -\kappa\varphi=\log\left(\frac{2\kappa}{\sigma^2}u\right)\Longrightarrow -\kappa d\varphi=d\log\left(\frac{2\kappa}{\sigma^2}u\right)\\
&\Longrightarrow -\kappa d\varphi=\frac{2\kappa}{\sigma^2}\frac{\sigma^2}{2\kappa u}du\Longrightarrow -\kappa d\varphi=\frac{1}{u}du\Longrightarrow d\varphi=\frac{1}{-\kappa u}du,
\end{aligned}$$

hence, we have

$$\int_0^\infty\left(\exp\left(\frac{\sigma^2}{2\kappa}e^{-\kappa\varphi}\right)-1\right)d\varphi=-\int_{\frac{\sigma^2}{2\kappa}}^0\frac{e^u-1}{\kappa u}du=\frac{1}{\kappa}\int_0^{\frac{\sigma^2}{2\kappa}}\frac{e^u-1}{u}du.$$

Note

$$\lim_{u\to0}\frac{e^u-1}{u}=\lim_{u\to0}\frac{e^u}{1}=1,$$

and $\left(e^{u}-1\right)/u$ strictly increases in u, hence,

$$1 \leq \frac{e^{u}-1}{u} \leq \frac{2\kappa}{\sigma^{2}}\left(e^{\frac{\sigma^{2}}{2\kappa}}-1\right), \text{ where } 0 \leq u \leq \frac{\sigma^{2}}{2\kappa}.$$

Therefore,

$$\int_{0}^{\frac{\sigma^{2}}{2\kappa}} \frac{e^{u}-1}{u} du < \infty \Longrightarrow \Upsilon_{int} < \infty.$$

Hence, based on the Mean-Square Ergodic Theorem (Finite Autocovariance Time), [7] we have that the process C_t/W_t is mean-square ergodic in the first moment, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\int_{s=0}^{t} \frac{C_s}{W_s} ds = \exp\left[\tau + \frac{\sigma^{2}}{4\kappa}\right],$$

converges in squared mean. According to the properties of mean-square ergodic convergence, we have

$$\lim_{t\to\infty} \mathrm{E}\left[\frac{1}{t}\int_{s=0}^{t} \frac{C_s}{W_s} ds\right] = \exp\left[\tau + \frac{\sigma^{2}}{4\kappa}\right], \tag{17}$$

$$\lim_{t\to\infty} \mathrm{Var}\left[\frac{1}{t}\int_{s=0}^{t} \frac{C_s}{W_s} ds\right] = 0. \tag{18}$$

By the definition of preservation of capital, to prove the spending rule preserves capital, we only need to prove

$$p\lim_{t\to\infty} \log \frac{W_t}{W_0} = \infty.$$

Note

$$W_t = W_0 \exp\left[\left(\mu - \frac{\sigma^{2}}{2}\right) t - \sigma Z_t - \int_{s=0}^{t} \frac{C_s}{W_s} ds\right],$$

hence, we have

$$\log \frac{W_t}{W_0} = \left(\mu - \frac{\sigma^{2}}{2} - \frac{1}{t}\int_{s=0}^{t} \frac{C_s}{W_s} ds\right) t - \sigma Z_t,$$

$$\Longrightarrow \frac{1}{t}\log \frac{W_t}{W_0} = \mu - \frac{\sigma^{2}}{2} - \frac{1}{t}\int_{s=0}^{t} \frac{C_s}{W_s} ds - \frac{\sigma}{t} Z_t.$$

[7] Original proof of ergodic theorem was in Neumann (1932). It is based on the spectral decomposition of unitary operators. Later a number of other proofs were published. The simplest is due to F. Riesz, see Halmos (1956).

According to the Chebyshev's inequality, we have for $\forall\epsilon > 0$,

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0} - \mathrm{E}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)\right| \geq \epsilon\right) \leq \frac{\mathrm{Var}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)}{\epsilon^2}. \tag{19}$$

Moreover, note

$$Z_t \sim N(0,t), \quad \text{and} \quad -\frac{\sigma}{t}Z_t \sim N\left(0,\frac{\sigma^2}{t}\right),$$

and

$$\frac{1}{t}\int_{s=0}^{t}\frac{C_s}{W_s}ds \xrightarrow{L^2} \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right],$$

hence, based on the results of (17) and (18), we have as $t \to \infty$,

$$\mathrm{E}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right) = \mu - \frac{\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right], \text{ and } \mathrm{Var}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right) = \frac{\sigma^2}{t}.$$

Then according (20), we have as $t \to \infty$,

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0} - \left(\mu - \frac{\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right]\right)\right| \geq \epsilon\right) \leq 0.$$

Since probability cannot be negative, hence, we have as $t \to \infty$, for $\forall\epsilon > 0$

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0} - \left(\mu - \frac{\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right]\right)\right| \geq \epsilon\right) = 0.$$

