Selling assets: When is the whole worth more than the sum of its parts?

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Abstract

When is it better to sell assets separately versus pooling them together? We study this issue in a setting where the number of potential buyers for an asset is endogenous and is a function not just of the costs associated with due diligence but also of the composition of the set of other possible buyers. We show that if buyers anticipate that there may be another buyer who in expectation has a higher value for the target, they will be less likely to enter since they anticipate being less likely to win, and having to pay a higher price when they do win. This reduces competition and lowers revenue for the seller of the asset. This reduction in competition, however, is different depending on how the assets are sold and is attenuated when assets are sold together relative to when they are sold separately. We characterize when this competitive effect leads the seller to prefer to sell the assets together rather than individually.

1 Introduction

Firms are frequently sold off piecemeal, with each asset auctioned off to the highest bidder. In fact, evidence suggests this practice was fairly commonplace during the 1970s and 1980s, the decades of the "corporate raiders" that would buy companies for the primary purpose of breaking them up and selling the assets individually, a practice often referred to as "asset stripping". The logic for stripping the firm in this way is presumably that there is a high likelihood that buyers with high values can be found for individual assets, but it is less likely that any given buyer values highly all assets that comprise the firm. The most value can therefore be generated though a piecemeal sale, so that the sum of the parts winds up being worth more than the whole.

Despite this often cited pattern, many firms regularly turn down opportunities to sell their own assets individually, even when it is apparent that there are buyers for the individual assets that value those assets highly. A salient recent example is the case of Blackberry Ltd., whose board in November of 2013 rejected proposals from several technologies companies for various assets, arguing that breaking up the assets was not in the best interest of the company's stakeholders. This action was considered striking by market observers given Blackberry's obvious need for cash and restructuring, and given the apparent interest in Blackberry's high value patents by companies such as Microsoft Corp. and Apple Inc. Blackberry's decision to remain whole and continue to pursue options to either recapitalize or sell the entire company as a going concern suggests a view that the greatest value may not always be obtained through a piecemeal sale or liquidation.

In this paper, we study the optimal way of selling a firm's assets, comparing the sale of the firm in it's entirety with the revenue obtained from selling the assets individually. Our starting point is the observation above that the external value of individual assets that can be redeployed to a better use is likely to be higher than the value of the collection of assets.

In other words, we assume that for each asset there may well be a buyer who values that asset highly, but it is less likely that the same buyer values multiple assets highly. We call such buyers "efficient" since they are able to make better use of the specific assets and hence (in expectation) value the assets more highly. The assets, whether individually or jointly, are sold using the standard mechanism of an ascending price auction. One important component of our analysis is that we endogenize the entry of possible buyers by assuming that they face a bidding or information acquisition cost: each buyer must incur an upfront and fixed cost in order to learn his value for the asset for sale and submit a bid. This cost can be interpreted as a due diligence cost, a filing fee when bidding for an entire company, a search cost, etc.

We first show that when an efficient buyer is present, other buyers may be less willing to enter the auction since doing so requires them to bear a cost. The reason is that buyers recognize that they are less likely to successfully acquire the asset when other buyers are likely to have higher values for the asset. The presence of an efficient buyer therefore endogenously reduces competition for the assets being sold. Strikingly, this reduction in competition, in turn, reduces the premium the seller receives for the sale of his asset. In other words, the reduction in competition is first order relative to the increase in overall value created by the presence of the efficient bidder.

However, while an increase in efficiency of a subset of bidders always reduces revenue, it has a larger effect in the case of an individual asset sale than when assets are sold jointly. The reason is that when assets are sold jointly, a bidder's efficiency for one of the assets has a smaller effect on other buyers' possibilities for winning the entire firm. Therefore, increases in efficiency reduce competition less when assets are sold jointly than when they are all sold separately. In fact, we show that when some buyers are sufficiently efficient, selling the assets jointly is optimal (i.e., it generates more revenue for the seller) rather than selling them individually.

Our analysis has implications for the optimal bundling of assets for sale. We abstract from other considerations that may drive acquisition premiums, such as complementarity across the assets, correlation in values, etc., in order to focus on how competition for the assets may be affected by the way in which they are packaged. The main implication is that competition may depend, endogenously, on whether the assets are sold together or separately, which, in turn, has important consequences for the premium a seller may expect to receive. Paradoxically, having high value bidders actually reduces revenue through the reduction in competition it engenders, but has a larger impact when assets are sold separately than when they are sold together since then the effect of a high value of an asset is muted.

The literature on the asset sales that focuses on joint versus separate sales is somewhat limited. In the context of bankruptcy auctions, Eckbo and Thorburn (2008) find empirically that "fire sale discounts" are observed when the the assets are liquidated in a piecemeal fashion, but not when they are sold jointly so that the firm is acquired as a going concern. For the case where the sale of assets are voluntary, Hite et al. (1987) find that partial firm sell offs lead to small abnormal returns for sellers. By contrast, the abnormal returns are much larger and significant for proposals to liquidate the entire firm. A different focus is found in Schlingemann et al. (2002), who study corporate divestitures as a function of the liquidity of the market for assets. However, to the best of our knowledge, no work has studied how bidder heterogeneity may be a determinant of how assets are sold. From a theory perspective, to the closest work to ours is Chakraborty (1999), who studies the question of when auctioning two "bundled" assets is preferable to conducting two separate second price auctions for the individual assets. Our work builds on his, as well as on Chakraborty (2006), to study the implications of having bidders with higher than average values participate in the auction.

An important component of our analysis is the fact that competition is endogenous and is determined by a standard free entry condition. Most analyses of auctions take the number of bidders as exogenous and analyze the consequences of varying the auction format, bidders' information sets, or the number of bidders. Moreover, most of the literature that analyzes the effects of changes in the number of bidders, even for cases where participation is endogenous, assumes that all bidders are symmetric and focuses either on limiting behavior as the number of bidders grows arbitrarily large, or on changes in the costs of participation.[1] This literature therefore ignores how the presence of stronger bidders may deter entry and thus have important consequences for equilibrium prices.

