Abstract

This paper studies the effects of the joint distribution of the stock of financial expertise and of financial wealth on the returns to risky assets which are traded in segmented markets. We emphasize the role of heterogeneity in the effective Sharpe Ratios investors face. We allow for endogenous entry and exit. Agents decide whether or not to invest in becoming an expert in the market for the risky asset, and entrants subsequently vary in their level of expertise. We show how parameters which describe the complexity of the asset class naturally determine the joint distribution of expertise and wealth, and thus drive excess returns and average market volatility. Finally, we study the dynamic reaction of the endogenously segmented market to return and volatility shocks, and demonstrate the model’s ability to generate persistently elevated excess returns following negative shocks.

Key Words: segmented markets, slow moving capital, risky arbitrage, hedge funds, industry equilibrium, financial expertise, intellectual capital, intermediary asset pricing.
1 Introduction

Complex investment strategies, such as those employed by hedge funds and other sophisticated investors, require investments in specific expertise. These investments can include the development of highly skilled human capital, such as trained financial engineers or computer scientists, the acquisition or construction of financial data, models, and information technology, as well as investments in relationships with brokers, dealers, or in other reputation capital such as a stable investor base. Indeed, such specialized capital may serve as the barrier to entry which potentially enables sophisticated investors to earn higher returns. This paper aims to understand the effects of such accumulated, specific capital on equilibrium asset prices. In particular, we are interested in explaining how prices for complex assets are determined by the joint distribution of expertise and financial wealth, and in turn by the deep parameters which determine these distributions in equilibrium.

In the cross section, we show how parameters which proxy for asset complexity determine the joint distribution of financial wealth and expertise, and hence equilibrium prices and excess returns across asset classes. In our model, we allow for endogenous entry into, and exit from, the segmented market for the risky asset. Returns to the risky asset are lower when more wealth is allocated to the segmented market, and both lower and less volatile when more wealth is concentrated in the hands of investors with high levels of expertise. The first effect arises when the same amount of risk is born by a smaller wealth base, despite free entry. The second effect, measured by the correlation between financial wealth and expertise, is a measure of the market’s efficiency. The same total endowment of expertise is more effective the more it is concentrated in high wealth investors.

In the time series, we demonstrate how prices for complex assets dynamically react to shocks to returns and to fundamental cash flow volatility. In particular, we illustrate how the joint distribution for expertise and financial wealth evolves after such shocks in a way that leads to persistently elevated excess returns. In doing so, we address why entry by new investors does not work to quickly eliminate such apparently profitable risky investment opportunities.
We develop a quantitative industry equilibrium model of investors who acquire expertise and invest in an endogenously segmented market in which the risky asset earns positive excess returns in equilibrium. There is a continuum of agents that choose to be either non-experts, which can invest only in the risk free asset, or experts, which can invest in both the risk free, and risky assets. In the stationary equilibrium, there is no aggregate risk, however idiosyncratic risk is priced, as discussed in Merton [1987], and as featured in common models of segmented markets or limits to arbitrage. Expertise is used by experts to shrink the volatility of the returns to the risky asset they receive, implying that more sophisticated experts face a higher Sharpe Ratio. Thus, expertise may be interpreted as a better ability to hedge risks either by developing a superior model or gathering superior information. Broadly interpreted, these risks may come either from the asset side, or from the liability or fund flow side through investments in a stable investor base. We abstract from the microfoundations of risks from the liability side of funds’ balance sheets, and model risk on the asset side.

Consider the market for a complex asset class. Investors who enter this market must make an initial investment in expertise. This expertise is comprised by the investor’s personnel, data, hedging and risk management technologies, back office operations and trade clearing processes, relationships with dealers, and relationships with clients. In our model, all investors in the market earn a common return in equilibrium, determined by market clearing. However, because each investor’s initial investment will result in a slightly different overall investment technology, their returns will be subject to idiosyncratic shocks. Thus, one can think of complex asset classes as requiring a joint investment in the asset, and in a hedging technology which requires financial expertise. Importantly, even given the same initial investment, individual investors’ expertise evolves stochastically, leading to cross sectional variation even within the class of experts.

To be precise, investors must pay a fixed entry cost to become experts, and incumbent entrants must pay a maintenance cost. These costs represent experts’

1See Sharpe [1966].
2Complex asset classes also have idiosyncratic risk because their constituents tend to be significantly heterogeneous.
investment and maintenance in their expertise. In addition to driving entry and exit
they induce a form of returns to scale. Once an agent becomes an expert, their
expertise evolves exogenously, and the persistence of the expertise process is also a
key parameter determining the joint distribution of financial wealth and expertise.
This is because, to a first order, experts’ wealth evolves via a stochastic process with
a drift that is determined by the investment rate, the fraction of investment allocated
to the risky asset, and the equilibrium excess return. Faced with their higher Sharpe
Ratio, experts with higher levels of expertise invest more in the risky asset, and hence
have a higher expected growth rate of wealth. Thus, positive persistence in expertise
reinforces the natural correlation between wealth and expertise.

In the cross section, then, parameters which proxy for asset complexity, such as
the entry cost, maintenance costs, the persistence of expertise and the quantitative
magnitude of its impact on volatility, naturally determine industry level returns and
volatility through their influence the joint distribution of expertise and wealth, and
on total participation in the segmented market. In “normal” times, investors earn a
return in excess of the risk free rate as compensation for bearing the idiosyncratic
risk that comes from the heterogeneity in investors’ combined investment in the asset
and in the expertise necessary to manage it.

In the time series, investors can earn even higher excess returns following a shock
which either reduces the wealth of experts, or increases the volatility of returns.
Note that depreciation shocks to expertise effectively increase volatility. Empirically,
when “experts” in a complex asset class experience a negative shock to their financial
wealth, the price of the asset declines and expected returns increase. It is realistic to
assume that some markets for complex assets are segmented, however clearly entry is
possible. The question is then, why, when the returns to entering a segmented market
appear to increase, do we not observe more entry, less exit, and “fast moving capital”
to take advantage of the greater profit opportunities? That is, all else equal wider
spreads should lead to more entry, but in practice we do not seem to observe this.
We note also that standard strategic barriers to entry, such as the excess capacity of
incumbents, are in fact particularly small following negative shocks.

Our work aims to explain entry and exit dynamics in order to shed light on the
dynamics of risky investment opportunities after shocks to the financial wealth of incumbents, and/or to the fundamental driving processes for risky asset returns. In our model, entry by new investors is not a foregone conclusion because the accumulated expertise of the incumbents gives incumbents an advantage which acts as a barrier to entry. In our model, expertise is a durable, capital-like good, which enables its investor owners to achieve a better tradeoff between risk and return. In addition to acting as a barrier to entry, it can also affect discount rates because it is complementary to financial wealth. Netting all of these effects, we show that persistent increases in excess returns following a negative shock naturally occur in equilibrium despite free entry.

We plan to estimate the key parameters of our model using the Simulated Method of Moments. We currently calibrate the model to match moments from the returns of asset backed fixed income (ABFI) hedge funds in the Hedge Fund Research Database. Asset backed fixed income hedge funds consist primarily of funds investing in mortgage backed securities (MBS), but also includes funds investing in other forms of securitized consumer or commercial credit. These funds must make substantial investments in hedging technologies in the form of their human and IT inputs. Duarte, Longstaff, and Yu [2006] find that, even within fixed income strategies, MBS strategies are amongst the most highly complex. They also find that such strategies earn consistent excess returns, possibly partially as compensation for the necessary investments in expertise. Models for hedging changes in consumer behavior, for example mortgage prepayments, are a key tool for such funds, and the quality of these models varies considerably. As such, we argue that this sector is a good laboratory in which to study our model. We match moments describing the size distribution of funds, the mean and cross sectional standard deviation of returns, and fund size conditional on being a new entrant or exiting firm. In the time series, we note that when there is a negative return shock to ABFI funds on average, that return dispersion in the cross section increases and remains persistently elevated. Given this finding, in addition to exploring the effects of temporary negative wealth shocks, we also explore the effect of persistent volatility shocks.
Our paper contributes to a large and growing literature on segmented markets and asset pricing. The existing literature typically falls into one or more of three main categories, namely financial constraints and limits to arbitrage, intermediary asset pricing, and segmented market models with alternative microfoundations to agency theory. Although our model is not one of arbitrage per se, we consider the return spread between two assets with identical systematic risk. Our model also shares the features of segmented markets and trading frictions with the limits to arbitrage literature. Gromb and Vayanos [2010b] provide a recent survey of the theoretical literature on limits to arbitrage, starting with the early work by Brenman and Schwartz [1990] and Shleifer and Vishny [1997]. Shleifer and Vishny [1997] emphasize that arbitrage is conducted by a fraction of investors with specialized knowledge, but similar to He and Krishnamurthy [2012], they focus on the effects of the agency frictions between arbitrageurs and their capital providers. Although we do not explicitly model risks to the liability side of investors’ balance sheets, one can interpret the shocks agents in our model face to include redemptions from risky assets.

Recently, the broader asset pricing impact of financially constrained intermediaries has been studied in the literature on intermediary asset pricing following He and Krishnamurthy [2012] and He and Krishnamurthy [2013]. This literature applies results from the literature on asset pricing with heterogenous agents, following Dumas [1989], to segmented markets with financial constraints. In doing so, the intermediary asset pricing literature both connects to empirical applications, and to the asset price dynamics which are the

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4For other models of risks stemming from redemptions and fund outflows and the resulting asset pricing implications, see Berk and Green [2004], and Liu and Mello [2011].

5See also, for example, Adrian and Boyarchenko [2013]. For empirical applications, see for example, Adrian, Etula, and Muir [Forthcoming] and Muir [2014].

6For closely related work on asset pricing with heterogeneous risk aversion and segmented markets, see also Basak and Cuoco [1998], Kogan and Uppal [2001], Chien, Cole, and Lustig [2011], and Chien, Cole, and Lustig [forthcoming].

Our model is an example of an “industry equilibrium” model in the spirit of Hopenhayn [1992a] and Hopenhayn [1992b]. These models study the important effects of firm dynamics, entry and exit in the heterogeneous agent framework developed in Bewley [1986]. This literature focuses in large part on explaining firm growth and moments describing the firm size distribution. Recent progress in the firm dynamics literature using continuous time techniques to solve for policy functions and stationary distributions include Miao [2005], Luttmer [2007], Gourio and Roys [2014], Moll Forthcoming, and Achdou, Han, Lasry, Lions, and Moll [2014]. We work in discrete time because our non-linear model features two state variables. However, we draw on intuition from these papers as well as discrete time models of firm dynamics, as in recent work by Clementi and Palazzo [2014], which emphasizes the role of selection in explaining the observed relationships between firm age, size, and productivity.

We use the asset backed fixed income (ABFI) segment of the hedge fund industry for empirical moments describing size and performance. As such, we draw from the literature on hedge funds performance and compensation. In particular, we motivate our use of ABFI funds as our main example of a complex strategy using the evidence in Duarte, Longstaff, and Yu [2006]. They provide evidence that MBS strategies are relatively complex and earn higher returns even in comparison to other sophisticated

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fixed income arbitrage strategies. Several papers provide evidence for the importance of idiosyncratic risk in the hedge fund returns, following the idea in Merton [1987] that idiosyncratic risk will be priced when there are costs associated with learning about or hedging a specific asset.8 Relatedly, Fung and Hsieh [1997] find that hedge fund returns have low and sometimes negative correlation with asset class returns. Our model features investors with constant relative risk aversion (CRRA) preferences. While we do this for tractability and parsimony to retain our focus on the effects of the joint wealth and expertise distribution, Panageas and Westerfield [2009] show that hedge fund compensation contracts with long horizons lead to portfolio choice which aligns perfectly with that of a CRRA investor. Drechsler [2014] extends these results to include variation in managers’ outside options and shows the CRRA result holds as long as such reservation values are neither too high nor too low. These results extend the analysis of the impact of high-water marks in Goetzmann et al. [2003].

The majority of the assets under management in the ABFI sector are mortgage backed securities (MBS). Cash flow risk in MBS securities typically comes primarily from prepayment risk, since the largest part of the market consists of agency securities. Gabaix, Krishnamurthy, and Vigneron [2007] provide convincing evidence that returns are driven in large part by limits to arbitrage. Importantly, they show that although prepayment risk is partly common within a class of MBS securities, the risk in MBS investing is negatively correlated with the aggregate risks born by a representative consumer. Recent work by Boyarchenko, Fuster, and Lucca [2014] extends these ideas and provides evidence that prepayment model risk explains the “smile” in MBS option adjusted spreads (OAS) and confirms that time series variation in returns is closely related to the MBS supply relative to the capital of MBS investors.9 That idiosyncratic risk is priced in MBS is supported by prior empirical studies. It is also consistent with the fact that different investors have different pric-8See Titman and Tu [2011] and Lee and Kim [2014], Jurek and Stafford Forthcoming emphasize that scarce and specialized knowledge may drive both hedge fund returns and put pricing.
ing and hedging models, and invest in different parts of the mortgage space. Some funds benefit from early prepayment (if they hold mainly principal-only securities, for example), and some funds are hurt by early prepayment (if they hold mainly interest-only securities, for example). We implement the “model risk” inherent in funds’ prepayment models via the variation in idiosyncratic risk faced by investors with varying amounts of expertise.