Therefore, according to the definition of convergence in probability, we have

$$p\lim_{t\to\infty}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right) = \mu - \frac{\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right].$$

By the condition (12)

$$\mu - \frac{\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right] > 0,$$

hence, we have

$$p\lim_{t\to\infty}\left(\log\frac{W_t}{W_0}\right) = \infty \Longrightarrow \lim_{t\to\infty}\Pr(W_t < W_0) = 0.$$

Given

$$\mu - \frac{\sigma^2}{2} - \exp\left[\tau + \frac{\sigma^2}{4\kappa}\right] < 0,$$

we have

$$p\lim_{t\to\infty}\left(\log\frac{W_t}{W_0}\right) = -\infty \Longrightarrow \lim_{t\to\infty}\Pr(W_t < W_0) = 1,$$

which completes the proof. ■

A.4 Proof of Theorem 4

By the definition of preservation of capital, to prove the spending rule preserves capital, we only need to prove

$$p\lim_{t\to\infty} \log \frac{W_t}{W_0} = \infty.$$

Note

$$dW_t = W_t\left(\mu_t dt + \sigma_t dZ\right) - C_t dt = \left(W_t\mu_t - C_t\right)dt + W_t\sigma_t dZ,$$

implies that

$$W_t = W_0 \exp\left[\int_{s=0}^{t}\left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right)ds - \int_{s=0}^{t}\sigma_s dZ_s\right].$$

hence, we have

$$\begin{aligned}\log\frac{W_t}{W_0} &= \int_{s=0}^{t}\left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right)ds - \int_{s=0}^{t}\sigma_s dZ_s\\ \Longrightarrow \frac{1}{t}\log\frac{W_t}{W_0} &= \frac{1}{t}\int_{s=0}^{t}\left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right)ds - \frac{1}{t}\int_{s=0}^{t}\sigma_s dZ_s.\end{aligned}$$

According to the Chebyshev's inequality, we have for $\forall\epsilon > 0$,

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0} - \mathrm{E}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)\right| \geq \epsilon\right) \leq \frac{\mathrm{Var}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right)}{\epsilon^2}. \tag{20}$$

and based on the assumption about the expectation that, as $t \to \infty$,

$$\begin{aligned}\mathrm{E}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right) &= \mathrm{E}\left[\frac{1}{t}\int_{s=0}^{t}\left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right)ds - \frac{1}{t}\int_{s=0}^{t}\sigma_s dZ_s\right]\\ &= \mathrm{E}\left[\frac{1}{t}\int_{s=0}^{t}\left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right)ds\right] \to B > 0,\end{aligned}$$

since as $t \to \infty$,

$$\mathrm{E}\left[\int_{s=0}^{t}\sigma_s dZ_s\right] = 0.$$

Moreover, we have the assumption about the variance that, as $t \to \infty$,

$$\mathrm{Var}\left[\frac{1}{t}\log\frac{W_t}{W_0}\right] = \frac{1}{t^2}\mathrm{Var}\left[\int_{s=0}^{t}\left(\mu_s - \frac{1}{2}\sigma_s^2 - c_s\right)ds - \int_{s=0}^{t}\sigma_s dZ_s\right] \to 0.$$

Therefore, we have as $t \to \infty$,

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0} - B\right| \geq \epsilon\right) \leq 0.$$

Since probability cannot be negative, hence, we have as $t \to \infty$, for $\forall \epsilon > 0$

$$\Pr\left(\left|\frac{1}{t}\log\frac{W_t}{W_0} - B\right| \geq \epsilon\right) = 0.$$

Therefore, according to the definition of convergence in probability, we have

$$p\lim_{t\to\infty}\left(\frac{1}{t}\log\frac{W_t}{W_0}\right) = B.$$

By the condition (12), if $B > 0$,hence, we have

$$p\lim_{t\to\infty}\left(\log\frac{W_t}{W_0}\right) = \infty \Longrightarrow \lim_{t\to\infty}\Pr\left(W_t < W_0\right) = 0.$$

Given $B < 0$,we have

$$p\lim_{t\to\infty}\left(\log\frac{W_t}{W_0}\right) = -\infty \Longrightarrow \lim_{t\to\infty}\Pr\left(W_t < W_0\right) = 1,$$

which completes the proof. ■

A.5 Derivations of HJB Equation, Dual Function, and Smooth Pasting Conditions

Let

$$M_t \equiv \int_{s=0-}^{t} e^{-\rho t}\left(\frac{C_t^{1-R}}{1-R} - \frac{a}{1-R}\left|d\left(C_t^{1-R}\right)\right|\right)dt + e^{-\rho t}V\left(W_t, C_t\right),$$

thus, we have

$$dM_t = e^{-\rho t}\left[\left(\frac{C_t^{1-R}}{1-R} - \frac{a}{1-R}\left|d\left(C_t^{1-R}\right)\right|\right)dt + V_C dC dt - \rho V dt + V_W dW + \frac{1}{2}V_{WW}\left(dW_t\right)^2\right]$$