Closer to our work is that of McAfee and McMillan (1987), who consider a setting where bidders must incur a cost in order to learn their private valuations and enter the bidding.[2] They show that in such a setting, the use of an entry fee by the seller is not optimal as it further reduces entry and leads to a lower sales price. Levin and Smith (1994) also analyze a similar setting, but study the case where there is a fixed number of potential bidders and focus on the symmetric entry equilibrium among the possible bidders, so that each bidder has a positive probability of bidding. They show that entry may sometimes be socially excessive (as in Mankiw and Whinston, 1986), and that increasing the number of potential bidders may actually reduce seller revenue because of the greater costs associated with "thicker" markets. These papers do not, however, study how even small bidder asymmetries can have exactly the opposite effect under free entry than when the number of bidders is fixed exogenously. Marquez and Singh (2013) analyze a related issue in the context of "club bidding" by private equity firms. They consider only the case of bidder values that are uniformly distribution, and of a club whose value is the sum of two uniforms. None of these papers, however, allow for the possibility that the seller may break up the asset into separate parts as a way of generating more revenue.

[1]Some examples are McAfee and McMillan (1987) and Levin and Smith (1994), among others. We discuss our relationship to this literature below.

[2]Harstad (1990) studies a similar setup for common value auctions. Lu and Ye (2013) consider the question of revenue maximization and auction design in the context of free entry, where bidders must pay "information acquisition" costs, much as in our setup. We discuss this paper further in Section **??**.

2 Model

Suppose that there is a firm that owns two assets, A and B. These assets can be interpreted as divisions of the firm, or as PPE or other productive assets. The owners (i.e., shareholders or the board) of the firm want to sell off the company and raise the largest revenue possible from the sale. The question is whether to sell the company as a whole, or to sell the assets separately.

For each asset $k \in \{A, B\}$, there is a large pool of risk neutral buyers potentially interested in buying it. All but two buyers are "regular", which means that buyer i's value x_k^i for asset k is drawn from the distribution $F(.)$, with support in $[0, 1]$.[3] Buyer i's value for the firm as a whole is therefore $X^i = x_A^i + x_B^i$. The other two potential buyers, which we denote by α and β, are "efficient" in the sense that the value y^α of bidder α for asset A is drawn from a distribution $G(.|\varphi)$ with support also in $[0, 1]$, and his value x^α for asset B is drawn from $F(.)$. Symmetrically, bidder β has a value y^β for asset B drawn from distribution $G(.|\varphi)$ and value x^β for asset A drawn from distribution $F(.)$. We assume that $G(z|0) = F(z)$, but that $\frac{\partial G}{\partial \varphi} < 0$, so that $G(z|\varphi) < F(z)$ for all $z \in (0, 1)$ when $\varphi > 0$ and G first order stochastically dominates (FOSD) F, with increases in the parameter φ increasing the degree of dominance G. Letting $\varphi^H \leq \infty$ be the maximal value of φ, we also assume that $\lim_{\varphi \to \varphi^H} G(z|\varphi) = 0$ for all $z < 1$, which simply means that at the limit an efficient bidder has a maximal draw of $y = 1$ with probability 1. None of the potential buyers know their true values for asset $k \in \{A, B\}$ and must first incur a cost c to learn their values and be able to make a bid to acquire the asset. Note that this implies that any bidder must pay a cost $2c$ to learn his value for the entire firm.

[3]Note that the normalization of the support of the buyers' values to be $[0, 1]$ implies that the asset could, in principle, sell for a zero or close to zero price. An equivalent interpretation of our setting is to assume that asset k has a value V_k to the seller, and buyer' i's value is $V_k + x_k^i$, where $x_k^i \in [0, 1]$ and represents the premium in the buyer's value over what it is worth to the seller. Competition to buy the asset thus represents competition over how large (or small) a premium to pay.

However the firm is sold - either as a whole or each asset separately - the sales mechanism is a standard (i.e., no reserve) ascending price auction or, equivalently, a second price auction.[4]

3 Preliminary analysis

3.1 Individual asset sales

We begin with a construction of the equilibrium profits for the regular bidders, as well as the seller's revenue from the auction for a given number of regular bidders N, for the case where the assets are sold individually. We later endogenize N.

We start with an individual asset sale for asset k. As is usual for a second price auction, a dominant strategy is for bidders to bid their actual value. In other words, letting $b_i\left(x_k^i\right)$ represent bidder i's strategy as a function of his value x_k^i for asset k, we have that $b_i = x_k^i$. We focus on this equilibrium in dominant strategies. Given a specific realization of values $\left(x_k^1, \ldots, x_k^N, y^\kappa\right)$, for $\kappa = \alpha$ when $k = A$ and $\kappa = \beta$ when $k = B$, the N^{th} bidder's profit is given by:

$$\hat{\pi}^N\left(x_k^1, \ldots, x_k^N, y^\kappa\right) = \max\left\{x_k^1, \ldots, x_k^N, y^\kappa\right\} - \max\left\{x_k^1, \ldots, x_k^{N-1}, y^\kappa\right\}.$$

Notice that if $x_k^N < \max\left\{x_k^1, \ldots, x_k^N, y^\kappa\right\}$ then bidder N does not win and $\hat{\pi}^N\left(x_k^1, \ldots, x_k^N, y^\kappa\right)$ as defined above is 0. Otherwise, it is equal to $x_k^N - \max\left\{x_k^1, \ldots, x_k^{N-1}, y^\kappa\right\}$, the profit of bidder N conditional on winning.