3 Model

3.1 Static Example

We present a static model, in which we can solve for the market clearing excess return in closed form, to build intuition. In the dynamic model, the equilibrium $\alpha$ is determined by the joint distribution of financial wealth and expertise, and this joint distribution is in turn driven by the deep preference and technology parameters of the model. We build intuition by working backwards, first taking the joint distribution of financial wealth and expertise as given in a simple static setting, and studying the comparative statics over parameters determining the market clearing excess return of the risky asset relative to the risk free rate, $E[R - R_f]$, which we call “$\alpha$” $^{[10]}$.

The advantage of the fully dynamic model, then, is that this distribution is itself determined endogenously.

Investors have constant relative risk aversion preferences over date 1 consumption, with coefficient of relative risk aversion $\gamma$. At date 0, they are endowed with financial wealth $W$ and expertise $E$. There is a riskless asset with gross return $R_f$, and a risky asset with gross return $R$, which are distributed log normally. We use lower case letters to denote logs. We assume that the log return on the risky asset for any given investor, which we denote by $r$, is distributed $\sim N(\mu - \frac{1}{2} \frac{\sigma^2}{E}; \frac{\sigma^2}{E})$, given the distribution of $W$ and $E$. $^{[11]}$ We denote the variance of log returns on the fundamental asset, before

$^{[10]}$Our use of the notation $\alpha$ is thus distinct from alpha as traditionally defined in asset management to assess relative performance, and as used in the empirical literature on pricing errors relative to systematic factor models. We nonetheless adopt this notation since our $\alpha$ is a somewhat analogous quantity in our model with priced idiosyncratic risk.

$^{[11]}$We assume this functional form for log return volatility for simplicity. It is straightfor-
expertise is applied, by $\sigma_v^2$. The variance of an investor’s return on the risky asset then decreases as expertise $E$ increases, while the innovation $\upsilon$ itself is independent from $W$ and $E$. Investing in the complex asset implies a joint investment in a common market clearing return, as well as a specific risk from hedging or asset specificities. Assuming that the mean of log returns is adjusted by the variance implies that the log of expected returns on the risky asset equals $\mu$. Conveniently, this in turn implies that expected level returns, which investors actually face, are the same for all investors, regardless of their individual expertise. Finally, note that in the static model, there is no substantive distinction between aggregate and idiosyncratic risk, however we stick to the idiosyncratic interpretation to be consistent with the dynamic model.

Using the approximation described in [Campbell and Viceira 2002a], and the associated appendix [Campbell and Viceira 2002b], which relates log individual-asset returns to log portfolio returns over short time intervals, the investor’s optimization problem becomes:

$$\max_{\theta} \left\{ \theta (\mu - r_f) - \frac{\gamma \sigma^2}{2} \frac{\sigma^2}{E} \right\}$$

(1)

where $r_f$ stands for the log return on the riskless asset. We use bold notation to denote equilibrium quantities. The solution for the optimal fraction of wealth allocated to the risky asset is:

$$\theta^* = \frac{(\mu - r_f)}{\gamma \sigma_v^2} E.$$  

(2)

Thus, portfolio choice in a lognormal model with power utility resembles that of a mean variance investor. The allocation to the risky asset is increasing in the equilibrium average excess return, decreasing in risk aversion, and decreasing in the fundamental shock variance. Moreover, the fraction of wealth that an investor allocate to show that our main conclusions for the static model are robust to a family of functions $\frac{\sigma^2}{k_0 + k_1 E + k_2 E^2 + ...}$, with all coefficients $k_0, k_1, k_2, ...$ being non-negative.

Because an individual investor’s return volatility depends on their expertise, for the approximation to be good given our specification for log return volatility, we have to impose a technical restriction that the majority of distribution of expertise $E$ is bounded away from zero. This assumption is unnecessary if one adopts the general functional form for volatility discussed in footnote 11.
cates to the risky asset strictly increases with expertise. The relationship is linear under our functional form assumptions.\textsuperscript{13}

We now describe how the equilibrium excess return depends on the parameters for preferences, technology, and the joint distribution of wealth and expertise. We focus on comparative statics over the equilibrium average excess return, market level Sharpe Ratio, and individual Sharpe Ratios. We normalize the mass of investors to one, define the supply of the risky asset to be $S$, determine the market clearing log expected return $\mu$, and then back out the equilibrium expected level return and therefore $\alpha$. We assume that $\log(W)$ and $\log(E)$ are jointly normally distributed. We denote the joint pdf of the log variables $f(w,e)$, with means and variances $\mu_w, \mu_e, \sigma^2_w$, and $\sigma^2_e$ respectively, and covariance $\rho_{w,e}\sigma_w\sigma_e$. Thus, an economy $\psi$ is described by $\psi \equiv \{r_f, \gamma, S, \sigma_v, \mu_w, \sigma_w, \mu_e, \sigma_e, \rho_{w,e}\}$. The equilibrium log expected return $\mu$ solves the market clearing condition:

$$\text{Supply} \equiv S = \text{Demand} = \int \int \exp(w)\theta^*(\exp(e))f(w,e)\,dw\,de = \frac{\mu - r_f}{\gamma \sigma^2_v} \mathcal{E} \quad (3)$$

where $\theta^*(\exp(e))$ is the portfolio choice given in Equation (2) and $\mathcal{E}$ is the wealth and population weighted average of expertise:

$$\int \int \exp(w + e)f(w,e)\,dw\,de = \exp\left(\frac{1}{2}(\sigma^2_v + \sigma^2_e + 2\rho\sigma_w\sigma_e + 2\mu_w + 2\mu_e)\right) \quad (4)$$

and we utilize the result for the expectation of log normally distributed variables.

Rearranging, we have:

$$\mu = r_f + \left(\frac{\sigma^2_v}{\mathcal{E}}\right) \gamma S. \quad (5)$$

The equilibrium log excess return is increasing in the amount of risk relative to the risk bearing capacity of investors. We decompose the inputs into two components. The first term is the effective risk in the market, namely the fundamental risk $\sigma^2_v$, scaled down by the wealth and population weighted average of expertise. The second

\textsuperscript{13}Without restrictions on the distribution of $E$, $\theta$ can be larger than one, implying borrowing at the risk free rate.
term is the risk aversion scaled supply of the risky asset which must be cleared. Conversely, the wealth and population weighted average of expertise, $\mathcal{E}$, scales $\mu$ down due to the positive impact of expertise on investors’ allocation to the risky asset.

The equilibrium mean of the level of the gross risky return over the level of the gross risk free rate, $\alpha$, is a monotonic transformation of $\mu$. In particular, we show in the Appendix that the equilibrium $\alpha$ is then given by:

$$\alpha = \exp(\mu) - R_f$$

which gives a one to one mapping from $\mu$ to $\alpha$ conditional on parameters. Note also that writing $\theta^*$ (Equation 3) as a function of either $\mu$ or $\alpha$ will always yield identical equilibrium outcomes.

**Lemma 1** Using Equation (6) describing the equilibrium market clearing $\alpha$, the following comparative statics can be directly calculated:

1. $\frac{\partial \alpha}{\partial \sigma} = \exp(\mu) \frac{2S}{\mathcal{E}} > 0$. $\alpha$ increases with fundamental risk.

2. $\frac{\partial \alpha}{\partial \gamma} = \exp(\mu) \frac{\sigma^2}{\mathcal{E}} S > 0$. $\alpha$ increases with the coefficient of relative risk aversion.

3. $\frac{\partial \alpha}{\partial S} = \exp(\mu) \frac{\sigma^2}{\mathcal{E}} \gamma > 0$. $\alpha$ increases with the supply of the risky asset investors must absorb.

4. $\frac{\partial \alpha}{\partial \mu_w} = -\exp(\mu) \frac{\sigma^2}{\mathcal{E}} \gamma S < 0$. $\alpha$ decreases as aggregate wealth increases.

5. $\frac{\partial \alpha}{\partial \mu_e} = -\exp(\mu) \frac{\sigma^2}{\mathcal{E}} \gamma S < 0$. $\alpha$ decreases as aggregate expertise increases.

6. $\frac{\partial \alpha}{\partial \rho_{w,e}} = -\exp(\mu) \frac{\sigma^2}{\mathcal{E}} \gamma S \sigma_w \sigma_e < 0$. As $\rho_{w,e}$ increases, there is a more efficient allocation of expertise and $\alpha$ decreases.

7. $\frac{\partial \alpha}{\partial \sigma_w} = -\exp(\mu) \frac{\sigma^2}{\mathcal{E}} \gamma S \left(\sigma_w + \rho_{w,e} \sigma_e\right)$

   * $> 0$ if $\rho_{w,e} < -\frac{\sigma_w}{\sigma_e}$, i.e. if wealth and expertise are strongly negatively correlated.
\[ \partial \alpha / \partial \sigma_e = -\exp (\mu) \frac{\sigma^2_e}{\sigma_w} \gamma S (\sigma_e + \rho_{w,e} \sigma_w) \]

- \( \rho_{w,e} > -\frac{\sigma_e}{\sigma_w} \), i.e. if wealth and expertise are positively or only weakly negatively correlated.

- \( \rho_{w,e} < -\frac{\sigma_e}{\sigma_w} \), i.e. if wealth and expertise are strongly negatively correlated.

- \( \rho_{w,e} > -\frac{\sigma_e}{\sigma_w} \), i.e. if wealth and expertise are positively or only weakly negatively correlated.

**Proof.** By direct calculation, see Appendix. ■

All comparative statics are intuitive. It is interesting to compare \( \partial \alpha / \partial \mu_w \) and \( \partial \alpha / \partial \mu_e \) with \( \partial \alpha / \partial \rho_{w,e} \). The effect of the correlation between \( w \) and \( e \) on \( \alpha \) is smaller than the effect of either the mean of \( w \) or the mean of \( e \). However, the fact that these parameters are of different scale, i.e. \( \rho_{w,e} \in [0, 1] \), while \( \mu_w \in (-\infty, \infty) \), implies that quantitative comparisons require calibration. The effect of an increase in \( \rho_{w,e} \) on the market clearing \( \alpha \) will be larger the larger is amount of fundamental risk, \( \sigma^2_w \), the larger is the coefficient of relative risk aversion, \( \gamma \), the larger is the supply of the risky asset, \( S \), the smaller is the mean of log wealth, \( \mu_w \), and the smaller is the mean of log expertise, \( \mu_e \).

We also derive results for the equilibrium market level, and investor specific, Sharpe Ratios. While, in our model with only idiosyncratic risk, the excess return has a relatively straightforward interpretation, as do investor specific Sharpe Ratios, the market level Sharpe Ratio requires a definition appropriate for our environment. Here, we define the equilibrium market level Sharpe Ratio to be the one that results from averaging over all investor specific Sharpe Ratios, weighting each investor equally. Equivalently, the weights are equal to the density at each level of expertise. Thus this market Sharpe Ratio can, for example, be interpreted as the

\[ \text{The comparison between } \frac{\partial \alpha}{\partial \mu_w} \text{ and } \frac{\partial \alpha}{\partial \mu_e} \text{ depends on the functional form assumption for effective risk. Under the specification in the main text, } \mu_w \text{ and } \mu_e \text{ will have the same impacts on equilibrium } \alpha, \text{ whereas if expertise scales variance by } E^2 \text{ or } \sqrt{E}, \text{ for example, the effect of a change in } \mu_e \text{ vs. } \mu_w \text{ will be greater, or lesser, respectively.} \]
expected Sharpe Ratio for an investor drawing from the distribution of possible levels of expertise, before the investment stage of the model. We call this Sharpe Ratio the equilibrium equally weighted market Sharpe Ratio. It results from all investor decisions and measures the *market level* risk return tradeoff.

\[
SR = \frac{1 - R_f \exp(-\mu)}{\sqrt{E \left[ \exp\left(\frac{\sigma_w^2}{E}\right) \right]}} - 1.
\]

(7)

Lemma 2 Using Equation (7) describing the equilibrium market clearing Sharpe Ratio, the following comparative statics can be directly calculated:

1. Let \( \eta \) denote any parameter \( \eta \in \{\gamma, S, \mu, \sigma, \rho\} \).
   Then, \( \text{Sign} \left( \frac{\partial (SR)}{\partial \eta} \right) = \text{Sign} \left( \frac{\partial \alpha}{\partial \eta} \right) \).

2. The signs for comparative statics with respect to parameters \( \hat{\eta} \in \{\sigma^2, \mu_e, \sigma_e\} \),
   are indeterminate.

Proof. By direct calculation, see Appendix. ■

Expected returns rise proportionally relative to the volatility of the risky asset return in our static model, so that the Sharpe Ratio improves with any parameter change that increases \( \alpha \). Thus, we confirm that, at the market level, parameter changes which lead to an increase in the equilibrium expected excess return in fact lead to better investment opportunities given the market risk in equilibrium.