According to the Martingale Principle of Optimal Control, M_t is martingale gives

$$\frac{\mathrm{E}\left[dM_t\right]}{e^{-\rho t}dt} = \frac{C_t^{1-R}}{1-R} - \frac{a}{1-R}\left|d\left(C_t^{1-R}\right)\right| - \rho V + V_C dC + V_W\left(rw_t + \theta_t\left(\mu - r\right) - C_t\right) + \frac{\sigma^2\theta_t^2}{2}V_{WW} = 0$$

$$\frac{C_t^{1-R}}{1-R} - \frac{a}{1-R}\left|(1-R)\,C_t^{-R}dC\right| - \rho V + V_C dC + V_W\left(rw_t + \theta_t\left(\mu - r\right) - C_t\right) + \frac{\sigma^2\theta_t^2}{2}V_{WW} = 0$$

The optimal strategy of consumption in the interior region is $dC = 0$ and optimal portfolio in stock is given as

$$\theta = -\frac{\mu - r}{\sigma^2}\frac{V_W}{V_{WW}} = -\frac{\kappa}{\sigma}\frac{V_W}{V_{WW}}$$

where $\kappa = \frac{\mu - r}{\sigma}$, thus we have

$$\begin{aligned}
\frac{C_t^{1-R}}{1-R} - \rho V + V_W\left(rW_t + \theta_t\left(\mu - r\right) - C_t\right) + \frac{\sigma^2\theta_t^2}{2}V_{WW} &= 0\\
\frac{C_t^{1-R}}{1-R} - \rho V + V_W\left(rW_t - \frac{\mu - r}{\sigma^2}\frac{V_W}{V_{WW}}\left(\mu - r\right) - C_t\right) + \left(\frac{\mu - r}{\sigma^2}\right)^2\frac{V_W^2}{V_{WW}}\frac{\sigma^2\theta_t^2}{2}V_{WW} &= 0\\
\frac{C_t^{1-R}}{1-R} - \rho V + V_W\left(rW_t - \kappa^2\frac{V_W}{V_{WW}} - C_t\right) + \frac{\kappa^2}{2}\frac{V_W^2}{V_{WW}} &= 0\\
\frac{C_t^{1-R}}{1-R} - \rho V + V_W\left(rW_t - C_t\right) - \frac{\kappa^2}{2}\frac{V_W^2}{V_{WW}} &= 0
\end{aligned}$$

we can simplify it by let $\omega \equiv W/C$, and conjecture $V\left(C, W\right) = C^{1-R}v\left(\omega\right)$. As a result, we have

$$\begin{aligned}
V_W &= C^{1-R}\frac{\partial v\left(\omega\right)}{\partial\omega}\frac{\partial\omega}{\partial W} = C^{-R}v_\omega,\\
V_{WW} &= \frac{1}{C}C^{1-R}v_\omega\frac{\partial v_\omega\left(\omega\right)}{\partial\omega}\frac{\partial\omega}{\partial W} = C^{-1-R}v_{\omega\omega},\\
V_C &= \left(1-R\right)C^{-R}v + C^{1-R}\frac{\partial v}{\partial\omega}\frac{\partial\omega}{\partial C} = \left(1-R\right)C^{-R}v - C^{-R}\frac{W}{C}v_\omega
\end{aligned}$$

Substitute into the above equation, we have

$$\begin{aligned}
\frac{C_t^{1-R}}{1-R} - \rho V + V_W\left(rW_t - C_t\right) - \frac{\kappa^2}{2}\frac{V_W^2}{V_{WW}} &= 0\\
\frac{C_t^{1-R}}{1-R} - \rho C^{1-R}v\left(\omega\right) + C^{-R}v_\omega\left(rW_t - C_t\right) - \frac{\kappa^2}{2}\frac{\left(C^{-R}v_\omega\right)^2}{C^{-1-R}v_{\omega\omega}} &= 0\\
\frac{C_t^{1-R}}{1-R} - \rho C^{1-R}v\left(\omega\right) + C^{1-R}v_\omega\left(r\frac{W_t}{C} - 1\right) - \frac{\kappa^2}{2}\frac{C^{1-R}v_\omega^2}{v_{\omega\omega}} &= 0\\
\frac{1}{1-R} - \rho v\left(\omega\right) + v_\omega\left(r\omega - 1\right) - \frac{\kappa^2}{2}\frac{v_\omega^2}{v_{\omega\omega}} &= 0
\end{aligned}$$

A.5.1 Dual Approach

Define the dual variable as $z = v_\omega$ and, thus, we have

$$\omega = -J_z, \quad v = J - zJ_z, \quad v_{\omega\omega} = \frac{dv_\omega}{d\omega} = \frac{dz}{d(-J_z)} = -\frac{1}{J_{zz}}. \tag{21}$$