[4] We focus on second price auctions because they are simple mechanisms that allow for dominant strategies in the sense that a bidder's optimal action does not depend on the number of competitors he faces or on their distributions of value. Given that the widely used ascending bid auctions are strategically equivalent to second price auctions for independent private values, our analysis has bearing on practice. Finally, second price auctions are also ex post efficient, an issue that has been highlighted as important in the context of auctions with free entry (see, e.g., Lu and Ye (2013) for a setting where bidders have asymmetric entry costs).

Taking expectations, the ex ante expected profit for bidder N (and by symmetry all other bidders) is

$$\pi_k = E\left[\max\left\{x_k^1, \ldots, x_k^N, y^\kappa\right\}\right] - E\left[\max\left\{x_k^1, \ldots, x_k^{N-1}, y^\kappa\right\}\right] \tag{1}$$
$$= \int_0^1 (1 - F(z))\, G(z)\, F(z)^{N-1}\, dz,$$

where the expectation is taken over the entire state space, i.e., $\{X_1, \ldots, X_N, Y\}$.

Similarly, for the efficient bidder we can express his profits as

$$\Pi_k = E\left[\max\left\{x_k^1, \ldots, x_k^N, y^\kappa\right\}\right] - E\left[\max\left\{x_k^1, \ldots, x_k^N\right\}\right] \tag{2}$$
$$= \int_0^1 (1 - G(z))\, F(z)^N\, dz.$$

The primary difference between (2) and (1) is that the profit for the efficient bidder when he wins is the difference between his valuation y^κ and the maximum of the other N regular bidders, $\max\left\{x_k^1, \ldots, x_k^N\right\}$. Hence the difference is only in the second term.

We can now calculate the seller's revenue, R_k, by recognizing that the sum of the seller's revenue plus the profits of all bidders - N regular bidders and 1 efficient bidder - equals the total surplus of the auction. Since the good is always sold, the total surplus must simply be equal to the maximum value of all bidders, $\max\left\{x_k^1, \ldots, x_k^N, y^\kappa\right\}$. Taking expectations and rewriting, we obtain

$$R_k = E\left[\max\left\{x_k^1, \ldots, x_k^N, y^\kappa\right\}\right] - \Pi_k - N\pi_k$$
$$= E\left[\max\left\{x_k^1, \ldots, x_k^N\right\}\right] - N\pi_k. \tag{3}$$

The seller's revenue is a simple function of the profits of the N regular bidders. Importantly, the effect of the efficient bidder's dominant distribution, G, only affects the seller through the regular bidders' profits.

From (3) and (1), one can show that, for a given N, the seller's revenue R_k is increasing in N, while a regular bidder's profit π_k is decreasing in N. Likewise, the seller's revenue is increasing in the dominance of the efficient bidder, as given by the distribution function G, whereas a regular bidder's profit decreases as the efficient bidder becomes more dominant.

3.2 Selling the assets together

When the assets are sold jointly, each regular bidder has a draw for each division, and these values are added together to form their value for the pooled assets. The two efficient bidders have a draw from a dominant distribution for one division, and from the regular distribution for the other.

Specifically, this means that the value of regular bidder i for the pool of assets A and B is simply $X^i = x_A^i + x_B^i$. Since both x_A^i and x_B^i are drawn from the same distribution F, the distribution of their sum can be obtained by taking the convolution as

$$j(z) = \int_0^1 f(z-y) f(y)\, dy.$$

Integrating to get the distribution function $J(z)$ yields

$$J(z) = \begin{cases} \int_0^z F(z-y) f(y)\, dy & \text{for } 0 \leq z \leq 1 \\ F(z-1) + \int_{z-1}^1 F(z-y) f(y)\, dy & \text{for } 1 < z \leq 2 \end{cases}.$$

For efficient bidder α, his value for the pool of assets is $Z^\alpha = y^\alpha + x^\alpha$, where y^α is drawn from $G(.|\varphi)$ and x^α is drawn from F. Therefore, the distribution H of the sum Z can again be obtained by taking the convolution as

$$h(z) = \int_0^1 g(z-y) f(y)\, dy.$$

Again integrating, we obtained the distribution function H as

$$H(z) = \begin{cases} \int_0^z G(z-y) f(y)\, dy & \text{for } 0 \leq z \leq 1 \\ F(z-1) + \int_{z-1}^1 (G(z-y)) f(y)\, dy & \text{for } 1 < z \leq 2 \end{cases}.$$

An identical expression obtains for efficient bidder β.

We can now use these distribution functions to find the profits for a regular bidder, which we denote as π_J for "joint" since the assets are being sold jointly.

$$\begin{aligned} \pi_J &= E\left[\max\left\{X^1, \ldots, X^N, Z^\alpha, Z^\beta\right\}\right] - E\left[\max\left\{X^1, \ldots, X^{N-1}, Z^\alpha, Z^\beta\right\}\right] \\ &= \int_0^2 (1 - J(z)) H(z)^2 J(z)^{N-1}\, dz. \end{aligned} \tag{4}$$

A similar expression obtains for the profit of an efficient bidder, Π_J (we drop the superscripts α and β since both efficient bidders are symmetric):

$$\begin{aligned} \Pi_j &= E\left[\max\left\{X^1, \ldots, X^N, Z^\alpha, Z^\beta\right\}\right] - E\left[\max\left\{X^1, \ldots, X^{N-1}, Z^\alpha\right\}\right] \\ &= \int_0^2 (1 - H(z)) H(z) J(z)^N\, dz. \end{aligned}$$

We can also obtain the expression for the seller's revenue as

$$\begin{aligned} R_J &= E\left[\max\left\{X^1, \ldots, X^N, Z^\alpha, Z^\beta\right\}\right] - 2\Pi_J - N\pi_J \\ &= E\left[\max\left\{X^1, \ldots, X^N, Z^\alpha\right\}\right] - \Pi_J - N\pi_J \\ &= 2 - \int_0^2 J(z)^N H(z) dz - \Pi_J - N\pi_J. \end{aligned} \tag{5}$$