However, since the log return volatility depends on an individual investors’ expertise, an observed increase in the market Sharpe Ratio does not necessarily guarantee a higher Sharpe Ratio for every investor in the market. Moreover, even if the Sharpe Ratio improves for each agent individually, the magnitude of the improvement an individual investor faces will not, in general, coincide with the market improvement. To see this, consider the investor-specific Sharpe Ratio. For an investor with wealth \( W \) and expertise \( E \), we show in the Appendix that this investor’s Sharpe Ratio is given by:

\[15\text{See Appendix for derivation. We also compute and analyze the market value weighted Sharpe Ratio in the Appendix.} \]
\[ SR(W,E) = \frac{1 - R_f \exp(-\mu)}{\sqrt{\exp\left(\frac{\sigma_w^2}{E}\right) - 1}}. \]  

Equation (8) clearly shows that the model can deliver considerable cross-sectional dispersion in investor-specific Sharpe Ratios.\(^{16}\) Investors with very low effective risk, \(\sigma^2_E\), face significantly higher Sharpe Ratios than their counterparts with low expertise. Although we are unable to compute the variance of investor-specific Sharpe Ratios directly, we can determine the signs of the following comparative statics:

**Lemma 3** Using Equation (8) describing the investor-specific Sharpe Ratio, and Equation (2) describing the portfolio allocation \(\theta^*\), the following comparative statics can be directly calculated. Let \(\eta\) denote any parameter \(\eta \in \{\gamma, S, \mu_w, \sigma_w, \rho_{w,e}\}\).\(^{17}\)

1. \(\frac{\partial SR(W,E)}{\partial E} > 0\). Higher expertise generates lower effective risk, and a correspondingly higher individual Sharpe Ratio.

2. \(\text{Sign}\left(\frac{\partial SR(W,E)}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial \alpha}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial SR}{\partial \eta}\right)\). All investor-specific Sharpe Ratios co-move with the equilibrium excess return and the market level equilibrium Sharpe Ratio.

3. \(\text{Sign}\left(\frac{\partial VAR(SR(W,E))}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial \alpha}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial SR}{\partial \eta}\right)\). Whenever a parameter change increases the market level equilibrium Sharpe Ratio, it leads to a larger cross-sectional dispersion in the investor-specific Sharpe Ratio.

4. \(\text{Sign}\left(\frac{\partial^2 SR(W,E)}{\partial \eta^2}\right) = \text{Sign}\left(\frac{\partial SR(W,E)}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial \alpha}{\partial \eta}\right) = \text{Sign}\left(\frac{\partial SR}{\partial \eta}\right)\). Whenever a parameter change increases the investor-specific Sharpe Ratio, it leads to a larger increase for high expertise investors relative to low expertise investors.

\(^{16}\)Note this would be the case even if one adopted a flexible functional form for effective risk.

\(^{17}\)Derivatives with respect to \(\mu_e\) and \(\sigma_e\) follow the same formulas as those that support parts 2 to 5 of lemma 3. However, the changes are not comparable to the market Sharpe Ratio, as we can’t determine the signs in lemma 3 part 2. Derivatives with respect to \(\sigma_v^2\) cannot be signed generally.
5. \( \text{Sign} \left( \frac{\partial^2 \theta^*(W,E)}{\partial \gamma \partial E} \right) = \text{Sign} \left( \frac{\partial \text{SR}(W,E)}{\partial \eta} \right) = \text{Sign} \left( \frac{\partial \alpha}{\partial \eta} \right) = \text{Sign} \left( \frac{\partial (\text{SR})}{\partial \eta} \right) \)

Whenever a parameter change increases the investor-specific portfolio allocation, it leads to a larger increase for high expertise investors relative to low expertise investors.

6. \( \exists \bar{E} > E > 0 \) such that \( \forall E > \bar{E}, \frac{\partial \text{SR}(W,E)}{\partial \sigma^2} > 0 \), and \( \forall E < \bar{E}, \frac{\partial \text{SR}(W,E)}{\partial \sigma^2} < 0 \).

An increase in the fundamental risk generates a higher Sharpe Ratio for high expertise investors and a lower Sharpe Ratio for low expertise investors.

**Proof.** By direct calculation. See Appendix.

Lemma (3) has rich implications. First, we emphasize the co-movement between cross sectional variation in investor-specific Sharpe Ratios and the level of the market Sharpe Ratio. Any parameter change with a determinate, positive effect on the market level Sharpe Ratio, is also associated with an increase in the cross sectional dispersion of investor specific Sharpe Ratios. Furthermore, because an increase in the market level Sharpe Ratio improves investment opportunities for high expertise investors by more than for low expertise investors, such an increase accordingly increases their allocation to the risky asset \( \theta^* \) by more. Thus, a seemingly improved market level risk return tradeoff in large part reflects the improved Sharpe Ratios faced by high expertise investors, and not by their low expertise counterparts. This is because any parameter change which increases the market level Sharpe Ratio increases the investor specific Sharpe Ratio for high expertise by more, and increases the influence of high expertise investors’ Sharpe Ratios on the market level risk return tradeoff. In short, measured improvements in the aggregate Sharpe Ratio are a misleading indicator of improvements in individual investors’ risk return tradeoff, and can indeed more accurately reflect changes in the Sharpe Ratio of higher expertise investors. The converse is also true.

Furthermore, part [6] states that, in fact, changes to fundamental risk can lead to changes in individual Sharpe Ratios that vary in sign. For example, if \( \sigma^2 \) increases,

\[18\text{Except for } \gamma, \text{ where } \frac{\partial^2 \theta^*(W,E)}{\partial \gamma \partial E} = 0.\]
all investors face the same increase in the equilibrium excess return, but investors with high expertise face a considerably smaller increase in risk.

The fact that the effect of any parameter change on individual Sharpe Ratios is magnified for high expertise investors also implies that, as the market level Sharpe Ratio varies, the Sharpe Ratio for high expertise investors varies more. Thus, in a model with systematic risk, there may be an interesting interaction between the intensive margin of participation, as determined by expertise, and the risk an investor faces. In contrast, low expertise investors have Sharpe Ratios which are less sensitive to the parameter changes, and instead are relatively stable, at low levels.

Although the example is static, we can use intuition from these comparative statics to consider related effects in a dynamic model. In particular, combining all the comparative statics, certain effects in the static model appear consistent with persistent sizable increases in risk adjusted market excess returns following a negative shock to investors’ wealth. For example, suppose an exogenous shock leads to an economy-wide contraction in funds available for investment (a negative shock to $\mu_w$). This shock drives up every investor’s Sharpe Ratio, but has a larger effect on investors with higher expertise. If these investors cannot increase their allocation to the risky asset (for example due to regulatory constraints, fund outflows, or borrowing constraints), then the market for the risky asset must clear through the demand of investors with less expertise. For this to happen, the equilibrium $\alpha$ must increase, and will stay elevated as long as the allocation of the risky asset is less efficient than it was prior to the shock. Persistence is reinforced by the higher growth of low expertise investors’ wealth relative to prior to the shock, due to the elevated $\alpha$ and their associated larger risky portfolio allocation. Hence, the persistence of seemingly abnormal returns, can depend crucially on potential constraints for high expertise investors as well as the speed of expertise accumulation for low expertise investors.

### 3.2 Dynamic Model

The static model takes the joint distribution of financial wealth and expertise as given. In this section, we develop a Bewley [1986] model of the asset management
industry that draws on Hopenhayn’s model of entry and exit. This dynamic model allows us to derive the joint distribution of financial wealth and expertise endogenously as a function of the deep model preference and technology parameters. However, the spirit and the general intuition of the two models are closely related. We use the dynamic model to understand how this distribution is determined, and to study the quantitative effects on equilibrium returns under plausible calibrations. We also perform comparative statics over parameters which describe asset complexity in the cross section. Finally, we also demonstrate that unanticipated aggregate negative return shocks lead to persistently elevated excess returns despite the fact that we allow for free entry.

Preferences and Endowments There are a continuum of investors of measure one with CRRA utility over consumption each period and coefficient of relative risk aversion $\gamma$.\footnote{See Panageas and Westerfield 2009 and Drechsler 2014 for theoretical motivations for modeling hedge fund managers as investors with CRRA preferences.} Investors discount the future at a rate $\beta < 1$ and face each period a constant probability $1 - p$ of being exogenously liquidated. In the event of liquidation, investors consume their current wealth. Liquidated agents are replaced with newly minted investors with a small initial wealth $w_0$. Exogenous liquidation ensures a stationary distribution of wealth despite linear asset returns.

Technology Investors can choose to be either experts or non-experts. Investors must pay an entry cost $f_{nx}$ to become an expert and to initialize their stock of financial expertise at level $e_0$. Incumbent experts must pay a smaller fixed cost $f_{xx}$ each period to maintain their expertise, and otherwise they revert to non-expert status. Subject to paying this maintenance cost, expertise evolves via an exogenous process. The fixed maintenance cost leads to a threshold rule for exit, so that, conditional on the equilibrium $\alpha$, agents with sufficiently low financial wealth and expertise choose to exit the market for the risky asset.

Non-experts can invest only in the risk free asset. Experts can invest in a risky trading strategy, as well as the riskless asset. The riskless asset delivers a fixed net return equal to $r_f$. We use $\alpha$ to denote the equilibrium excess return on the
risky asset which equates aggregate demand for the risky asset to the fixed aggregate supply, \( S \). The risky trading technology delivers the following stochastic returns per unit of wealth invested:

\[
r_{t+1} = r_f + \alpha + \nu_{t+1}\sigma(e_t),
\]

where \( \nu \sim N(0, \sigma^2_{\nu}) \) and is i.i.d. across investors, \( e \) denotes the (agent specific) stock of financial expertise for expert investors, and we assume that \( \sigma(e) > 0 \ \forall e \) and \( \sigma'(e) < 0 \). Investing in the complex asset implies a joint investment in a common market clearing return, as well as a specific risk from hedging or asset specificities amongst the heterogeneous constituents of the complex asset class. Investors vary in the expertise they can apply to reduce hedging or asset specific risk.

The investor specific Sharpe Ratio from operating the risky trading technology at time \( t \) is given by:

\[
\frac{\alpha}{\sigma(e_t)},
\]

which increases in the agent’s stock of financial expertise \( e \), and decreases in the total demand for the risky asset. As in the static example, aggregate demand will depend on the aggregate wealth of experts, their savings rate, and the experts’ portfolio allocation to the risky asset. This allocation, as in the static example, is approximately a constant fraction of wealth which depends positively on \( \alpha \) and negatively on the investor specific variance and investor risk aversion. Investors with higher expertise face a higher Sharpe Ratio, and will therefore allocate a larger fraction of their investment to the risky asset. As in the static model, there is a crucial difference between the market Sharpe Ratio, and investor specific Sharpe Ratios.

**Recursive Optimization Problem: Bellman Equations**  An investor’s individual state is given by their previous period type \( t \in \{ x, n \} \), where \( x \) denotes expert and \( n \) denotes non-expert; their financial wealth \( w \); and their expertise level \( e \).

Each period, investors choose whether it is better for them to operate as an expert with value \( v_x(t, w, e) \), or non-expert with value \( v_n(w) \), by solving the following recursive problem:

\[
v(t, w, e) = \max_{x,n} \{ v_x(t, w, e), v_n(w) \}
\]
where \( t \) is type expert (\( x \)) or non-expert (\( n \)), \( w \) is financial wealth, and \( e \) is financial expertise.

Agents who choose to be experts maximize their value by choosing their savings rate and their portfolio allocation to the risky asset. They solve the Bellman Equation for experts given by:

\[
v_x(t, w, e) = \max_{i, \theta} \frac{c^{1-\gamma}}{1-\gamma} + \beta E \left[ pv(t', w', e') + (1 - p) \frac{w'^{1-\gamma}}{1-\gamma} \right] (t, w, e)
\]

subject to

\[
e' = \xi + \rho e + \eta'
\]

\[
c + f_{xx} \mathbb{I}_{t=x} + f_{nx} \mathbb{I}_{t=n} = w(1 - i)
\]

\[
w' = iw(1 + \theta r' + (1 - \theta) r_f)
\]

\[
r' = r_f + \alpha + \nu' \sigma(e)
\]

\[
t' = x
\]

where we use \( t \) to denote next period values. The law of motion for log expertise is given by (13) with persistence parameter \( \rho < 1 \) and mean \( \frac{\xi}{1-\rho} \geq e_0 \) where \( e_0 \) is the initial value of log expertise, which applies to new experts. We assume that \( \eta \sim N(0, \sigma_\eta) \). The budget constraint (14) states that consumption plus entry and maintenance costs must be less than investors’ wealth after applying their chosen savings rate \( i \). \( \mathbb{I} \) is an indicator function, \( f_{xx} \) is the maintenance cost for current experts, and \( f_{nx} \) is the entry cost for new experts. We assume that \( f_{nx} >> f_{xx} \). The law of motion for wealth (15) indicates that wealth grows at a rate that is increasing in the savings rate and the allocation to the risky asset \( \theta \) as long as \( r' > r_f \), which will be true in an equilibrium. The return for the risky asset is given by equation (16). To ensure that \( \sigma(e) > 0 \), we assume that \( \sigma(e) = \sqrt{\frac{\sigma^2}{1+\exp(e)}} \).