With above transformations, we can rewrite the ODE with dual variable as

$$\begin{aligned}
\frac{1}{1-R} - \rho(J - zJ_z) + z(r\omega - 1) - \frac{\kappa^2}{2}\frac{z^2}{-\frac{1}{J_{zz}}} = 0 &\iff \\
\frac{1}{1-R} - \rho(J - zJ_z) - z(rJ_z + 1) + \frac{\kappa^2}{2}z^2 J_{zz} = 0 &\iff \\
\frac{1}{1-R} - \rho J + \rho z J_z - rzJ_z - z + \frac{\kappa^2}{2}z^2 J_{zz} = 0 &\iff \\
\frac{\kappa^2}{2}z^2 J_{zz} + (\rho - r) zJ_z - \rho J = z - \frac{1}{1-R} &\iff \\
J_{zz} + \frac{2(\rho - r)}{\kappa^2 z}J_z - \frac{2\rho}{\kappa^2 z^2}J = \frac{2}{\kappa^2 z^2}\left(z - \frac{1}{1-R}\right)&
\end{aligned}$$

which is a Euler-Cauchy ODE.

A.5.2 Solving for Linear Second Order ODE in Dual

All possible solutions to a linear second-order ODE can be obtained from two linearly independent solutions to the homogeneous problem and any particular solution. Hence, the procedure for solving linear second-order ODE has two steps. First, find the general solution of a homogeneous problem and, second, find a particular solution of a nonhomogeneous problem. We first solve for the following homogeneous problem:

$$J_{zz} + \frac{2(\rho - r)}{\kappa^2 z}J_z - \frac{2\rho}{\kappa^2 z^2}J = 0.$$

Note it is the Euler-Cauchy ODE: let

$$p = \frac{2(\rho - r)}{\kappa^2}, \text{ and } q = \frac{2\rho}{\kappa^2},$$

thus the characteristic polynomial equation of the ODE is

$$\varpi^2 + (p-1)\varpi - q = 0.$$

Thus, the two roots are

$$\beta_1 = \frac{-(p-1) + \sqrt{(p-1)^2 + 4q}}{2}, \text{ and } \beta_2 = \frac{-(p-1) - \sqrt{(p-1)^2 + 4q}}{2}.$$

Since it is a maximizing problem, we have V is increasing and concave and, thus, $V_W = C^{-R} v_\omega > 0 \Longrightarrow z = v_\omega > 0$. Therefore, $J_1(z) = z^{\beta_1}$ and $J_2(z) = z^{\beta_2}$ are linearly independent solutions to the original ODE and the general solution to the ODE is:

$$J_H(z) = C_1 J_1(z) + C_2 J_2(z) = C_1 z^{\beta_1} + C_2 z^{\beta_2},$$

and we will determine the constant C_1 and C_2 in the following subsections of boundary conditions.

We now find a particular solution of the nonhomogeneous problem:

$$J_{zz} + \frac{2(\rho - r)}{\kappa^2 z} J_z - \frac{2\rho}{\kappa^2 z^2} J = \frac{2}{\kappa^2 z^2}\left(z - \frac{1}{1-R}\right)$$

The particular solution is any solution of the nonhomogeneous problem and is denoted $J_p(z)$. Variation of parameters is a method for computing a particular solution to the nonhomogeneous linear second-order ODE, we will employ the approach.

Particular Solution First let

$$f(z) = \frac{2}{\kappa^2 z^2}\left(z - \frac{1}{1-R}\right),$$

and the Wronskian $g(z)$, is defined by

$$g(z) = J_1(z) J_2'(z) - J_2(z) J_1'(z) = (\beta_2 - \beta_1) z^{\beta_2 + \beta_1 - 1},$$

then the particular solution is given by

$$\begin{aligned}
J_p(z) &= -J_1(z) \int \frac{J_2(z) f(z)}{g(z)} dz + J_2(z) \int \frac{J_1(z) f(z)}{g(z)} dz \\
&= -z^{\beta_1} \int \frac{z^{\beta_2} \frac{2}{\kappa^2 z^2}\left(z - \frac{1}{1-R}\right)}{(\beta_2 - \beta_1) z^{\beta_2 + \beta_1 - 1}} dz + z^{\beta_2} \int \frac{z^{\beta_1} \frac{2}{\kappa^2 z^2}\left(z - \frac{1}{1-R}\right)}{(\beta_2 - \beta_1) z^{\beta_2 + \beta_1 - 1}} dz \\
&= -z^{\beta_1} \int \frac{2\left(z - \frac{1}{1-R}\right)}{\kappa^2 z^2 (\beta_2 - \beta_1) z^{\beta_1 - 1}} dz + z^{\beta_2} \int \frac{2\left(z - \frac{1}{1-R}\right)}{\kappa^2 z^2 (\beta_2 - \beta_1) z^{\beta_2 - 1}} dz
\end{aligned}$$