4 Optimal bundling of assets

In this section we study when selling assets individually is optimal (i.e., raises more revenue for the seller) versus when the seller would do better by bundling the two assets together. As a first step, we endogenize the number of bidders through a standard free entry condition for the regular bidders, that entry in the first stage should take place as long as an entering bidder's profit are at least as large as the cost of learning one's value for the asset and submitting an offer. Ignoring integer constraints, at equilibrium with free entry the profits of each regular bidder, net of the cost of entry, should be zero. For the case of individual sales, we can use the following condition to characterize the equilibrium number of regular bidders, N^*:

$$\pi_k (N^* | G) = c. \tag{6}$$

When the assets are sold jointly, a similar expression holds for the equilibrium number of regular bidders, which we denote N_J:

$$\pi_J (N_J | H) = 2c. \tag{7}$$

We now establish that in the absence of an efficient bidder, selling the assets individually is superior to bundling them together for a large class of distribution functions of values, $F(.)$, and for a broad range of parameter values.

Proposition 1 *For $\varphi = 0$, and for any symmetric distribution function F satisfying $\frac{\partial}{\partial x} \frac{f(x)}{1-F(x)} \geq 0$ (i.e., non-decreasing hazard rate), there is a always a value $\overline{c}$ small enough such that, for $c \leq \overline{c}$, the revenue from selling both assets individually, $2R_k$, is greater than when selling them jointly, R_J.*

Proof: First, note that $N_J \leq N^*$ since, conditional on entering and there being the same number of bidders, the expected profits to a regular bidder are higher, per asset, when the assets are sold individually. This means that more bidders will enter when the assets are sold separately.

The result now follows directly from the arguments in Chakraborty (1999): for c small enough, the free entry number of bidders when the assets are sold individually, N^*, will be sufficiently large such that $2R_k(N^*) > R_J(N^*)$. Since $N_J \leq N^*$, the result then follows. $\square$

The proposition shows that, for a broad range of circumstances, the seller would prefer to sell the assets individually when an efficient bidder for any of the individual assets is not anticipated to be present. In this case, joint asset sales would not be observed, and each asset should be auctioned off independently.

The intuition is that, when selling the assets individually, a standard auction like a second price auction allocates the good efficiently: the buyer with the highest value acquires the asset, paying the value of the next highest buyer. By contrast, selling the assets jointly introduces a distortion and may fail to allocate the assets to those who value them most. In particular, when sold jointly the winning party may well have a high value for one asset, but not a particularly high value for the other asset. In fact, it is certainly possible for the winner to not have the highest value for either of the two assets, but to have the highest value for the sum of the assets. When there is no efficient bidder, so that there is no reduction in competition associated with his presence, selling the assets in the most efficient way also maximizes the seller's revenue when the auction is sufficiently competitive (i.e., when c is sufficiently small).[5]

It is also worth noting that the threshold value $\bar{c}$ need not be particularly small relative to the expected profits of bidders. In fact, a sufficient condition for Proposition 1 to hold is that, at equilibrium, the number of bidders is just three: $N^* \geq 3$. In other words, when costs

[5] It is worth noting that Proposition 1 presents sufficient but not necessary conditions for the optimality of individual sales to hold. Given that $N_J < N^*$ so that, in equilibrium, competition is lower under joint sales, individual sales can be optimal even for larger values of the entry cost c, of the underlying distribution F is not symmetric.

are sufficiently low that there is competition by as few as three bidders, individual assets sales dominate joint sales when there is no dominant bidder. This condition is established formally in Chakraborty (1999), who shows that when the distribution function F is symmetric, all one needs is for there to be three or more bidders present in order for individual asset sales to generate greater revenue than "bundled" sales.

We now show that as the dominance of the efficient bidders increases, selling the assets jointly becomes optimal. To do this, we first note that given that each regular bidder's profit π_k is decreasing in the dominance of the efficient bidder, φ, to satisfy (6) the equilibrium number of regular bidders, N^*, must decrease as φ increases. We summarize this in the following result.

Lemma 2 *The free entry number of regular bidders, N^*, is decreasing in φ.*

In other words, the (expected) higher value of one of the bidders has a depressing effect on competition, leading fewer other bidders to be willing to incur the cost of entry. This endogenous reduction in competition turns out, however, to have a very important implication for the equilibrium revenue for the seller under free entry.

Lemma 3 *Under free entry, the seller's revenue from selling asset k individually, $R_k\left(N^*|G\right)$, is decreasing in the dominance of the efficient bidder.*

Proof: The seller's revenue can be written as

$$R_k = E\left[\max\left\{x_k^1, \ldots, x_k^N\right\}\right] - N\pi_k = 1 - \int_0^1 F\left(z\right)^N dz - N\pi_k.$$

The equilibrium number of bidders, N^*, is a function of φ: $\pi_k\left(N^*\left(\varphi\right)|G(\varphi)\right) = c$. We can therefore calculate the change in revenue when φ increases:

$$
\begin{aligned}
\frac{dR_k}{d\varphi} &= \frac{d}{d\varphi}\left(1-\int_0^1 F(z)^{N^*(\varphi)}\,dz - N^*(\varphi)\,\pi_k\left(N^*(\varphi)\,|G(\varphi)\right)\right)\\
&= -\frac{dN^*(\varphi)}{d\varphi}\int_0^1 F(z)^{N^*(\varphi)}\ln F(z)\,dz - \frac{dN^*(\varphi)}{d\varphi}\pi_k\left(N^*(\varphi)\,|G(\varphi)\right)
\end{aligned}
$$