For agents which choose to be non-experts in the following period, we have:

\[
v_n(w) = \max_{c} \frac{c^{1-\gamma}}{1-\gamma} + \beta E \left[ pv(t', w', e') + (1 - p) \frac{w'^{1-\gamma}}{1-\gamma} \right] (t, w, e)
\]

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subject to

\[ c = w(1 - i) \quad (19) \]
\[ w' = iw(1 + r_f) \quad (20) \]
\[ e' = e_0 \quad (21) \]
\[ t' = n \quad (22) \]

Non-experts have a zero allocation \( \theta \) to the risky asset, and have a constant (latent) level of expertise \( e_0 \), which they can access by choosing to become an expert, and making an investment in this initial level of expertise by paying the entry cost \( f_{nx} \).

**Definition 1 Stationary Equilibrium** Define economy \( \Psi \) to be the economy described by equations (11) to (22) along with parameters describing investors’ subjective discount factor, \( \beta \), coefficient of relative risk aversion, \( \gamma \), exogenous liquidation rate, \( p \), the entry cost, \( f_{nx} \), maintenance cost, \( f_{xx} \), risky asset supply, \( S \), risk free rate, \( r_f \), fundamental asset risk, \( \sigma_\nu \), as well as the parameters describing the expertise process, \( e_0, \xi, \rho, \) and \( \sigma_\eta \). \( \Psi = \{ \beta, \gamma, p, f_{nx}, f_{xx}, S, r_f, \sigma_\nu, e_0, \xi, \rho, \sigma_\eta \} \). An equilibrium for economy \( \Psi \) consists of a market clearing excess return \( \alpha \), policy functions for investor types, \( t \), savings rates \( i \), and portfolio allocations, \( \theta \), and a stationary distribution over individual states \( \Lambda(t, w, e) \), such that the policy functions solve the investors’ problem described in (11) to (22), and the market for the risky asset clears according to:

\[
\text{Supply} = \text{Demand} = \int_\Lambda i(t, w, e)w(t, w, e)\theta(t, w, e)d\Lambda(t, w, e). \quad (23)
\]

We plan to estimate the parameters of the model using Simulated Method of Moments (see Lee and Ingram [1991]), and discuss the model solution at current, calibrated parameters, along with numerical comparative statics in Section 4. It is worthwhile noting some general properties of the solution.

**Policy Functions: Entry, Exit, and Portfolio Allocation** Since we assume that all entrants have identical initial expertise levels, the entry decision rule takes the
form of a threshold rule in wealth above which non-experts enter. The exit threshold is a cutoff for wealth, conditional on an investor’s expertise, below which experts exit, and it is decreasing in expertise.\(^{20}\)

We can show using variational arguments\(^{21}\) and numerically that:

- \(\exists \ w(\alpha, \sigma_v)\) above which non-experts enter, with
  - \(w'_{\alpha} < 0\). The entry wealth threshold is lower when \(\alpha\) is higher.
  - \(w'_{\sigma_v} > 0\). The entry wealth threshold is higher when \(\sigma_v\) is higher.

- \(\exists \ w(\alpha, \sigma_v, e)\) below which experts exit,
  - \(w'_{\alpha|e} < 0\). The exit wealth threshold, conditional on expertise, is lower when \(\alpha\) is higher.
  - \(w'_{\sigma_v|e} > 0\). The exit wealth threshold, conditional on expertise, is higher when \(\sigma_v\) is higher.
  - \(w'_{e} < 0\). The exit wealth threshold is lower when expertise is higher.

- CRRA preferences imply a portfolio allocation fraction \(\theta\) which is an approximately constant fraction of wealth, subject to:
  - \(\theta\) is conditional on an investor’s level of expertise
  - \(\theta\) may be distorted by the distance to exit threshold

Cross Section: Effect of complexity on \(\alpha\)  Intuition from comparing stationary equilibria in the dynamic model can be used to evaluate the quantitative importance of some key effects of segmented markets with expertise on asset prices. First, there is a large effect on \(\alpha\) from the supply of the risky asset relative to the aggregate wealth of experts, supporting the importance of the “cash-in-the-market” pricing mechanism described in Allen and Gale [2005]. Second, there is also an important effect from the amount of risk in the market, and from investors’ coefficient of relative risk aversion.

\(^{20}\)Both can be shown using a variational argument (to be completed).

\(^{21}\)To be completed.
Finally, the heterogeneity in risk bearing capacity also affects equilibrium excess returns. We hope to document the relative importance of these mechanisms, which have all played a role in the literature described in Section 2. We provide numerical comparative statics to this effect in Section 4.

The market clearing excess return $\alpha$ depends on the joint distribution over wealth and expertise, and also the fraction of investors who choose to be experts, as described in equation (25). The key inputs from this joint distribution are the amount of wealth commanded by experts, along with their savings rate and their portfolio allocation, "$iw\theta$". Thus, $\alpha$ will be decreasing in the wealth of experts, in their savings rate, and in their fractional allocation to the risky asset. There are a few deep parameters which are key to aggregate investor demand for the risky asset.

First, the entry cost will determine the fraction of investors who choose to be experts. Absent the general equilibrium effect on $\alpha$, a higher entry cost will lead to a higher wealth cutoff for entry, and ceteris paribus, fewer expert investors with less wealth in aggregate. The aggregate wealth of entrants also decreases in the entry cost, due to the dead weight loss of the cost. Unless $\alpha$ increases, experts will not allocate a larger fraction of wealth to the risky asset. Thus, in order for the market to clear, $\alpha$ must increase. That $\alpha$ is increasing in the entry cost is consistent with a higher equilibrium excess return in markets which are more complex and thus require larger fixed investments in developing expertise.

Several parameters jointly determine the amount of risk, relative to risk bearing capacity, conditional on the participation rate. The coefficient of relative risk aversion, $\gamma$, determines investors’ utility costs of risk bearing. The amount of risk is determined by the supply of the risky asset, $S$, along with the fundamental volatility $\sigma_v$. The effective risk faced by investors is determined by the quality of their hedging technologies, as governed by the expertise parameters $e_0, \xi, \rho$, and $\sigma_{\eta}$. Markets with a large supply, substantial fundamental risk, and in which it is costly or hard to develop expertise, such as that for mortgage backed securities, are likely to have the highest equilibrium excess returns.

The risk bearing capacity of the economy also increases with improvements to the allocation of wealth across expertise levels, which determines the efficiency of the
Equations (15) and (16) imply that the expected growth rate of wealth is 
\[ i(1 + r_f + \theta \alpha), \]
or the savings rate, times the gross risk free rate plus \( \theta \) times the 
equilibrium \( \alpha \). To a first order, experts’ portfolio allocation is a constant fraction 
of wealth which is increasing in expertise, as can be shown in a model with propor-
tional entry and maintenance costs (See Appendix). Savings rates can theoretically 
be slightly decreasing in expertise, due to the wealth effect from higher expertise and 
the associated larger present value of investment opportunities.\(^{22}\) However, under 
reasonable parameters, this effect tends to be dominated by the portfolio choice ef-
fact. As a result, investors with higher expertise have higher growth rates of wealth 
on average. A higher average growth rate of wealth for investors with greater ex-
pertise tends to lead to a more efficient allocation of the risky asset, and a higher 
correlation between expertise and financial wealth. It is worth considering, then, 
what would ameliorate this effect. First, if there is less variation in the amount or 
impact of expertise, portfolio allocations will be less disperse, as will wealth growth 
rates. Similarly, if expertise is less persistent, there will be greater mean reversion 
in portfolio allocations and wealth growth rates. Finally, greater turnover from exit 
effectively lowers expertise persistence and has a similar effect.

Note that the fact that investors’ allocation to the risky asset \( \theta \) is to a first order 
independent of wealth (with deviations due to fixed costs and the entry and exit 
boundaries), implies that the size distribution of investors per se does not matter. 
What matters is the total amount of wealth commanded by experts, and the way 
in which that wealth is allocated across expertise levels. The more wealth which 
lies in the hands of investors with high expertise, the lower will be equilibrium ex-
cess returns. This implication is relaxed for preferences or technologies assumptions 
which lead to portfolio allocations which are non-linear in wealth. For example if 
there are decreasing returns to scale, a more concentrated market would have higher 
equilibrium excess returns than a less concentrated market, all else equal. Note also 
that efficiency is higher when more wealth is commanded by investors with greater 
expertise. This is a different efficiency tradeoff than that in other studies of the

\(^{22}\)This can be shown analytically in the static model with a consumption savings decision (to be 
completed).
benefits provided by large financial institutions, and one which we plan to explore further.

Table 1 presents simple summary statistics from the Hedge Fund Research (HFR) database for a selection of the larger strategy classification codes, or “industries”, as listed in Table 2. We order industries by their “pseudo Sharpe Ratio”, which is the average over time of the industry level return divided by the average over time of the industry cross sectional variation.

Figure 4.3 plots the share of assets held by each size decile for eight of the larger hedge funds strategies in the HFR database. All strategies have at least 40% of assets held by the largest decile of funds, and eight out of nine have more than 50% of assets held by the largest decile. Seen through the lens of our model, whether these size distributions are important for equilibrium strategy returns depends on whether size and expertise are correlated, or, whether portfolio allocations vary with size conditional on expertise. It is interesting that the industry with the largest contribution of smaller funds is asset backed fixed income, which is also the industry with the highest pseudo Sharpe Ratio.

Time Series: Effect of shocks on $\alpha$  We study transition dynamics for economies in stationary equilibrium which are hit by unanticipated aggregate shocks. We study two shocks. First, we examine the effects of a transitory negative shock to financial wealth, emanating from a negative shock to the risky asset return. Second, we investigate the effects of a permanent positive shock to the fundamental asset volatility, which is equivalent to a symmetric depreciation shock to all investors’ expertise. We motivate our study of these two shocks using data from the Hedge Fund Research Database describing Asset Backed Fixed Income (ABFI) hedge funds. We document that both average performance, and the cross sectional dispersion in performance, become persistently elevated subsequent to a negative return shock. Thus, we posit that the fundamental volatility of the risky asset may be higher following a negative shock to the risky asset return. There are theoretical motivations for studying such a shock to volatility (or an expertise depreciation shock) as well, and we discuss these below.
Figure 2 plots the average performance of asset backed fixed income hedge funds (ABFI) from the HFR database from 1996-present, along with the cross sectional standard deviation of performance at each date. Recall that Gabaix, Krishnamurthy, and Vigneron [2007] show that MBS prepayment risk is, if anything, negatively correlated with aggregate consumption, and thus argue that fundamental cash flow risk is not systematic. Thus, we report raw performance. Figure 3 plots percentiles of the return distribution, along with normalized entry and exit rates. Taken together, Figures 2 and 3 show that, following the negative average returns in the late 1990’s, early 2000’s, and in 2008, both average returns, and the cross section standard deviation of returns, became persistently elevated. Figure 3 also shows that some funds benefited more than others from the elevated average returns following these negative shocks. This is consistent with variation in subsequent exposure to the risky asset, perhaps due to variation in expertise. Figure 6 plots the share of assets by size decile, and consistent with this idea shows that the largest funds tend to have a higher share subsequent to negative shocks. Thus, although past research has emphasized tail risk, the tail benefits of being a surviving incumbent seem to be an equally important feature of the returns to complex assets. These tail benefits result in our model from the dynamics of equilibrium excess returns in the endogenously segmented market. After a negative shock, both total expert wealth and the efficiency of its distribution relative to the distribution of expertise take time to recover, despite free entry.

Figure 3 provides some evidence that there were greater liquidations following the negative shocks in 2007-2008. For exit rates, we use only funds that report liquidation as the reason for ceasing to report. On the other hand, the stylized facts for entry are harder to discern for a few reasons. First, the ABFI hedge fund industry is growing over time, as shown in Figure 4. Figure 5 shows the actual less predicted log number of funds from a regression on a linear time trend, and shows that the growth rate of funds has stabilized since the early 2000’s. This data is also suggestive evidence that entry rates were somewhat lower after the financial crisis, however the log number of funds is higher than predicted by a linear time trend during the crisis itself. Entry is particularly hard to measure in self reported data, especially for recent time periods, since funds tend to begin reporting well after inception.
To gain intuition for the overall effect of unanticipated aggregate shocks on the resulting new stationary equilibrium, it is useful to consider the effects of the shocks on entry and exit, and then the overall effect on $\alpha$. We proceed by considering three effects, namely, the effect on the entry and exit thresholds, $w(\alpha, \sigma)$ and $w(\alpha, \sigma, e)$, and the effect on incumbent experts’ allocation to the risky asset $iw\theta$. We consider these effects numerically, and compute the transition dynamics, in section 4.