$$= \frac{2}{\kappa^2(\beta_2-\beta_1)}\left[-z^{\beta_1}\int \frac{z-\frac{1}{1-R}}{z^{\beta_1+1}}dz + z^{\beta_2}\int \frac{z-\frac{1}{1-R}}{z^{\beta_2+1}}dz\right]$$
$$= \frac{2}{\kappa^2(\beta_2-\beta_1)}\left[\begin{array}{l} -z^{\beta_1}\left(\int \frac{z}{z^{\beta_1+1}}dz - \frac{1}{1-R}\int \frac{1}{z^{\beta_1+1}}dz\right) \\ +z^{\beta_2}\left(\int \frac{z}{z^{\beta_2+1}}dz - \frac{1}{1-R}\int \frac{1}{z^{\beta_2+1}}dz\right)\end{array}\right]$$
$$= \frac{2}{\kappa^2(\beta_2-\beta_1)}\left[\begin{array}{l} -z^{\beta_1}\left(\int z^{-\beta_1}dz - \frac{1}{1-R}\int z^{-(\beta_1+1)}dz\right) \\ +z^{\beta_2}\left(\int z^{-\beta_2}dz - \frac{1}{1-R}\int z^{-(\beta_2+1)}dz\right)\end{array}\right]$$

$$= \frac{2}{\kappa^2(\beta_2-\beta_1)}\left[\begin{array}{l} -z^{\beta_1}\left(\frac{1}{1-\beta_1}z^{1-\beta_1} - \frac{1}{1-R}\frac{1}{1-(\beta_1+1)}z^{1-(\beta_1+1)}\right) \\ +z^{\beta_2}\left(\frac{1}{1-\beta_2}z^{1-\beta_2} - \frac{1}{1-R}\frac{1}{1-(\beta_2+1)}z^{1-(\beta_2+1)}\right)\end{array}\right]$$
$$= \frac{2}{\kappa^2(\beta_2-\beta_1)}\left[\frac{-z}{1-\beta_1} - \frac{1}{1-R}\frac{1}{\beta_1} + \frac{z}{1-\beta_2} + \frac{1}{1-R}\frac{1}{\beta_2}\right]$$
$$= \frac{2}{\kappa^2(\beta_2-\beta_1)}\left[z\left[\frac{1}{1-\beta_2} - \frac{1}{1-\beta_1}\right] + \frac{1}{1-R}\left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right)\right]$$
$$= \frac{2}{\kappa^2(\beta_2-\beta_1)}\left[z\frac{(1-\beta_1)-(1-\beta_2)}{(1-\beta_2)(1-\beta_1)} + \frac{1}{1-R}\frac{\beta_1-\beta_2}{\beta_2\beta_1}\right]$$
$$= \frac{2}{\kappa^2(\beta_2-\beta_1)}\left[z\frac{\beta_2-\beta_1}{(1-\beta_2)(1-\beta_1)} + \frac{1}{1-R}\frac{\beta_1-\beta_2}{\beta_2\beta_1}\right]$$
$$= \frac{2}{\kappa^2}\left(\frac{z}{(1-\beta_2)(1-\beta_1)} - \frac{1}{1-R}\frac{1}{\beta_2\beta_1}\right).$$

Note

$$\beta_1 + (p-1)\beta_1 - q = 0, \quad \text{and} \quad \beta_2 + (p-1)\beta_2 - q = 0,$$

thus

$$\beta_1 + (p-1)\beta_1 = \beta_2 + (p-1)\beta_2 \Rightarrow \beta_1 - \beta_2 = (p-1)(\beta_2-\beta_1)$$
$$\Rightarrow \frac{\beta_1-\beta_2}{(\beta_2-\beta_1)} = (p-1) \Rightarrow -\frac{\beta_2-\beta_1}{(\beta_2-\beta_1)} = (p-1)$$
$$\Rightarrow p-1 = -(\beta_2+\beta_1),$$

and

$$p = 1 - (\beta_2+\beta_1),$$
$$q = \beta_2 + (p-1)\beta_2 = \beta_2 - (\beta_1+\beta_2)\beta_2 = -\beta_1\beta_2,$$

therefore,

$$\frac{1}{(1-\beta_2)(1-\beta_1)} = \frac{1}{1-\beta_1-\beta_2+\beta_2\beta_1} = \frac{1}{1-\beta_1-\beta_2+\beta_2\beta_1} = \frac{1}{p-q}$$
$$= \frac{\kappa^2}{2(\rho-r)-2\rho} = -\frac{\kappa^2}{2r}$$

hence

$$\begin{aligned}
J_p(z) &= \frac{2}{\kappa^2}\left(\frac{z}{(1-\beta_2)(1-\beta_1)} - \frac{1}{1-R}\frac{1}{\beta_2\beta_1}\right) = \frac{2}{\kappa^2}\left(-\frac{z\kappa^2}{2r} + \frac{1}{1-R}\frac{1}{q}\right) \\
&= \frac{2}{\kappa^2}\left(\frac{1}{1-R}\frac{\kappa^2}{2\rho} - \frac{z\kappa^2}{2r}\right) = \frac{1}{\rho(1-R)} - \frac{z}{r}
\end{aligned}$$