Given that $\frac{dN^*(\varphi)}{d\varphi} < 0$, we have that

$$
\begin{aligned}
\frac{dR_k}{d\varphi} &\propto \int_0^1 F(z)^{N^*}\ln F(z)\,dz + \int_0^1 (1-F(y))\,G(y)\,F(y)^{N^*-1}\,dy\\
&< \int_0^1 F(y)^{N^*}\ln F(y)\,dy + \int_0^1 (1-F(y))\,F(y)^{N^*}\,dy\\
&= \int_0^1 \left[1-F(y)+\ln F(y)\right]F(y)^{N^*}\,dy. \qquad (8)
\end{aligned}
$$

Examine the expression in the square bracket as a function of F: $1-F+\ln F$. This expression is concave in F and it attains its maximum at $F=1$. Moreover, since $F(y)$ is monotonic in y, this implies that the expression is maximized at $y=1$. At $y=1$, the expression is zero. Hence it is negative everywhere else. Therefore,

$$
\int_0^1 \left[1-F(y)+\ln F(y)\right]F(y)^{N^*}\,dy < 0,
$$

which implies that $\frac{dR_k}{d\varphi} < 0$, as desired. □

Lemma 3 establishes that under free entry, the presence of the efficient bidder leads to a reduction in the expected sales price, with the depression in the price becoming worse the more dominant is the bidder. It bears noting as well that Lemma 3 holds for any entry cost $c > 0$, even if arbitrarily small so that the potential for competition is very large. As long as the entry cost is not strictly equal to zero, increases in the dominance of the efficient bidder will reduce the revenue to the seller.

The key reason for Lemma 3 to hold is the reduction in competition (i.e., reduced entry) due to the presence of the efficient bidder, as highlighted in Lemma 2. Since an auction is a competitive bidding process, the reduced competition, which arises endogenously, becomes first order relative to the possibly higher value of the efficient bidder and leads to a lower expected sales price. Moreover, since competition is reduced further the more dominant is the efficient bidder, increases in the degree of dominance lead to yet greater reductions in the equilibrium price.

We can now use Lemma 3 to establish the following result, which represents the converse of that in Proposition 1.

Proposition 4 *There is a value $\overline{\varphi}$ such that, for $\varphi > \overline{\varphi}$, the total revenue obtained from sellling both assets individually is less than the revenue from selling the assets jointly:* $2R_k < R_J$.

Proof: To establish this result, note from (3) that the seller's revenue when the assets are sold individually is positive only if the equilibrium number of regular bidders, N^*, is strictly greater than zero. However, from (1) it is straightforward to see that $\lim_{\varphi \to \varphi^H} \pi_k = 0$, implying that for any $c > 0$, N^* will converge to zero as $\varphi \to \varphi^H$. Therefore, R_k also converges to zero.

By contrast, the revenue when selling the assets jointly, R_J, given in (5), may be positive even for $N = 0$ because of the presence of the two efficient buyers. Even ignoring that, note that for a regular bidder, keeping N constant,

$$\begin{aligned} \lim_{\varphi \to \varphi^H} \pi_J &= \lim_{\varphi \to \varphi^H} \int_0^2 (1 - J(z)) H(z)^2 J(z)^{N-1} dz \\ &= \int_1^2 (1 - J(y)) F(y-1)^2 J(y)^{N-1} dy > 0. \end{aligned}$$

Therefore, $\lim_{\varphi \to \varphi^H} \pi_J$ for any finite N, implying that the equilibrium number of regular bidders will be bounded away from zero even as $\varphi \to \varphi^H$. As a result, $\lim_{\varphi \to \varphi^H} R_J > 0$ as well, establishing the result. $\square$

The proposition establishes that when the efficient bidders are sufficiently dominant for their respective preferred assets, selling the assets jointly rather than separately will raise the greatest revenue. In other words, when φ is high, the whole is worth more than the sum of the parts. This is true precisely because the number of bidders is endogenous and depends on bidders' decisions whether or not to pay the entry or due diligence c for the case of an individual asset sale, and $2c$ in the case the firm is sold as a whole.

What is most surprising about the result in Proposition 4 is that selling the assets jointly is optimal precisely when the highest social value would be obtained by selling the assets individually. As argued above, when sold individually, each asset is allocated to the party that values it most. The total surplus, therefore, is maximal when both assets are sold separately. Pooling the assets to sell them, therefore, introduces a distortion in the efficient allocation of the assets. The tradeoff, however, is that by doing so it reduces the impact of improved efficiency on the other bidders, and limits the extent to which competition is reduced when φ increases. At some point, the revenue from selling the assets separately goes down too much because there is too little competition, whereas competition is not quite so dramatically reduced when the assets are sold jointly. Put differently, increases in φ have a larger impact in reducing competition in the case of an individual asset sale than when the assets are sold jointly, where the effect is attenuated. For φ large enough, the reduction in competition dominates so that selling the assets jointly is optimal.

5 Numerical example

In this section we present numerical results to help understand when selling the firm's assets jointly is optimal, showing that this occurs when the efficient bidder's advantage is high.

Conversely, the revenue from breaking up the firm and selling the assets individually is higher either when all potential bidders are symmetric or when the efficient bidder's advantage is small.

To parametrize the model in order to obtain numerical solutions, we assume that the regular bidder's value for each asset is drawn from a uniform distribution in the unit interval, i.e., $F(x) = x \ \forall x \in [0,1]$. The efficient bidder is assumed to sample his value from distribution $G(x) = x^{1+\varphi} \ \forall x \in [0,1]$, where $\varphi \geq 0$. Notice $G(x)$ dominates $F(x)$ in the first order stochastic sense, as per our assumption in the general model.