First, recall that there is an entry wealth threshold, above which non-experts enter. The following effects on the entry threshold result from the comparative statics on $w(\alpha, \sigma)$ with respect to changes in $\alpha$ and $\sigma$:

- **Wealth shock only**: No direct effect on entry threshold. General equilibrium increase in $\alpha$ leads to a decrease in $w(\alpha, \sigma)$, and thus more entry.

- **Risk shock only**: Two effects.
  - Direct effect: $\sigma \upsilon$ increases $\Rightarrow w(\alpha, \sigma)$ increases, leading to less entry.
  - General equilibrium effect: $\alpha$ increases with increase in risk $\Rightarrow w(\alpha, \sigma)$ decreases, leading to more entry.

Thus, ceteris paribus, entry will increase following a negative return shock, due to the resulting increase in the market clearing excess return on the risky asset over the risk free rate. Non-experts’ wealth is unaffected, and when the entry threshold decreases due to the higher $\alpha$, all agents with wealth above the new cutoff will enter immediately. On the other hand, an increase in $\sigma \upsilon$ can suppress entry, as in the static model, if the direct effect of the increase in risk on potential new entrants with low expertise, outweighs the increase in $\alpha$ required to clear the market amongst incumbent experts with higher expertise.

Likewise, the following effects on the exit threshold result from the comparative statics on $w(\alpha, \sigma, e)$ with respect to changes in $\alpha$ and $\sigma$:

- **Wealth shock only**: Two effects.
  - Direct effect: Expert wealth lower $\Rightarrow w(\alpha, \sigma, e)$ unchanged, but more exit as the negative return shock pushes investors’ wealth below the exit threshold.
General equilibrium effect: $\alpha$ increases $\Rightarrow w(\alpha, \sigma, e)$ decreases, resulting in less exit.

• Risk shock only: Two effects
  
  - Direct effect: $\sigma_v$ higher $\Rightarrow w(\alpha, \sigma, e)$ increases, resulting in more exit.
  
  - General equilibrium effect: $\alpha$ increases $\Rightarrow w(\alpha, \sigma, e)$ decreases, resulting in less exit.

Thus, the effect of either shock on exit is indeterminate, as the direct and general equilibrium effects work in opposite directions.

Finally, we consider the effect of a negative return shock and a positive risk shock on $iw\theta$. We consider the general equilibrium effects of changes in $\alpha$ on both $i$ and $\theta$, however quantitative exercises indicate that effects on $\theta$ are much larger and tend to outweigh wealth effects on the savings rate, $i$.

• Wealth shock only: Two effects.
  
  - Direct effect: Expert wealth lower $\Rightarrow iw\theta$ decreases due to the decrease in $w$.
  
  - General equilibrium effect: $\alpha$ increases $\Rightarrow iw\theta$ increases, due to the positive effect of the more favorable risk return tradeoff on $\theta$, unless the wealth effect of the improved investment opportunity causes the effect of a decrease in $i$ to more than offset the increase in $\theta$.

• Risk shock only: Two effects
  
  - Direct effect: $\sigma_v$ higher $\Rightarrow iw\theta$ decreases, due to the effect of the less favorable risk return tradeoff on $\theta$.
  
  - General equilibrium effect: $\alpha$ increases $\Rightarrow iw\theta$ increases, due to the the positive effect of the more favorable risk return tradeoff on $\theta$, unless the wealth effect of the improved investment opportunity causes the effect of a decrease in $i$ to more than offset the increase in $\theta$. 

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Summing up, and considering the overall effect of the two shocks individually can be described as:

- Wealth shock to incumbents:

  ⇒ Risk concentrated on smaller wealth base, $\alpha \uparrow$

  ⇒ More entry, exit can increase or decrease. Holding the mass and distribution of experts constant, incumbents’ demand for the risky asset $iw\theta$ displays two effects. There is a direct negative effect from lower wealth $w$. As $\alpha$ increases to clear the market, $\theta$ increases. However, it cannot be that $\theta$ increases enough to result in an ex post lower $\alpha$, as that would imply a higher $\theta$ but lower $\alpha$ than was required to clear the market ex ante. Thus, it must be that to clear the market for the risky asset amongst incumbents, holding entry and exit fixed, $\alpha$ increases.

- Risk shock:

  ⇒ There is more risk in the economy, $\alpha \uparrow$

  ⇒ Entry, exit can increase or decrease. Holding the mass and distribution of experts constant, incumbents’ demand for the risky asset would decrease, absent an increase in $\alpha$, and thus the market clearing excess return must increase to clear the market. Because investor specific Sharpe Ratios can increase for high expertise investors while declining for low expertise investors, as shown analytically in the static example, it is possible that exit can increase but entry is still depressed.

Thus, in our model, one would not observe that exit increases and entry decreases after a negative wealth shock. On the other hand, it is possible that exit might increase while entry decreases following a positive risk shock. This can happen as long as the general equilibrium effect on the Sharpe Ratio favors high expertise incumbents enough.
4 Detailed Numerical Results

This section presents results from a calibrated model. We plan to estimate the model using simulated method of moments (SMM). Details describing the computation of the stationary equilibrium are described in the Appendix. We first solve the value function, and simulate the model with 5,000 agents over 1,000 periods. We drop the first 100 periods to avoid any dependence on initial values. We then compute moments using the remainder of the simulated data.

4.1 Baseline Model

We parameterize the model assuming that the time period is one year. All parameter values are summarized in Table 3. We assume that log expertise process follows an AR(1) process. The value of the persistence and variance parameters are calibrated to match moments describing fund performance and size in the asset backed fixed income (ABFI) sector, using data from the Hedge Fund Research Database. The unconditional variance of the risky asset return is 50%. Expertise scales shocks to the risky asset returns according to \(\frac{1}{1+\exp(e)}\), where \(\exp(e)\) is a positive number with mean 1. Thus, for an investor with the average level of expertise, the effective volatility of the risky asset is half of fundamental risk. The AR(1) process is approximated using standard Gauss-Hermite quadrature techniques. We discretize the continuous process with a four state Markov chain. We assume that the expertise level for new entrants is the second lowest level to ensure positive endogenous exit. We choose a grid which is large enough to ensure that there is very little mass in the stationary distribution close to the boundary. To offset the unbounded wealth growth resulting from the linear return technology, we assume investors face an exogenous death probability. This ensures a stationary distribution.

We plan to estimate the remaining four key parameters, \(f_{nx}, f_{xx}, p, S\), simultaneously using SMM. The parameters will be estimated to minimize the weighted sum of squared errors between the following model moments and their empirical counter-

\[23\text{See Lee and Ingram [1991].}\]
\[24\text{See Tauchen and Hussey [1991].}\]
parts: 1) relative size of new entrants to average size of experts; 2) relative size of exiters to average size of experts; 3) entry/exit rate; 4) average excess returns. Currently, we calibrate these parameters to roughly match these moments. The SMM algorithm solves the value function and simulates the model by constructing a large sample of investors over a long time periods. The algorithm searches for the set of parameters which minimize the sum of squared errors between moments from simulated data from the model, and their empirical counterparts. We plan to use a bootstrapping method to construct standard errors and the weighting matrix.25

We minimize \( g(b)'Wg(b) \), where \( g(b) \) is a vector of difference between the model moments and data moments, \( b \) is the vector of parameters we are estimating here, including the entry cost, \( f_{nx} \), maintenance cost, \( f_{xx} \), survival probability, \( p \), and aggregate supply of the risky asset, \( S \). \( W \) is the weighting matrix, which is set to be the inverse of variance-covariance matrix of the empirically estimated moments. We check the comparative statics of different parameters and moments. The moments selected are those for which the parameters to be estimated show maximum sensitivity.

Figures 8 to 12 display policy functions and stationary distributions from the model at the baseline calibration from Table 3. Figure 8 shows portfolio choices for experts with different levels of expertise and wealth. Ignoring wealth levels near the boundaries, at which there is very little mass, the portfolio choice is constant over wealth for a given expertise level. Moreover, experts with higher expertise clearly allocate a higher fraction of investment to the risky asset. The boundary effects are as follows. Experts choose more risky assets when their wealth is close to 0 because there is no downside risk, as we bound wealth from below at zero, consistent with limited liability. Conversely, experts choose less risky assets when their wealth is close to the upper bound since the upper bound on wealth essentially leads to decreasing returns. To minimize the effects of these distortions, we choose an appropriate upper bound for wealth. In the stationary equilibrium, we ensure that there is no mass point, and very little overall mass, close to the close to this boundary. Figure 9 graphs the policy function of savings decisions for experts, which is almost constant.

25See Hansen [1982] and Hansen and Singleton [1982].
over different wealth and expertise levels. Thus, any wealth effects of higher expertise on the consumption savings decision appear to be quantitatively small.

Figure 10 plots the joint distribution of wealth and expertise. Figure 11 shows a histogram of wealth distributions conditional on different expertise levels. Figure 12 is kernel distribution of wealth conditional on expertise level. Both graphs show that there is more mass at lower wealth levels for low expertise levels.

Tables 4 and 9 present equilibrium cross sectional and aggregate moments from the model at the baseline calibration. The first row of Table 4 shows that the effective risk investors face decreases with expertise, and the second row displays the resulting increasing portfolio allocations to the risky asset. Combining these with the distribution of the population and wealth, displayed in rows four through six yields a summary of the composition of demand for the risky asset.

### 4.2 Cross Section: Comparative Statics

We perform two experiments in order to illustrate the effects of changes in parameters related to asset complexity on the equilibrium returns to the risky asset. Define the baseline calibration as Economy 1. In Economy 2, we double the entry cost relative to that of Economy 1. More complex asset classes presumably have higher required initial investments in expertise. Increasing the entry cost increases the wealth threshold for entry, and in this way limits the amount of wealth that can be allocated to the risky asset. As such, the effect is like a cash-in-the-market effect, but we allow for free entry.26 Tables 6, 5, and Table 9 describe the results, and demonstrate that an increase in the entry cost indeed leads to an increase in the equilibrium $\alpha$. Logically, this must be the case since when the entry threshold increases, a smaller fraction of the measure one of investors will be experts. Thus, the only way that those agents will either have more wealth in aggregate, or allocate more to the risky asset, is if $\alpha$ increases. Note that the higher $\alpha$ in turn lowers the entry and exit thresholds. In particular, in Table 5, there is less expert wealth overall, and correspondingly less than market clearing demand for the risky asset.

---

26See Allen and Gale [2005].
Table 6 then includes the general equilibrium effect on $\alpha$. As $\alpha$ increases, so does expert wealth, however on net the market clearing excess return is higher when the entry cost increases.

In Economy 3, we eliminate persistence in expertise, and instead specify an i.i.d. process. This should reduce the efficiency of the market by reducing the correlation between wealth and expertise since for any agent the expected growth rate of wealth should be uncorrelated over time. The results of this experiment are in Tables 8, 7 and Table 9. Again, $\alpha$ increases as the economy’s efficiency declines. Table 8 shows that average wealth is more similar across expertise levels when expertise is not persistent.

These two experiments essentially describe the two main effects of the joint wealth and expertise distribution on the risky asset return. The two key determinants of the equilibrium $\alpha$ are the total wealth of experts, and the correlation between wealth and expertise. Any parameter change which reduces expert wealth or the correlation between wealth and expertise should increase $\alpha$.

### 4.3 Time Series: Aggregate Shocks

Figure 7 plots the dynamics for $\alpha$ following a temporary negative return shock ($\nu = -10\%$), which we term a “wealth shock”, a permanent shock to fundamental return volatility $\sigma_\nu$ (an increase of 2%), which we term a “risk shock”, and the two shocks combined. We use a standard “reverse shooting” algorithm to solve for the transition dynamics. Table 10 describes the parameters used for these experiments. Clearly, the model can generate persistently elevated excess returns despite free entry. However, if the shock is to investor wealth only, the resulting increase in $\alpha$ will lead to a large jump in the entry rate in the period following the negative shock. On the other hand, if the shock also affects the volatility of asset returns, either directly through fundamental asset volatility, or by depreciating the effect of expertise (perhaps due to a new environment), then entry can decline even as $\alpha$ increases.
References


Figure 1: Share of Assets by Size Decile Within Strategies, 10=Largest

- Relative value fixed income - asset backed (21)
- Relative value fixed income - convertible arbitrage (18)
- Macro active trading (40)

- Equity hedge equity market neutral (7)
- Relative value volatility (19)
- Macro discretionary thematic (13)

- Macro systematic diversified (14)
- Macro currency systematic (15)
- Equity hedge quantitative directional (27)
Figure 2: Mean and Cross Sectional Standard Deviation of Asset Backed Fixed Income Hedge Fund Returns

![Graph showing mean and cross sectional standard deviation of asset backed fixed income hedge fund returns.]

Figure 3: Percentiles of Asset Backed Fixed Income Hedge Fund Returns, Entry and Liquidation Rates

![Graph showing percentiles of asset backed fixed income hedge fund returns, entry and liquidation rates.]