Therefore, the general solution of the full nonhomogeneous problem is

$$J(z) = C_1 J_1(z) + C_2 J_2(z) + J_p(z) = C_1 z^{\beta_1} + C_2 z^{\beta_2} + \frac{1}{\rho(1-R)} - \frac{z}{r}.$$

and thus

$$\begin{aligned}
J_z &= C_1\beta_1 z^{\beta_1-1} + C_2\beta_2 z^{\beta_2-1} - \frac{1}{r} \\
J_{zz} &= C_1\beta_1(\beta_1-1) z^{\beta_1-2} + C_2\beta_2(\beta_2-1) z^{\beta_2-2}
\end{aligned} \tag{22}$$

A.5.3 Boundaries for Ratio of Wealth to Spending

Value Functions and Smooth Pasting Conditions Assume the lower boundary and the upper boundary for ω are ω_* and ω^* respectively. For $\omega < \omega_*$, the endowment decreases C immediately so that $\omega = \omega_*$, i.e., move C to W/ω_*. Thus, the value function for the lower boundary $\underline{V}(W_t, C_t)$ is given as

$$\begin{aligned}
\underline{V}(C_t, W_t) &= V\left(\frac{W}{\omega_*}, W_t\right) - \frac{a}{1-R}\left(C^{1-R} - \left(\frac{W}{\omega_*}\right)^{1-R}\right) \\
&= \left(\frac{W}{\omega_*}\right)^{1-R} v(\omega_*) - \frac{a}{1-R}\left(C^{1-R} - \left(\frac{C\omega}{\omega_*}\right)^{1-R}\right) \\
&= \left(\frac{C\omega}{\omega_*}\right)^{1-R} v(\omega_*) - \frac{a}{1-R}\left(C^{1-R} - \left(\frac{C\omega}{\omega_*}\right)^{1-R}\right) \\
&= C^{1-R}\left(\left(\frac{\omega}{\omega_*}\right)^{1-R} v(\omega_*) - \frac{a}{1-R}\left(1 - \left(\frac{\omega}{\omega_*}\right)^{1-R}\right)\right),
\end{aligned}$$

therefore, for $\omega < \omega_*$,

$$v(\omega) = \left(\frac{\omega}{\omega_*}\right)^{1-R}\left(v(\omega_*) + \frac{a}{1-R}\right) - \frac{a}{1-R}.$$

For $\omega > \omega_*$, the endowment increases C immediately so that $\omega = \omega^*$, i.e., move C to W/ω^*. Thus, the value function for the upper boundary $\bar{V}(W_t, C_t)$ is given as

$$\begin{aligned}\bar{V}(C_t, W_t) &= V\left(\frac{W}{\omega^*}, W_t\right) - \frac{a}{1-R}\left(\left(\frac{W}{\omega^*}\right)^{1-R} - C^{1-R}\right) \\ &= C^{1-R}\left(\left(\frac{\omega}{\omega^*}\right)^{1-R}\left(v(\omega^*) - \frac{a}{1-R}\right) + \frac{a}{1-R}\right),\end{aligned}$$

therefore, for $\omega > \omega_*$,

$$v(\omega) = \left(\frac{\omega}{\omega^*}\right)^{1-R}\left(v(\omega^*) - \frac{a}{1-R}\right) + \frac{a}{1-R}.$$

Moreover, outside the lower boundary we have

$$V_C = -aC^{-R},$$

while inside the boundary we have

$$V_C = (1-R)\,C^{-R}v - C^{-R}\frac{W}{C}v_\omega.$$

Note V_C matches at the boundary, yielding

$$\begin{aligned}-aC^{-R} &= (1-R)\,C^{-R}v - C^{-R}\frac{W}{C}v_\omega \\ -a &= (1-R)\,v(\omega_*) - \omega_* v_\omega(\omega_*).\end{aligned}$$

10Outside the upper boundary we have $V_C = aC^{-R}$, matching yields

$$a = (1-R)\,v(\omega^*) - \omega^* v_\omega(\omega^*).$$

Moreover, matching V_{CC} give rise to

$$
\begin{aligned}
-aC^{-R} &= (1-R)\,C^{-R}v - C^{-R}\frac{W}{C}v_{\omega} \\
RaC^{-R-1} &= (1-R)\left(-RC^{-R-1}v + C^{-R}v_{\omega}\frac{\partial\omega}{\partial C}\right) - W\left((-R-1)\,C^{-R-2}v_{\omega} + C^{-R-1}v_{\omega\omega}\frac{\partial\omega}{\partial C}\right) \\
&= (1-R)\left(-RC^{-R-1}v - \omega C^{-R-1}v_{\omega}\right) - W\left((-R-1)\,C^{-R-2}v_{\omega} - \omega C^{-R-2}v_{\omega\omega}\right)
\end{aligned}
$$