With this, we can explicitly write the joint cdf for the combined firm values as

$$J(z) = \begin{cases} \frac{z^2}{2} & \text{for } 0 \leq z \leq 1 \\ z - \frac{1}{2} - \frac{(z-1)^2}{2} & \text{for } 1 < z \leq 2 \end{cases}$$

for the regular bidders, and

$$H(z) = \begin{cases} \frac{z^{2+\varphi}}{2+\varphi} & \text{for } 0 \leq z \leq 1 \\ z - \frac{1+\varphi}{2+\varphi} - \frac{(z-1)^{2+\varphi}}{2+\varphi} & \text{for } 1 < z \leq 2 \end{cases}$$

for the efficient bidders.

Using the expressions derived in the previous section, we can also write the expected profit for a regular bidder when bidding for the combined assets as

$$\pi_J(\varphi, N) = \int_0^2 (1 - J(z)) H^2(z) J^{N-1}(z)\, dz.$$

The equilibrium number of bidders can be determined by using the endogenous entry condition, $\pi_J(\varphi, N) = 2c$. For a given level of efficiency, φ the expected profit of the regular bidder is strictly decreasing in N. This allows us to pin down the unique number of regular bidders in equilibrium.

The expected revenue for the case of selling the assets jointly is given by

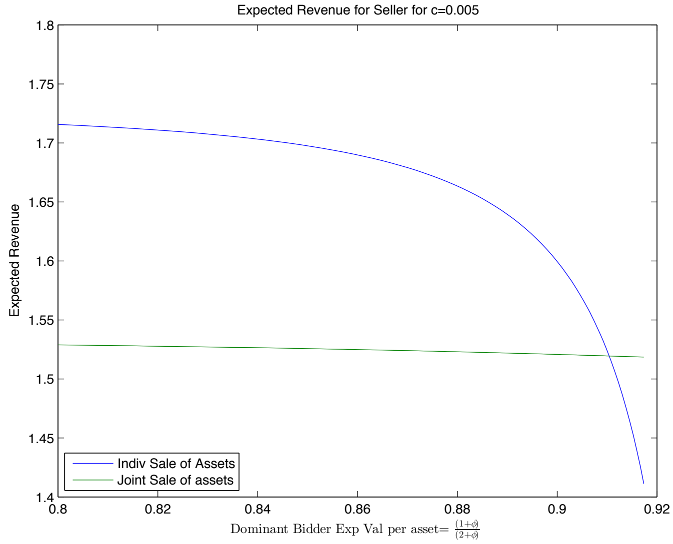
$$R_J = 2 - \int_0^2 J(z)^N H(z)dz - \Pi_J - N\pi_J,$$

which can also be calculated numerically using the given values of φ and c, and the imputed value of N.

Figure 1 graphs the seller's revenue assuming a cost of bidding of $c = 0.005$ for various values of the the efficiency parameter φ. We represent the efficient bidder's level of dominance along the x-axis by calculating his expected value of the asset for a given value of φ. Specifically, for the assumed parametrization (i.e., a uniform distribution) the expected value of the dominant bidder for his preferred asset is given by $\frac{1+\varphi}{2+\varphi}$. Thus, for $\varphi = 0$, the dominant bidder's expected value collapses to that of the regular bidder, $\frac{1}{2}$, while for $\varphi = 3$ corresponds to an expected value of $\frac{1+\varphi}{2+\varphi} = 0.8$ (the lowest level of dominance in the figure). The red line represents the revenue obtained when selling both assets individually (so the combined revenue), while the blue line represents the revenue from selling the two assets jointly. While both revenue curves decrease as efficiency φ increases because the number of regular bidders decreases, it is clear that the revenue when the assets are sold individually is much more sensitive to changes in φ than the revenue when the assets are sold jointly. For low values of φ, the firm is clearly better off selling the assets individually. However, as φ increases the difference between the two revenue curves decreases. At around $\frac{1+\varphi}{2+\varphi} = 0.91$ the two curves intersect, and for larger values the firm is better off selling the two assets jointly.

Figure 2 and 3 illustrate a similar pattern for higher values of c, the entry cost. Interestingly, as c increases from 0.005, as in Figure 1, to 0.01 (Figure 2) and 0.015 (Figure 3), the value of $\frac{1+\varphi}{2+\varphi}$ and, as a consequence, φ, at which selling the assets jointly becomes optimal decreases - it is approximately 0.875 when $c = 0.01$ and 0.845 for $c = 0.015$. In other words, increases in the cost of bidding, so that there are greater barriers to competition, favor selling

Figure 1: Expected revenue as a function of the degree of dominance by the efficient bidder(s) for $c = 0.005$.



the assets jointly since they reduce the point at which joint asset sales are optimal.

This pattern can be seen very clearly from Figure 4, which plots the expected value of the dominant bidder per asset, $\frac{1+\varphi}{2+\varphi}$, in the horizontal axis against the entry cost c on the vertical axis. The green diamonds represent configurations of (c, φ) for which selling the assets individually raises greater revenue and is thus optimal for the seller. The blue circles represent configurations for which selling the assets jointly is optimal. Since the dominant bidder's expected value is monotonically increasing in φ, Figure 4 clearly illustrates that as c increases, the threshold value of φ beyond which selling the assets jointly is optimal (Proposition 4) decreases.

One interesting aspect illustrated by the numerical example in this section is the stark contrast between how sensitive seller revenue is to increases in bidder efficiency φ in the joint versus individual asset sale cases. As φ increases, revenue when the assets are sold separately decreases quite dramatically, moving quickly toward providing a vanishingly small premium to the seller. By contrast, in the joint sale case the seller's revenue is very nearly flat over a broad range of values for φ. The reason is twofold. First, as argued above, the dominance

Expected revenue as a function of the degree of dominance by the efficient bidder(s).