The lowest value of the 10th percentile of the return distribution (not shown) is -54%, on 09/30/2002.
Figure 4: Total and Average Assets, and Number of, Asset Backed Fixed Income Hedge Fund Returns

Figure 5: Actual Less Predicted Log Number of Asset Backed Fixed Income Hedge Funds

Residuals from a regression of log(number of ABFI funds) on a linear time trend.
Figure 6: Asset Backed Fixed Income Hedge Funds: Share of Assets by Size Decile

Figure 7: Transition Dynamics for $\alpha$ following a temporary negative return shock ($\nu = -10\%$), a permanent shock to fundamental return volatility $\sigma_\nu$ (an increase of 2%), and the two shocks combined.
Figure 8: Policy Function for Portfolio Choice $\theta$ by increasing levels of expertise
Figure 9: Policy Function of Investment Decision $i$ by increasing levels of expertise
Figure 10: Joint Distribution of Wealth and Expertise by increasing levels of expertise
Figure 11: Distribution of Wealth by Expertise Level
Figure 12: Distribution of wealth by increasing levels of expertise (kernel)
Figure 13: Distribution of Savings Rates $i$ by increasing levels of expertise
Table 1: Summary Statistics by HFR Strategy Classification.∗

<table>
<thead>
<tr>
<th>Strategy Code</th>
<th>TS AVG of Industry Returns</th>
<th>XS AVG of TS Return by Fund</th>
<th>TS AVG of XS STD of Returns</th>
<th>XS AVG of TS STD of Returns by Fund</th>
<th>Pseudo Sharpe Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>9.12 %</td>
<td>10.32 %</td>
<td>8.45 %</td>
<td>8.42 %</td>
<td>1.08</td>
</tr>
<tr>
<td>18</td>
<td>7.44 %</td>
<td>3.96 %</td>
<td>9.53 %</td>
<td>9.42 %</td>
<td>0.78</td>
</tr>
<tr>
<td>40</td>
<td>6.24 %</td>
<td>3.36 %</td>
<td>11.05 %</td>
<td>11.71 %</td>
<td>0.56</td>
</tr>
<tr>
<td>7</td>
<td>4.56 %</td>
<td>0.20 %</td>
<td>8.49 %</td>
<td>8.42 %</td>
<td>0.54</td>
</tr>
<tr>
<td>19</td>
<td>5.76 %</td>
<td>-0.60 %</td>
<td>11.67 %</td>
<td>15.03 %</td>
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<tr>
<td>13</td>
<td>6.60 %</td>
<td>1.68 %</td>
<td>16.80 %</td>
<td>14.27 %</td>
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</tr>
<tr>
<td>14</td>
<td>6.24 %</td>
<td>1.80 %</td>
<td>16.14 %</td>
<td>15.59 %</td>
<td>0.39</td>
</tr>
<tr>
<td>15</td>
<td>3.84 %</td>
<td>1.08 %</td>
<td>12.61 %</td>
<td>11.22 %</td>
<td>0.30</td>
</tr>
<tr>
<td>27</td>
<td>5.64 %</td>
<td>4.56 %</td>
<td>20.30 %</td>
<td>20.44 %</td>
<td>0.28</td>
</tr>
</tbody>
</table>

∗We use TS to denote “time series”, XS to denote “cross section”, AVG to denote “average”, and STD to denote “standard deviation”.

Table 2: HFR Strategy Classifications

<table>
<thead>
<tr>
<th>Strategy Code</th>
<th>Strategy Name</th>
<th>HFR Industry AUM in Millions</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>Relative Value Fixed Income - Asset Backed</td>
<td>$12,524</td>
</tr>
<tr>
<td>18</td>
<td>Relative Value Fixed Income - Convertible Arbitrage</td>
<td>$12,243</td>
</tr>
<tr>
<td>40</td>
<td>Macro Active Trading</td>
<td>$2,922</td>
</tr>
<tr>
<td>7</td>
<td>Equity Hedge Equity Market Neutral</td>
<td>$23,817</td>
</tr>
<tr>
<td>19</td>
<td>Relative Value Volatility</td>
<td>$5,672</td>
</tr>
<tr>
<td>13</td>
<td>Macro Discretionary Thematic</td>
<td>$16,521</td>
</tr>
<tr>
<td>14</td>
<td>Macro Systematic Diversified</td>
<td>$35,291</td>
</tr>
<tr>
<td>15</td>
<td>Macro Currency Systematic</td>
<td>$9,216</td>
</tr>
<tr>
<td>27</td>
<td>Equity Hedge Quantitative Directional</td>
<td>$13,481</td>
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Table 3: Baseline Calibration

<table>
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<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Target</th>
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<tbody>
<tr>
<td>Discount factor</td>
<td>( \beta )</td>
<td>0.95</td>
<td>Annual interest rate</td>
</tr>
<tr>
<td>Risk aversion</td>
<td>( \gamma )</td>
<td>2</td>
<td>Data/mean portfolio choice</td>
</tr>
<tr>
<td>Entry cost</td>
<td>( fnx )</td>
<td>0.1</td>
<td>Size of new entrants over average size</td>
</tr>
<tr>
<td>Maintenance cost</td>
<td>( fxx )</td>
<td>0.05</td>
<td>Size of exiters over average size</td>
</tr>
<tr>
<td>Survival probability</td>
<td>( p )</td>
<td>0.98</td>
<td>entry rate</td>
</tr>
<tr>
<td>Risky asset supply</td>
<td>( S )</td>
<td>0.2</td>
<td>Average annual excess return 3-4%</td>
</tr>
<tr>
<td>Volatility of risky asset return</td>
<td>( \sigma_v )</td>
<td>50%</td>
<td>Data</td>
</tr>
<tr>
<td>Persistency of expertise process</td>
<td>( \rho )</td>
<td>0.7</td>
<td>Correlation between wealth and expertise</td>
</tr>
<tr>
<td>Volatility of expertise process</td>
<td>( \sigma_e )</td>
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<td>vol of wealth</td>
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Table 4: Policy Functions and Moments Conditional on Expertise Level, Economy 1: Baseline Model

<table>
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<td></td>
<td>Level 1</td>
<td>Level 2</td>
</tr>
<tr>
<td>Volatility of Shock</td>
<td>0.3709</td>
<td>0.2915</td>
</tr>
<tr>
<td>Mean ( \theta )</td>
<td>0.2213</td>
<td>0.3232</td>
</tr>
<tr>
<td>Mean investment</td>
<td>0.9490</td>
<td>0.9201</td>
</tr>
<tr>
<td>Fraction of population</td>
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<td>0.0174</td>
</tr>
<tr>
<td>Mean wealth</td>
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<td>Total wealth</td>
<td>0.6867</td>
<td>0.0588</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0.0816</td>
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</table>
Table 5: Policy Functions and Moments Conditional on Expertise Level, Economy 2: $f_{nx} = 0.2$, PE

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<tr>
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<td>Level 1</td>
<td>Level 2</td>
</tr>
<tr>
<td>Volatility of Shock</td>
<td>0.3709</td>
<td>0.2915</td>
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<tr>
<td>Mean $\theta$</td>
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<td>Mean investment</td>
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<tr>
<td>Excess Return</td>
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<tr>
<td>Demand for Risky Asset</td>
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Table 6: Policy Functions and Moments Conditional on Expertise Level, Economy 2: $f_{nx} = 0.2$, GE

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<td>Level 2</td>
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<tr>
<td>Volatility of Shock</td>
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<tr>
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<td>Mean investment</td>
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<td>Total wealth</td>
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<td>Sharpe Ratio</td>
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</table>
Table 7: Policy Functions and Moments Conditional on Expertise Level, Economy 3: $\rho = 0$, PE

<table>
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<td>Level 2</td>
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<td>Level 4</td>
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<td>Mean $\theta$</td>
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<td>0.5441</td>
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<td>0.9183</td>
<td>0.9184</td>
<td>0.9190</td>
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<td>Mean wealth</td>
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<td>Excess Return</td>
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<td></td>
<td>0.0300</td>
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<td>Demand for Risky Asset</td>
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<td></td>
<td>0.1815</td>
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Table 8: Policy Functions and Moments Conditional on Expertise Level, Economy 3: $\rho = 0$, GE

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<th></th>
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<tr>
<td></td>
<td>Level 1</td>
<td>Level 2</td>
<td>Level 3</td>
<td>Level 4</td>
<td></td>
</tr>
<tr>
<td>Volatility of Shock</td>
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<td>0.2915</td>
<td>0.2085</td>
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<tr>
<td>Mean $\theta$</td>
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<td>Demand for Risky Asset</td>
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Table 9: Numerical Comparative Statics

<table>
<thead>
<tr>
<th>Model</th>
<th>Economy 1</th>
<th>Economy 2: $f_{nx} = 0.2$</th>
<th>Economy 3: $\rho = 0$</th>
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<tr>
<td></td>
<td>Fixed $\alpha$</td>
<td>Fixed $S$</td>
<td>Fixed $\alpha$</td>
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<tr>
<td>Mean wealth of experts</td>
<td>3.3318</td>
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<tr>
<td>Fraction of experts</td>
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</tr>
<tr>
<td>Total wealth of experts</td>
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<td>0.5028</td>
<td>0.5632</td>
</tr>
<tr>
<td>Mean wealth of non-experts</td>
<td>0.8274</td>
<td>0.9478</td>
<td>0.9389</td>
</tr>
<tr>
<td>Relative size of entry over average</td>
<td>37.51%</td>
<td>65.83%</td>
<td>54.71%</td>
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<tr>
<td>Weighted Sharpe Ratio</td>
<td>0.1155</td>
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Table 10: Parameterizations for IRF

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<th>Parameter</th>
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<th>Value</th>
<th>Target</th>
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<tr>
<td>Discount factor</td>
<td>$\beta$</td>
<td>0.95</td>
<td>Annual interest rate</td>
</tr>
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<td>Risk aversion</td>
<td>$\gamma$</td>
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<td>Data/mean portfolio choice</td>
</tr>
<tr>
<td>Entry cost</td>
<td>$f_{nx}$</td>
<td>0.1</td>
<td>Size of new entrants over average size</td>
</tr>
<tr>
<td>Maintenance cost</td>
<td>$f_{xx}$</td>
<td>0.05</td>
<td>Size of exiters over average size</td>
</tr>
<tr>
<td>Survival probability</td>
<td>$p$</td>
<td>0.98</td>
<td>entry rate</td>
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<td>Volatility of risky asset return</td>
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<td>Correlation between wealth and expertise</td>
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<tr>
<td>Volatility of expertise process</td>
<td>$\sigma_e$</td>
<td>0.2</td>
<td>vol of wealth</td>
</tr>
</tbody>
</table>
Appendix

Optimal Portfolio Choice

This section describes how to solve the optimal portfolio allocation problem in the static model. We also use upper case letters for level variables, and lower case letters for log variables. Under the assumptions in the main text, the optimization problem for an investor with wealth $W$ and expertise $E$, can be written as:

$$v(W, E) = \max_{\theta} E \left[ \frac{(WR_p)^{1-\gamma}}{1-\gamma} \right]$$

subject to

$$R_p = \theta R + (1 - \theta) R_f,$$
$$r| (W, E) \sim N \left( \mu - \frac{1}{2} \frac{\theta^2 \sigma_v^2}{E}, \frac{\sigma_v^2}{E} \right).$$

Campbell and Viceira [2002a] and Campbell and Viceira [2002b] shows that the log portfolio return $r_p$ can be approximated by:

$$r_p \approx r_f + \theta (r - r_f) + \frac{1}{2} \theta (1 - \theta) \frac{\sigma_v^2}{E}.$$

As a result,

$$r_p | (W, E) \sim N \left( r_f + \theta (\mu - r_f) - \frac{1}{2} \theta^2 \frac{\sigma_v^2}{E}, \theta^2 \frac{\sigma_v^2}{E} \right).$$

Then the value function equals:

$$v(W, E) = \max_{\theta} \frac{W^{1-\gamma}}{1-\gamma} \exp \left( (1 - \gamma) r_f + (1 - \gamma) \theta (\mu - r_f) - \frac{1}{2} \gamma (1 - \gamma) \theta^2 \frac{\sigma_v^2}{E} \right).$$

Hence, The investor’s optimization problem becomes:

$$\max_{\theta} \left\{ \theta (\mu - r_f) - \frac{\gamma \theta^2 \sigma_v^2}{2 E} \right\}.$$
**Equilibrium Market Excess Return**

This section describes how to derive the equilibrium market excess return, $\alpha$, from log expected return, $\mu$, given all parameters. Because

$$ r \mid (W, E) \sim N \left( \mu - \frac{1}{2} \frac{\sigma_r^2}{\sigma_E^2} , \frac{\sigma_r^2}{\sigma_E^2} \right), $$

Then

$$ E (R \mid W, E) = \exp (\mu). $$

In addition,

$$ E [R] = E [E (R \mid W, E)]. $$

Hence,

$$ E [R] = \exp (\mu). $$

Finally,

$$ \alpha = \exp (\mu) - R_f. $$
Equilibrium Equally Weighted Market Sharpe Ratio