that is

$$
\begin{aligned}
Ra &= (1-R)\,(-Rv-\omega v_{\omega}) - W\left((-R-1)\,C^{-1}v_{\omega} - \omega C^{-1}v_{\omega\omega}\right) \\
&= (1-R)\,(-Rv-\omega v_{\omega}) - \left((-R-1)\,\omega v_{\omega} - \omega^2 v_{\omega\omega}\right) \\
&= -R\,(1-R)\,v + 2R\omega v_{\omega} + \omega^2 v_{\omega\omega}
\end{aligned}
$$

that is

$$
\begin{aligned}
R\left((1-R)\,v\,(\omega^*) - \omega^* v_{\omega}\,(\omega^*)\right) &= -R\,(1-R)\,v + 2R\omega v_{\omega} + \omega^2 v_{\omega\omega} \\
-R\,(1-R)\,v + R\omega^* v_{\omega}\,(\omega^*) &= -R\,(1-R)\,v + 2R\omega v_{\omega} + \omega^2 v_{\omega\omega} \\
0 &= Rv_{\omega} + \omega v_{\omega\omega} \\
-\omega v_{\omega\omega} &= Rv_{\omega}
\end{aligned}
$$

and this condition is identical for both boundaries.

Solve for Optimal Boundaries For the upper boundary, we need to determine the $z^* = v_{\omega}\,(\omega^*)$ by

$$
\begin{aligned}
a &= (1-R)\,v\,(\omega^*) - \omega^* v_{\omega}\,(\omega^*) = (1-R)\,(J\,(z^*) - z^* J_z\,(z^*)) + z^* J_z\,(z^*)\,, \\
&= (1-R)\left(C_1\,(z^*)^{\beta_1} + C_2\,(z^*)^{\beta_2} + \frac{1}{\rho\,(1-R)} - \frac{z^*}{r}\right) + Rz^*\left(C_1\beta_1\,(z^*)^{\beta_1-1} + C_2\beta_2\,(z^*)^{\beta_2-1} - \frac{1}{r}\right),
\end{aligned}
$$

and

$$
-\omega v_{\omega\omega} = Rv_{\omega} \Longleftrightarrow -J_z\,(z^*) = Rz^* J_{zz}\,(z^*) \Longleftrightarrow
$$
$$
-\left(C_1\beta_1\,(z^*)^{\beta_1-1} + C_2\beta_2\,(z^*)^{\beta_2-1} - \frac{1}{r}\right) = Rz^*\left(C_1\beta_1\,(\beta_1-1)\,(z^*)^{\beta_1-2} + C_2\beta_2\,(\beta_2-1)\,(z^*)^{\beta_2-2}\right).
$$

For the lower boundary, we need to determine the $z_* = v_\omega(\omega_*)$ by

$$\begin{cases} -a = (1-R)\, J(z_*) + R z_* J_z(z_*), \\ -\frac{J_z(z_*)}{J_{zz}(z_*)} = R z_*. \end{cases}$$

i.e,

$$\begin{cases} -a = (1-R)\left(C_1 (z_*)^{\beta_1} + C_2 (z_*)^{\beta_2} + \frac{1}{\rho(1-R)} - \frac{z_*}{r}\right) + R z_* \left(C_1\beta_1 (z_*)^{\beta_1 - 1} + C_2\beta_2 (z_*)^{\beta_2-1} - \frac{1}{r}\right), \\ -\left(C_1\beta_1 (z_*)^{\beta_1-1} + C_2\beta_2 (z_*)^{\beta_2-1} - \frac{1}{r}\right) = R z_* \left(C_1\beta_1(\beta_1-1)(z_*)^{\beta_1-2} + C_2\beta_2(\beta_2-1)(z_*)^{\beta_2-2}\right), \end{cases}$$

then we obtain the equations

$$\begin{cases} a = (1-R)\left(C_1 (z^*)^{\beta_1} + C_2 (z^*)^{\beta_2} + \frac{1}{\rho(1-R)} - \frac{z^*}{r}\right) + R z^* \left(C_1\beta_1 (z^*)^{\beta_1 - 1} + C_2\beta_2 (z^*)^{\beta_2-1} - \frac{1}{r}\right) \\ -\left(C_1\beta_1 (z^*)^{\beta_1-1} + C_2\beta_2 (z^*)^{\beta_2-1} - \frac{1}{r}\right) = R z^* \left(C_1\beta_1(\beta_1-1)(z^*)^{\beta_1-2} + C_2\beta_2(\beta_2-1)(z^*)^{\beta_2-2}\right) \\ -a = (1-R)\, C_1 (z_*)^{\beta_1} + (1-R)\, C_2 (z_*)^{\beta_2} + \frac{1-R}{\rho(1-R)} - \frac{(1-R)z_*}{r} \\ \quad + C_1 R z_* \beta_1 (z_*)^{\beta_1-1} + C_2 R z_* \beta_2 (z_*)^{\beta_2-1} - \frac{R z_*}{r} \\ \frac{(1-R)z_*}{r} - \frac{1-R}{\rho(1-R)} - a + \frac{R z_*}{r} = (1-R+R\beta_1)(z_*)^{\beta_1} C_1 + (1-R+R\beta_2)(z_*)^{\beta_2} C_2 \end{cases}$$