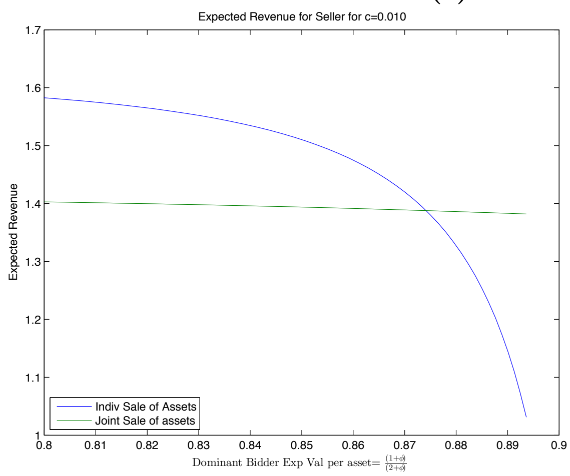


Figure 2: $c = 0.010$

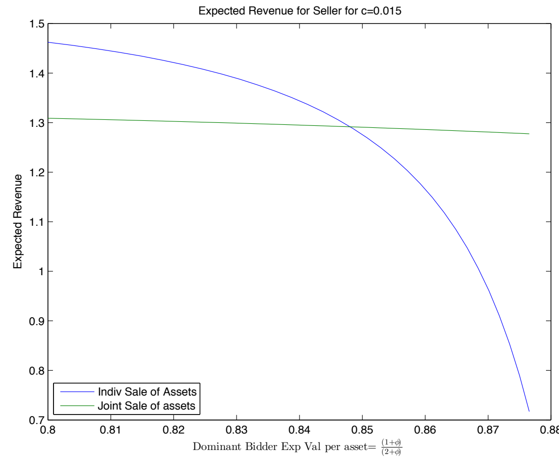


Figure 3: $c = 0.015$

of any given efficient bidder is more diffuse when the assets are sold as a package, leading to a smaller reduction in competition. Second, since with two assets there are actually two efficient bidders, these bidders continue to compete against each other in a more symmetric fashion even as the other regular bidders become more handicapped.

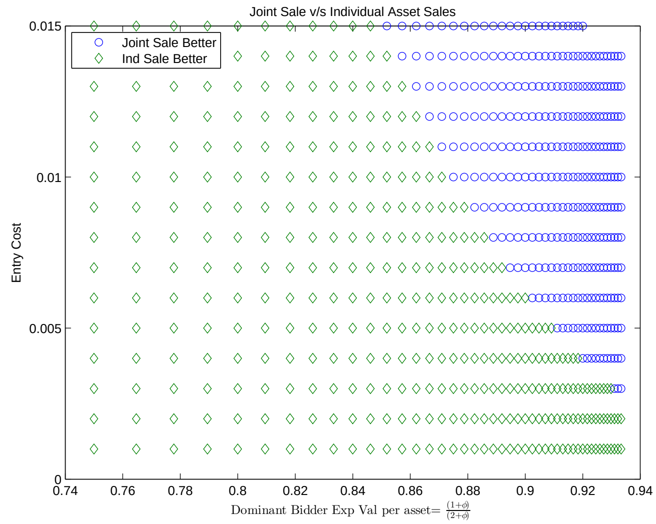
6 Additional considerations

Here, we analyze how various other factors, such as financial constraints or synergies obtained from buying the assets together, affect the optimality of joint sales relative to individual asset sales.

6.1 Due diligence costs

Up to now we have assumed that a bidder's cost associated with the due diligence process for learning the value of the combined assets is simply double the cost of learning the value of a single asset. While likely sensible in many circumstances, the primary reason for this

Figure 4: The figure displays whether joint or individual sales of assets generate higher revenue for the seller, for various pairs of (c, φ).



assumption was to allow us to focus the analysis entirely on the role played by endogenous entry and bidder heterogeneity, excluding other factors, such as differential due diligence costs, that may favor one type of asset sale versus the other. In practice, however, it is likely that in many circumstances, and particularly for larger transaction, information acquisition costs may in fact be convex in the number of assets being acquired, thus increasing faster as more assets, and hence more complexity, is added. For instance, this would be the case if a potential bidder has limited time to learn about any possible assets for sale, and evaluating two (or more) assets rather than one comes at an increasingly higher opportunity cost of the bidder's time.[6]

Suppose that, as before, the information acquisition cost to a bidder of learning his value for an asset, either A or B, is c. However, the information acquisition cost for learning the value of the combined assets is $C > 2c$. Otherwise, the model is unchanged.

Define the threshold value $\overline{C}$ as the largest value of C such that an efficient bidder

[6]One may imagine that some instances may be characterized instead by economies of scale, such as if the assets are similar and learning about one then reduces the cost associated with learning about the other. In that case, the cost of acquiring information for the combined assets would likely be less than $2c$. We discuss that case briefly at the end of the section.

would always find it optimal to participate in the auction for asset j, when there are no regular bidders participating: $\Pi_j(\varphi_H, N = 0) = \overline{C}$. We can now state the following result, which extends Proposition 4 to the case where there are diseconomies of scale in information acquisition.

Proposition 5 *For $C < \overline{C}$, there is a value $\overline{\varphi}(C)$ such that, for $\varphi > \overline{\varphi}(C)$, the total revenue obtained from sellling both assets individually is less than the revenue from selling the assets together: $2R_k < R_J$.*

The proposition establishes that despite the diseconomy of scale in information acquisition, so that learning about the combined assets is proportionally more difficult and costly that learning about each individual asset, selling the assets jointly is still preferable to the seller when the efficient bidders are sufficiently dominant. The reason is similar to that in Section 4: as the degree of dominance of the efficient bidders increases, the number of regular bidders when assets are sold individually goes to zero, thus driving the sales price to zero as well. When assets are sold jointly, however, for any given N the profit of a regular bidder is bounded above zero for any value of φ, so that entry will still occur. The higher cost of information acquisition, C, does reduce entry by regular bidders relative to the case we considered above, but does not eliminate competition entirely as long as the cost is not so high that not even a single bidder ($N = 1$) could stand to make a profit. Finally, the constraint on C, that it not be larger than $\overline{C}$, ensures that the diseconomy of scale is not so large as to completely prevent the participation of even the efficient bidders in the sale. For C greater than $\overline{C}$, joint asset sales generate too little competition because the cost of evaluating the assets is too high.