This section describes how to derive the equilibrium equally weighted market Sharpe Ratio, $SR$, from the log expected return, $\mu$, given all parameters. Because

$$r| (W,E) \sim N \left( \mu - \frac{1}{2} \frac{\sigma^2}{E}, \frac{\sigma^2}{E} \right),$$

Then

$$VAR (R|W, E) = \exp (2\mu) \left( \exp \left( \frac{\sigma^2}{E} \right) - 1 \right).$$

In addition, we have proven that

$$E (R|W, E) = E [R] = \exp (\mu).$$

Therefore, the equally weighted variance of the risky asset, is given by:

$$VAR [R] = E (VAR (R|W, E)) = \exp (2\mu) \left[ E \left[ \exp \left( \frac{\sigma^2}{E} \right) \right] - 1 \right].$$

Hence, the equally weighted market Sharpe Ratio, can be written as:

$$SR = \frac{1 - R_f \exp (-\mu)}{\sqrt{E \left[ \exp \left( \frac{\sigma^2}{E} \right) \right] - 1}},$$
Equilibrium Investor-Specific Sharpe Ratio

This section describes how to derive the equilibrium investor-specific Sharpe Ratio, $SR(W, E)$, from log expected return, $\mu$, given all parameters. For an investor with wealth $W$ and expertise $E$, Because

$$r| (W, E) \sim N \left( \mu - \frac{1}{2} \frac{\sigma_\nu^2}{E}, \frac{\sigma_\nu^2}{E} \right),$$

Then

$$E(R|W, E) = \exp(\mu),$$

And

$$VAR(R|W, E) = \exp(2\mu) \left( \exp \left( \frac{\sigma_\nu^2}{E} \right) - 1 \right).$$

Hence, the investor-specific Sharpe Ratio is given by:

$$SR(W, E) = \frac{1 - R_f \exp(-\mu)}{\sqrt{\exp \left( \frac{\sigma_\nu^2}{E} \right) - 1}}$$
Proof of Lemma 2 and 3

This section describes how to prove lemma 2 and 3. From equations (2), (6), (7) and (8), we can derive that, if $\eta$ denotes any parameter $\eta \in \{\gamma, \mu, \sigma, \rho, e\}$:

1. $\frac{\partial \alpha}{\partial \eta} = \exp(\mu) \frac{\partial \mu}{\partial \eta}$;
2. $\frac{\partial (SR)}{\partial \eta} = \frac{R_f \exp(-\mu)}{\sqrt{E}) \exp(\sigma^2 \nu E)}^{-1} \frac{\partial \mu}{\partial \eta}$;
3. $\frac{\partial SR(W,E)}{\partial \eta} = \frac{R_f \exp(-\mu)}{\sqrt{\exp(\sigma^2 \nu E)} - 1} \frac{\partial \mu}{\partial \eta}$;
4. $\frac{\partial VAR(SR(W,E))}{\partial \eta} = 2 (1 - R_f \exp(-\mu)) R_f \exp(-\mu) V A R \left( \frac{1}{\sqrt{\exp(\sigma^2 \nu E)} - 1} \right) \frac{\partial \mu}{\partial \eta}$;
5. $\frac{\partial^2 SR(W,E)}{\partial \eta \partial E} = \frac{\partial}{\partial \eta} \left( \frac{R_f \exp(-\mu)}{\sqrt{\exp(\sigma^2 \nu E)} - 1} \right) \frac{\partial^2 \mu}{\partial E \partial \eta}$;
6. $\frac{\partial^2 \theta^*(W,E)}{\partial \eta \partial E} = \frac{1}{\gamma \sigma^2 \nu E} \frac{\partial \mu}{\partial \eta}, \forall \eta \neq \gamma$, and $\frac{\partial \theta^*(W,E)}{\partial \gamma} = 0$;
7. $\frac{\partial SR(W,E)}{\partial \sigma^2 \nu} = \frac{R_f \exp(-\mu) \frac{\nu - 1}{\sigma^2 \nu} - \frac{1}{2} (1 - R_f \exp(-\mu)) \frac{\exp(\sigma^2 \nu E)}{\exp(\sigma^2 \nu E)}^{-1}}{\exp(\sigma^2 \nu E)}^{-1}$.

Hence, $\text{Sign} \left( \frac{\partial \mu}{\partial \eta} \right) = \text{Sign} \left( \frac{\partial \alpha}{\partial \eta} \right) = \text{Sign} \left( \frac{\partial (SR)}{\partial \eta} \right) = \text{Sign} \left( \frac{\partial SR(W,E)}{\partial \eta} \right) = \text{Sign} \left( \frac{\partial VAR(SR(W,E))}{\partial \eta} \right) = \text{Sign} \left( \frac{\partial^2 SR(W,E)}{\partial \eta \partial E} \right) = \text{Sign} \left( \frac{\partial^2 \theta^*(W,E)}{\partial \eta \partial E} \right)$
In addition, we have:

1. Because \(\frac{\exp\left(\frac{\sigma^2}{E}\right)}{\exp\left(\frac{\sigma^2}{E}\right)} > \frac{1}{E}, \forall E\),

then \(\frac{\partial SR(W,E)}{\partial \sigma^2} < \frac{R_f \exp(-\mu) \rho - \frac{1}{2} \left(1 - R_f \exp(-\mu)\right) \frac{1}{E}}{\exp\left(\frac{\sigma^2}{E}\right)} < 0, \forall E < \bar{E}\),

where \(\bar{E} = \frac{1}{2} \left(1 - R_f \exp(-\mu)\right) \rho > 0;\)

2. \(0 = \frac{1}{\sigma^2} \left(1 + \frac{\sigma^2}{E}\right) - \frac{1}{E} - \frac{1}{\sigma^2} \exp\left(\frac{\sigma^2}{E}\right) - \frac{1}{\sigma^2} = \left(\exp\left(\frac{\sigma^2}{E}\right) - 1\right) \left(\frac{1}{E} + \frac{1}{\sigma^2}\right) - \exp\left(\frac{\sigma^2}{E}\right) \frac{1}{E},\)

then \(\frac{\exp\left(\frac{\sigma^2}{E}\right)}{\exp\left(\frac{\sigma^2}{E}\right)} \frac{1}{E} < \frac{1}{E} + \frac{1}{\sigma^2}, \forall E,\)

and \(\frac{\partial SR(W,E)}{\partial \sigma^2} > \frac{R_f \exp(-\mu) \rho - \frac{1}{2} \left(1 - R_f \exp(-\mu)\right) \left(\frac{1}{E} + \frac{1}{\sigma^2}\right)}{\exp\left(\frac{\sigma^2}{E}\right)} > 0, \forall E > \bar{E},\)

where \(\bar{E} = \frac{1}{2} \left(1 - R_f \exp(-\mu)\right) \rho \frac{1}{\sigma^2}\).

Since we have proved \(\exists E > 0\) such that \(\forall E < \bar{E}, \frac{\partial SR(W,E)}{\partial \sigma^2} < 0\), and \(E\) is bounded above 0, then we must have \(\bar{E} > 0\). Otherwise, for \(\bar{E} \leq 0 < \forall E < \bar{E}, \frac{\partial SR(W,E)}{\partial \sigma^2} > 0\) and \(\frac{\partial SR(W,E)}{\partial \sigma^2} < 0\), contradiction. Then it’s obvious that \(\bar{E} > E\).

In sum, \(\exists \bar{E} > \bar{E} > 0\) such that \(\forall E > \bar{E}, \frac{\partial SR(W,E)}{\partial \sigma^2} > 0, \forall E < \bar{E}, \frac{\partial SR(W,E)}{\partial \sigma^2} < 0\). The signs for comparative statics for \(E < \forall E < \bar{E}\) are indeterminate. The general functional form for effective risk yields similar results.
Equilibrium Value-Weighted Market Sharpe Ratio

This section shows that our main conclusions still hold with respect to the value-weighted equilibrium market Sharpe Ratio. Because

\[ r| (W, E) \sim N \left( \mu - \frac{1}{2} \frac{\sigma_v^2}{E}, \frac{\sigma_v^2}{E} \right), \]

Then

\[ E (R|W, E) = \exp (\mu) . \]

Then the value-weighted market expected return also equals to:

\[ \exp (\mu) , \]

In addition,

\[ \text{VAR} (R|W, E) = \exp (2\mu) \left( \exp \left( \frac{\sigma_v^2}{E} \right) - 1 \right) . \]

Therefore, the value-weighted variance of the risky asset, is given by:

\[
\int \int \text{VAR} (R|W, E) \frac{\exp(w)\theta^*(\exp(e)) f(w, e)}{\int \int \exp(w)\theta^*(\exp(e)) f(w, e) dw de} dw de,
\]

Which equals to

\[
\exp (2\mu) \frac{E \left[ \left( \exp \left( \frac{\sigma_v^2}{E} \right) - 1 \right) \exp(w + e) \right]}{E} .
\]

Hence, the value-weighted market Sharpe Ratio, can be written as:

\[
\frac{1 - R_f e^{-\mu}}{\sqrt{E \left[ \exp \left( \frac{\sigma_v^2}{E} \right) - 1 \right] \exp(w + e)}} .
\]

Then, if \( \eta \) denotes any parameter \( \eta \in \{ \gamma, S \} \),

\[
\frac{\delta (SR)}{\delta \eta} = \frac{R_f e^{-\mu}}{\sqrt{E \left[ \exp \left( \frac{\sigma_v^2}{E} \right) - 1 \right] \exp(w + e)}} \frac{\delta \mu}{\delta \eta} .
\]

We can't prove lemma 2 and 3 for \( \eta \in \{ \mu_w, \sigma_w, \rho_{w,e} \} \), as the derivatives are indeterminate.
Wealth effect of Expertise

This section shows that while savings rates can theoretically be slightly decreasing in expertise, due to the wealth effect from higher expertise and the associated larger present value of investment opportunities, this effect tends to be dominated by the portfolio choice effect.

The static model with a consumption savings decision can be written as:

\[ v(W, E) = \max_{(I, \theta)} \left( \frac{W - I}{1 - \gamma} + \beta I^{1-\gamma} E \left[ \frac{R_{p}^{1-\gamma}}{1 - \gamma} \right] \right) \]

subject to:

\[ R_{p} = \theta R + (1 - \theta) R_{f}, \]

\[ r \mid (W, E) \sim N \left( \mu - \frac{1}{2} \frac{\sigma_{v}^{2}}{E}, \frac{\sigma_{v}^{2}}{E} \right). \]

Clearly, the portfolio choice problem is independent from the consumption savings decision, and the solution to the portfolio choice problem coincides with that of the static model without the consumption saving decision. For any choice of investment \( I \), the optimal portfolio allocation always solves the same problem, maximizing the expected utility derived from the chosen investment level, given the return process for the riskless and risky assets. Therefore, we can plug the optimal portfolio choice back into the value function, and then derive the optimal investment. Finally we get:

\[ I^{*} = W \frac{(\beta E \left[ R_{p}^{1-\gamma} \right])^{\frac{1}{\gamma}}}{1 + (\beta E \left[ R_{p}^{1-\gamma} \right])^{\frac{1}{\gamma}}} \]

where

\[ E \left[ R_{p}^{1-\gamma} \right] = \exp \left( (1 - \gamma) r_{f} + \frac{1}{2} \frac{(1 - \gamma)(\mu - r_{f})^{2}}{\sigma_{v}^{2}/E} \right). \]

Then, we can show that:

\[ \frac{\partial I^{*}}{\partial E} = W \frac{(\beta E \left[ R_{p}^{1-\gamma} \right])^{\frac{1}{\gamma}}}{\left( 1 + (\beta E \left[ R_{p}^{1-\gamma} \right])^{\frac{1}{\gamma}} \right)^{2}} \frac{1}{2} \frac{(\gamma - 1)(\mu - r_{f})^{2}}{\left( \sigma_{v}^{2}/E \right)^{2}} \frac{\partial \left( \sigma_{v}^{2}/E \right)}{\partial E}. \]

Observe that the saving rate decreases with the expertise if and only if \( \gamma > 1 \).
However, for the investment in the risky asset, $I^*\theta^*$, we have:

$$I^*\theta^* = W \frac{(\beta E [R_{p}^{1-\gamma}])^{\frac{1}{\gamma}} (\mu - r_f)}{1 + (\beta E [R_{p}^{1-\gamma}])^{\frac{1}{\gamma}} \gamma \frac{\sigma^2}{E}}.$$  

Then,

$$\frac{\partial (I^*\theta^*)}{\partial E} = W \frac{(\beta E [R_{p}^{1-\gamma}])^{\frac{1}{\gamma}} (\mu - r_f) \left( \frac{1}{2} (\gamma - 1) \left( \frac{\sigma^2}{E} \right)^2 \frac{1}{\gamma (\beta E [R_{p}^{1-\gamma}])^{\frac{1}{\gamma}} - 1 \right) \frac{\partial (\sigma^2)}{\partial E} \right)}{1 + (\beta E [R_{p}^{1-\gamma}])^{\frac{1}{\gamma}} \gamma \frac{\sigma^2}{E}}.$$  

There are two cases, depending on the coefficient of relative risk aversion:

1. If $\gamma < 1$, the saving rate does not fall with the expertise, neither does the investment in the risky asset.

   We have $\frac{1}{2} (\gamma - 1) \left( \frac{\sigma^2}{E} \right)^2 \frac{1}{1 + (\beta E [R_{p}^{1-\gamma}])^{\frac{1}{\gamma}}} - 1 < 0$.