We can obtain analytical expressions for C_1 and C_2 as

$$\begin{cases} C_1 = \frac{\left(\frac{(1-R)z_*}{r} - \frac{1-R}{\rho(1-R)} - a + \frac{R z_*}{r}\right)\beta_2 - \frac{z_*}{r}}{(\beta_2-\beta_1)(1-R+R\beta_1)(z_*)^{\beta_1}}, \\ C_2 = \frac{\frac{z_*}{r} - (R\beta_1 - R + 1)\beta_1 (z_*)^{\beta_1} C_1}{(R\beta_2 - R + 1)\beta_2 (z_*)^{\beta_2}} = \frac{\frac{z_*}{r} - (R\beta_1 - R + 1)\beta_1 (z_*)^{\beta_1}}{(R\beta_2 - R + 1)\beta_2 (z_*)^{\beta_2}} \frac{\left(\frac{(1-R)z_*}{r} - \frac{1-R}{\rho(1-R)} - a + \frac{R z_*}{r}\right)\beta_2 - \frac{z_*}{r}}{(\beta_2-\beta_1)(1-R+R\beta_1)(z_*)^{\beta_1}}. \end{cases}$$

Substitute C_1 and C_2 into

$$\begin{cases} a = (1-R)\left(C_1 (z^*)^{\beta_1} + C_2 (z^*)^{\beta_2} + \frac{1}{\rho(1-R)} - \frac{z^*}{r}\right) + R z^* \left(C_1\beta_1 (z^*)^{\beta_1 - 1} + C_2\beta_2 (z^*)^{\beta_2-1} - \frac{1}{r}\right), \\ -\left(C_1\beta_1 (z^*)^{\beta_1-1} + C_2\beta_2 (z^*)^{\beta_2-1} - \frac{1}{r}\right) = R z^* \left(C_1\beta_1(\beta_1-1)(z^*)^{\beta_1-2} + C_2\beta_2(\beta_2-1)(z^*)^{\beta_2-2}\right), \end{cases}$$

then we can solve for z^* and z_* numerically.

A.6 Algorithm of Simulation

Step 1. With the given parameters, we calculate the value of boundaries of the ratio of wealth to spending, which are given as ω_* and ω^*.
Step 2. With assumed initial wealth W_0, let $C^* = W_0/\omega_*$, $C_* = W_0/\omega^*$, and assume the initial spending $C_0 = (C^* + C_*)/2$.
Step 3. Given wealth and spending at time t, i.e., W_t, C_t, solve the following equation for

the dual variable z_t,

$$W_t/C_t = -J_z = -\left(C_1\beta_1 z^{\beta_1-1} + C_2\beta_2 z^{\beta_2-1} - \frac{1}{r}\right).$$

Step 4. With the solved dual variable z_t, calculate the portfolio in stock by the following formula,

$$\begin{aligned}\theta_t &= -\frac{\mu - r}{\sigma^2}\frac{V_W}{V_{WW}} = -\frac{\kappa}{\sigma}\frac{Cv_\omega}{v_{\omega\omega}} = \frac{\kappa C}{\sigma} z_t J_{zz} \\ &= \frac{\kappa C}{\sigma}\left(C_1\beta_1\left(\beta_1 - 1\right) z^{\beta_1-1} + C_2\beta_2\left(\beta_2 - 1\right) z^{\beta_2-1}\right).\end{aligned}$$

Step 5. With the wealth invested in stock, we can obtain the change in wealth according to

$$dW_t = rw_t dt + \theta_t\left(\left(\mu - r\right) dt + \sigma dZ_t\right) - C_t dt,$$

and obtain $W_{t+1} = W_t + dW_t$.

Step 6. If wealth at time $t+1$, $W_{t+1} \le 0$, terminate program. If $W_{t+1} > 0$, calculate $C^*\left(t+1\right) = W_{t+1}/\omega_*$, $C_*\left(t+1\right) = W_{t+1}/\omega^*$.

Step 7. If $C_*\left(t+1\right) \le C_t \le C^*\left(t+1\right)$, then spending at time $t+1$, $C_{t+1} = C_t$. If $C_t < C_*\left(t+1\right)$, then $C_{t+1} = C_*\left(t+1\right)$. If $C^*\left(t+1\right) < C_t$, then $C_{t+1} = C^*\left(t+1\right)$.

Step 8. With wealth and spending at time $t+1$, turn to step 2.

Step 9. When terminal date is reached, terminate the program.