6.2 Financial constraints

An important consideration for many instances of asset sales is the ability of potential bidders to finance the acquisition. This is particularly true in the context of divisional sales for companies, where the dollar value of the acquisition is likely high and a substantial fraction of the total value of the acquiring party. Moreover, in those instances the valuation is likely a function of projected future cash flows derived from the ownership of the assets, making the information acquisition cost, and hence the endogeneity of bidder entry, an important consideration. In this section we study how financial constraints bidders may face affect a seller's preference between bundling assets or not.

To study this issue, we consider financing costs that increase in the amount of financing that might be required. Specifically, we assume that every dollar that is bid has a cost of $\alpha > 0$ associated with raising it, so that if an offer of W is made for an asset, the bidder must incur αW in fund raising costs.

We can now see how such financial constraints affect seller revenue and the optimal sales mechanism. Suppose that bidder i has value x_A^i for asset A, and offers a price b_A^i. If he wins the auction, his value is $x_A^i - (1+\alpha)\max\left\{b_A^j | j \neq i\right\}$. The bidder's payoff will be positive if $x_A^i - (1+\alpha)\max\left\{b_A^j | j \neq i\right\} > 0 \Leftrightarrow \frac{x_A^i}{1+\alpha} - \max\left\{b_A^j | j \neq i\right\} > 0$. Therefore, redefining the bidder's adjusted value for the asset as $\widetilde{x}_A^i = \frac{x_A^i}{1+\alpha}$, it is a dominant strategy for the bidder to bid his adjusted value $\widetilde{x}_A^i$: $b_A^i = \widetilde{x}_A^i = \frac{x_A^i}{1+\alpha}$. Other than this change, however, all other aspects remain the same so that, qualitatively, Proposition 4 continues to hold for the case where bidders face financial constraints that make it costly to raise financing to pay for the assets being auctioned. In other words, while financial constraints make large (i.e., joint) acquisitions proportionately more costly, increases in dominance by the efficient bidders reduces entry to a greater extent when assets are sold individually than when they are sold jointly. For sufficiently high levels of dominance φ, joint sales become optimal despite the higher financing costs associated with them.

6.3 Asset synergies

So far, we have assumed that the value of the assets when combined is simply the sum of their individual values for each possible bidder. As above, the primary reason for this assumption was so as not to introduce a bias toward either selling the assets individually or, more likely if there are synergies across the assets, selling them jointly. In practice, however, it is likely that assets belonging to one seller may naturally go together, so that there value may be higher when purchased together than simply the sum of their individual values. In other words, the intrinsic value of the whole may be higher than the sum of the parts. Such synergies would naturally make a joint sale relatively more profitable for the seller, all things equal.

Consider now a small change to the model to reflect the possibility of synergies across the two assets: assume that the combined value of assets A and B to any given bidder i is $X^i = x_A^i + x_B^i + S$, where $S \geq 0$ represents the possible synergy associated with purchasing the assets together. We will also assume that if the assets are sold separately, they are sold in entirely segmented markets so that no bidder can acquire both assets and as a result enjoy the synergy. This assumption should, if anything, bias sellers in favor of selling the assets jointly, and we discuss its role further below. The model is otherwise unchanged.

While the existence of a synergy between assets A and B, such that their combined value is greater than the sum of their individual values, clearly creates an incentive to sell the assets jointly, it need not induce sellers to move away entirely from individual sales, as the following result establishes.

Proposition 6 *For $N \geq 3$, there exists a value $\overline{S}(N) > 0$ such that, for $\varphi = 0$ and $S \in \left(0, \overline{S}\right)$, the revenue from selling the assets individually, $2R_k$, is greater than when selling them jointly, R_J.*

The proposition establishes that, as long as the synergy is not too large, selling the assets individually may still be optimal despite the fact that *all* bidders value the individual assets less than they do the joint assets. The intuition is similar to that for Proposition 1: when $\varphi = 0$ and no bidder is efficient relative to the other bidders, a regular bidder finds it more profitable to participate in an individual auction than in a joint auction, all things equal. There is therefore more competition when selling the assets individually, giving rise to higher prices for the seller. As long as the synergy is not too large, this effect will continue to hold and revenue will be higher when selling the assets individually. At some point, however, once the synergy becomes very large, the revenue obtained from selling the assets jointly becomes greater despite the lower participation. At that point, joint sales becomes optimal, even when there is no efficient bidder(s).

7 Conclusion

Our analysis focuses on how endogenous entry and the form in which assets are sold - either bundled together or sold separately - affects the revenue that might be obtained from their sale. We study how the presence of an efficient bidder who values some subset of the assets highly actually depresses prices through its entry deterrence effect and, as a result, reduces the premium a seller might receive. This has implications for whether assets should be sold together as a bundle or separated and sold individually since the effect on bidder entry is lower when assets are bundled. As a consequence, selling assets as a bundle is optimal when some bidders are very dominant since it is precisely then that entry will be lowest when assets are sold individually and hence revenue would also be very low.

The analysis extends to incorporate other issues that might be relevant in the selling of assets. For instance, one can well imagine that dis-economies of scale in screening, performing due diligence, and fixed costs of putting together an offer may increase the costs of entry when assets are sold jointly relative to when sold individually. We show that while this tends to favor individual asset sales, joint sales are still optimal when bidders are very efficient because of the entry deterrence effect highlighted above. An aspect that we have not considered, but which may well be relevant in practice, is that asset values may well be correlated, either positively or negatively, introducing additional reasons why assets may be bundled to exploit such correlations. These issues are likely important but, we believe, separate from the competitive concerns identified here.

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