   Therefore, $\frac{\partial (I^*\theta^*)}{\partial E} > 0$, $\forall E$.

2. If $\gamma > 1$, the saving rate falls with the expertise, while the investment in the risky asset doesn't, as long as the expertise level is not too high.

   We have $\frac{1}{2} (\gamma - 1) \left( \frac{\sigma^2}{E} \right)^2 \frac{1}{1 + (\beta E [R_{p}^{1-\gamma}])^{\frac{1}{\gamma}}} - 1 < \frac{1}{2} (\gamma - 1) \left( \frac{\sigma^2}{E} \right)^2 - 1$, $\forall E$.

   Then $\frac{1}{2} (\gamma - 1) \left( \frac{\sigma^2}{E} \right)^2 \frac{1}{1 + (\beta E [R_{p}^{1-\gamma}])^{\frac{1}{\gamma}}} - 1 < 0$, $\forall E < \bar{E}$, where $\bar{E} = \frac{2\gamma^2}{(\gamma - 1) (\mu - r_f)^2}$.

   Therefore, $\frac{\partial (I^*\theta^*)}{\partial E} > 0$, $\forall E < \bar{E}$. The signs for comparative statics for $\forall E > \bar{E}$ are indeterminate.

In sum, investment in the risky asset increases with expertise, as long as the expertise level is not too high. The general functional form for effective risk yields similar results.
Computation of the Steady State

This section outlines the computation of the stationary joint distribution of wealth and expertise.

1. Set discrete grids on $w$ and $e$. Set the Markov transition matrix for $e$. Set the realization shocks of risky asset returns, $\varepsilon$.

2. Optimization loop. Objects: value functions $v_x(w, e), v_n(w), v(w, e)$, policy functions $c_x(w, e), c_n(w), i_x(w, e), i_n(w), \theta(w, e)$

   (a) Given the initial value of $v_x(w, e), v_n(w), v(w, e)$, where $w$ and $e$ are the realization in the last period.

   (b) Calculate value of being non-expert

   $v_n(w) = \max_{c,i} \frac{c^{1-\gamma}}{1-\gamma} + \beta E \left[ pv(w', e') + (1-p) \frac{w'^{1-\gamma}}{1-\gamma} \right]$ 

   subject to

   $c = w(1-i), \quad w' = iw(1+r_f), \quad e' = e_0.$

   record the decision rules for $c_n(w), i_n(w)$.

   (c) Calculate value of being expert

   $v_x(w, e) = \max_{c,i,\theta} \frac{c^{1-\gamma}}{1-\gamma} + \beta E \left[ pv(w', e') + (1-p) \frac{w'^{1-\gamma}}{1-\gamma} \right]$ 

   subject to

   $e' = \xi + \rho e + \eta' \quad c + f_{xx}\mathbb{I}_{t=x} + f_{nx}\mathbb{I}_{t=n} = w(1-i)$

   $w' = iw(1+\theta r' + (1-\theta)r_f)$

   $r' = r_f + \alpha + v'\sigma(e)$

   record the decision rules for $c_x(w, e), i_x(w, e), \theta(w, e)$

   (d) Compute entry and exit decision rules

   entry iff $v_x(w, e) \geq v_n(w)$

   exit iff $v_x(w, e) \leq v_n(w)$
(e) Update value functions

\[ v'(w,e) = \max \left\{ v_x(w,e), v_n(w) \right\}. \]

Until

\[ \|v' - v\| \leq 10^{-4}. \]

3. Solving equilibrium \( \alpha \). Given policy functions, we simulate an economy with 5,000 agents for 300 periods. We dropped first 100 periods to compute the demands of risky assets.

(a) Initialize matrix for wealth, expertise level, death shocks, and shocks of risky asset returns

(b) Compute the portfolio returns given the realization of shocks and update wealth of each agent, expert and non-expert.

(c) Interpolate policy functions of \( i_x(w,e) \), \( i_n(w) \), \( \theta(w,e) \), for given \( w \) and \( e \), get simulated \( i_x^*(w,e) \), \( i_n^*(w) \), \( \theta^*(w,e) \)

(d) Update entry/exit decisions and record indicator for being expert

(e) If non-experts enter the market, update simulated expertise process for them with initial value \( e_0 \)

(f) If agent dies, replace with a new agent with initial wealth 90% of entry threshold.

(g) Calculate the simulated stationary joint distribution of wealth and expertise, \( f^*(w,e) \). Obtain the aggregate demand of risky asset

\[ S' = \int \int f(w,e) \theta^*(w,e) i_x^*(w,e) \, dw \, de. \]

(h) If \( S' - S > 10^{-2} \), decrease \( \alpha \); If \( S' - S < -10^{-2} \), increase \( \alpha \); Back to Step 2.
Model with Homogeneity in Wealth

We consider a version of the dynamic model in which the maintenance cost and entry cost are proportional to wealth: $F_{xx} = f_{xx}w$ and $F_{nx} = f_{nx}w$. As a result, the value functions are homogenous in wealth.

**Proposition 2** In the model with homogeneous entry and maintenance costs, the value functions are given by

$$v_e(t, w, e) = \frac{w^{1-\gamma}}{1-\gamma} f(t, e),$$

$$v_n(t, w) = \frac{w^{1-\gamma}}{1-\gamma} g(t, e),$$

and the optimal policy functions are given by: $i_e(t, w, e) = x(t, e), i_n(t, w) = y(t)$, $c_e(t, w, e) = (1 - x(t, e) - f_x) w$, $c_n(t, w) = (1 - y(t)) w$.

**Corollary 3** The optimal policy functions $\theta(e), x(t, e), y(t, e)$ and the value functions $g(t, e)$ and $f(t, e)$ are determined by the following conditions, respectively:

$$0 = E \left[ (1 + r_f + \theta r')^{-\gamma} r'_s \right],$$

$$x(e) \gamma (1 - x(e) - f_{xx})^{-\gamma} = \beta E \left[ (p \min \{ f(e'), g(e') \} + (1 - p)) \right] E \left[ (1 + r_f + \theta(e) r'_s)^{1-\gamma} \right],$$

$$y(e) \gamma (1 - y(e))^{-\gamma} = \beta E \left[ (p \min \{ f(e'), g(e') \} + (1 - p)) \right] E \left[ (1 + r_f)^{1-\gamma} \right],$$

$$g(e) = (1 - y(e))^{-\gamma},$$

$$f(e) = (1 - x - f_{xx} \eta_{t=x} - f_{nx} \eta_{t=n})^{-\gamma}.$$  

**Proof.** We prove this Proposition by guess and verify. The value function of an investor entering the period as an expert is given by the following functional equation:

$$v_e(t, w, e) = \max_{i, \theta} \frac{(w - iw - f_{xx}w \eta_{t=x} - f_{nx}w \eta_{t=n})^{1-\gamma}}{1-\gamma} + \beta E[pv(t', w', e') + (1 - p) \frac{w^{1-\gamma}}{1-\gamma}]$$

s.t. $w' = wi(1 + r_f + \theta r')$

$$r'_s = \alpha(I) + v' \sigma(e)$$

$$e' = \xi + pe + \eta$$

The value function of an investor entering the period as a non-expert is given by the following functional equation:

$$v_n(t, w) = \max_{i, \theta} \frac{w^{1-\gamma} (1 - i)^{1-\gamma}}{1-\gamma} + \beta E[pv(t', w', e') + (1 - p) \frac{w^{1-\gamma}}{1-\gamma}]$$

s.t. $w' = i(1 + r_f)$.
Finally, the investor’s value function is determined by the maximum of the expert and the non-expert value function:

\[ v(t, w, e) = \max \{ v_e(t, w, e), v_n(t, w) \} . \]

Given the homogeneity of this optimization problem, we conjecture that the value functions take the following form:

\[ v_e(t, w, e) = \frac{w^{1-\gamma}}{1-\gamma} f(t, e), \quad v_n = \frac{w^{1-\gamma}}{1-\gamma} g(t), \]

and we conjecture that the policy functions take the following form:

\[ i^e = x(t, e), \quad i^n = y(t). \]

Given these conjectured expressions, the law of motion for wealth, the first order condition for \( x \), the first order condition for \( \theta \), the first order condition for \( y \), the envelope condition for \( x \) and the envelope condition for \( y \) take the following form:

\[
\begin{align*}
\beta E \left[ (1 + r_f + \theta r_s') \right. & = x w (1 + r_f + \theta r_s'), \\
\beta E \left[ (1 + r_f + \theta r_s') \right. & = (1 - x - f_{xx} \gamma^{-\gamma} w^{-\gamma}, \\
\beta E \left[ (1 + r_f + \theta r_s') \right. & = 0, \\
\beta E \left[ (1 + r_f + \theta r_s') \right. & = (1 - y)^{\gamma} w^{-\gamma}, \\
\beta E \left[ (1 + r_f + \theta r_s') \right. & = (1 - y)^{\gamma}, \\
\beta E \left[ (1 + r_f + \theta r_s') \right. & = (1 - x - f_{xx} \gamma^{-\gamma} w^{-\gamma}).
\end{align*}
\]

Substitute for \( w' \) by \( x w (1 + r_f + \theta r_s') \), we obtain the following expressions

\[
\begin{align*}
E \left[ (1 + r_f + \theta r_s')^{-\gamma} r_s' \right. & = 0, \\
x^\gamma (1 - x - f_{xx} \gamma^{-\gamma} & = \beta E \left[ (p \min \{ f(e'), g(e') \} + (1 - p) \right) E \left[ (1 + r_f + \theta r_s')^{1-\gamma} \right], \\
y^\gamma (1 - y)^{-\gamma} & = \beta \gamma E \left[ (p \min \{ f(e'), g(e') \} + (1 - p) \right) E \left[ (1 + r_f)^{1-\gamma} \right], \\
g(e) & = (1 - y)^{-\gamma}, \\
f(e) & = (1 - x - f_{xx} \gamma^{-\gamma} w^{-\gamma}).
\end{align*}
\]

The first equation determines \( \theta \), which is independent of wealth. The second and the third equations solve the investment choices \( x \) and \( y \) for experts and non-experts respectively, and it is independent of wealth. The fourth and the fifth equations determine the value function terms \( f \) and \( g \). Note that, given the conjectured value and policy functions, none of these conditions depend on wealth \( w \). q.e.d. ■
Given the homogeneity of the value functions, the investors’ entry and exit decisions do not depend on wealth. Current expert exits the market iff
\[
\frac{f(e)}{1-\gamma} < \frac{g(e)}{1-\gamma}.
\]
Non-experts become experts iff
\[
\frac{f(e)(1-f_{xx})^{1-\gamma}}{1-\gamma} \geq \frac{g(e)}{1-\gamma}.
\]

**Equilibrium** We define \( \mathcal{W} = [0, \infty) \) to be the set of possible asset holdings. Define \( Z = \mathcal{W} \times E \). The joint measure \( \Phi \) over wealth and expertise will live on \( M = (Z, \mathcal{B}(Z)) \), where \( \mathcal{B}(Z) = \mathcal{P}(E) \times \mathcal{B}(\mathcal{W}) \). We can define the transition function
\[
Q((e,w),(E,W)) : (Z, \mathcal{B}(Z)) \rightarrow [0,1]
\]

Let \( \Phi(t,w,e) \) denote the stationary distribution over wealth and expertise. \( W \) denotes the total wealth invested in the risky asset:
\[
W = \int \Phi \times \theta \times w(t,w,e) d\Phi(t,w,e).
\]

The production of \( \alpha \) is subject to decreasing returns to scale. We posit that \( \alpha = \Psi(W) \) is a decreasing function of the aggregate amount of wealth invested in the risky technology.

**Definition 4 Stationary Equilibrium** An equilibrium consists of an \( \alpha \), policy functions for investor types, \( t \), savings rates \( i \), and portfolio allocations, \( \theta \), and a stationary measure over individual states \( \Phi(t,w,e) \), such that the policy functions solve the investors’ problem described in (11) to (22), and such that the equilibrium \( \alpha \) is generated by the equilibrium amount of aggregate wealth invested in the risky technology
\[
\Psi \left( \int \Lambda \ w(t,w,e)\theta \ (t,w,e) \ d\Phi(t,w,e) \right) = \alpha.
\]
and such that for all $\mathcal{E}, \mathcal{W} \in \mathcal{B}(Z)$, the stationary measure reproduces itself:

$$\Phi(\mathcal{T}, \mathcal{W}, \mathcal{E}) = \int Q((t, w, e), (\mathcal{T}, \mathcal{W}, \mathcal{E})) d\Phi,$$

where $Q$ is the transition functions implied by the optimal policy functions.