How much for a haircut?
Illiquidity, the secondary market, and the value of private equity

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Abstract
Limited partners of private equity funds commit capital with extreme restrictions on liquidity and a large degree of uncertainty regarding the timing of capital calls and payoffs. Secondary markets have emerged which alleviate some of the associated cost of illiquidity. This paper develops a subjective valuation model that incorporates these institutional features. We find that for a risk-averse limited partner the value of a private equity investment is highly sensitive to the time-varying discount observed in secondary market transactions. Model-implied breakeven returns required to compensate the limited partner for liquidity costs, market risk, and fees of investing in private equity funds generally exceed empirically observed returns. Only for highly risk-tolerant LPs facing an efficient secondary market and able to access above-average funds are the costs of private equity justified by performance.

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Introduction

Institutional investors now routinely allocate 15-30% of their portfolios to private equity (PE) funds. This capital is committed with extreme restrictions on liquidity. PE funds are organized as 10-12 year partnerships, with investors as limited partners (LPs) and the fund manager as the general partner (GP). Importantly, during the partnership’s life there is no option to redeem a stake with the fund. Instead, LPs requiring an early exit from the partnership must turn to the secondary market to sell their stakes, usually at a substantial discount from fund NAV. These haircuts can reach fire sale levels in times of broader market dislocations, such as during the recent global financial crisis. Moreover, even while invested in the fund, LPs face a large degree of uncertainty regarding the timing and magnitude of capital calls for investments in portfolio companies and payoffs from exited portfolio companies. These liquidity arrangements intuitively affect the ex-ante value of commitments to PE funds. Yet, despite the importance of PE to institutional portfolios, to date no academic research provides guidance.

In this paper, we develop a valuation model for commitments to private equity funds from the perspective of a risk-averse LP subject to liquidity shocks, such as that brought on by the crisis, which force the LP to seek an early exit from the partnership. In the model, the market is incomplete and private equity cannot be fully spanned by public equity, so present values are not well defined and an LP’s subjective (given its risk aversion, etc.) valuation is relevant. The model explicitly incorporates both the secondary market, which we assume the LP accesses when a liquidity shock occurs, and uncertainty about the timing of capital calls and distributions. We show that each of these features has a quantitatively important impact on the value of commitments to PE funds.

The model allows us to evaluate several closely related questions that are central to assessing any alternative asset class. How large are the returns required to compensate for the special liquidity risks borne by LPs, and how do these compare to actual returns to PE funds in the data? How do required returns depend on an LP’s tolerance for risk, the efficiency of the secondary market, and the fees charged by the GP? What is the impact of diversification across public and private equity and how does an LP’s allocation to private equity affect the return it requires? Equivalently, how does an LP’s allocation to PE depend on the returns it expects?
In the spirit of the real options literature, we model the joint evolution of private and public equity asset values using a trinomial lattice in discrete time, which allows us to efficiently accommodate a number of path dependencies. In addition to forced secondary sales and uncertain capital call and distribution timing, our framework allows the LP to voluntarily sell on the secondary market (early exercise), and this decision incorporates the probability of a future liquidity shock and the subsequent haircut applied in the secondary market.

The model is solved for the certainty-equivalent value of the LP’s portfolio at fund inception using backward recursion, exploiting the fact that PE funds have a finite contracted life. In most of our analyses, we solve for the return and CAPM alpha of the PE assets (portfolio companies) required to make the LP’s certainty equivalent at least equal to the certainty equivalent from a benchmark portfolio consisting only of public equities and the riskfree security. From this gross-of-fee breakeven required return on PE assets we compute the corresponding net-of-fee LP performance measures commonly used in the academic literature and in practice: the IRR, TVPI, and PME.¹ We also conduct several analyses in which we solve for an LP’s allocation to PE given a particular expected return on PE investments. Naturally, the results depend on the model inputs. While we calibrate parameters to the data wherever possible, individual LPs can use our procedure to generate their own benchmarks.

We begin by considering the case of a two-asset LP portfolio consisting of a PE fund and the riskfree asset. The purpose is to illustrate the impact of the PE institutional features novel to our approach in the simplest possible setting. Our first finding is that relative to a benchmark in which capital calls and distributions occur at known times, uncertain capital calls and distributions reduce the LP’s valuation by 5%-10% of its initial wealth.

Next we introduce liquidity shocks which force an LP to sell his stake on the secondary market at some discount to NAV. Empirical evidence of secondary markets indicates that the size of haircuts varies systematically over time, and for this reason we allow for different haircuts in future secondary market transactions. When haircuts are large, i.e. the secondary market is inefficient, private equity investments are worth as much as 15% to 25% less than

¹ The TVPI, or total value to paid-in capital, is the undiscounted ratio of the payoffs received by the LP, which come from distributions to the LP of proceeds from exited or liquidated portfolio companies, to the capital calls paid by the LP to the fund. The PME, or public market equivalent, is the ratio of discounted distributions to discounted capital calls, using the realized public market return as the discount rate.
when haircuts are small, especially to LPs with higher risk aversion. Highly risk-tolerant LPs are much less sensitive to the magnitude of secondary market haircuts. Consistent with this, required breakeven PE asset returns and net-of-fee LP returns are more highly sensitive to the efficiency of the secondary market when the LP is more risk averse. The required breakeven returns that emerge from this analysis far exceed the documented empirical performance of PE funds, suggesting that standalone PE investments are unwise for most LPs.

A more realistic case is one in which the LP holds private equity in a portfolio together with public equities and the risk-free asset. Here we ask, for a given allocation to private equity, what PE returns are required to generate the same certainty equivalent as a portfolio consisting of 80% public equity and 20% riskless bonds. Given the diversification benefit achieved when mixing public and private equity, as well as the smaller allocation to private equity compared to the prior analysis, we find more modest breakeven asset and fund returns.

Nevertheless, breakeven returns are still greater than empirical estimates of typical fund performance for most parameter values, indicating that for many LPs the returns they receive in practice are insufficient to compensate for the risks of private equity. Recent work finds an average buyout PME of approximately 1.2-1.3, and finds that venture capital has roughly equaled public equities since 2000. In our model, these returns are insufficient even for relatively risk-tolerant LPs facing a relatively efficient secondary market. At the same time, such LPs who are also able to access or select above-average funds will find private equity attractive even at allocations up to 60%. In particular, top-quartile funds provide attractive returns even to more moderately risk-averse LPs.

The portfolio analysis also illustrates the impact of diversification. For the most risk-tolerant LPs, increasing the PE allocation up to about 40% lowers breakeven returns because of diversification. Risk-tolerant LPs are also relatively insensitive to the level of secondary market haircuts. Overall, the model suggests that the relatively high allocations to private equity made by large endowments (e.g., Yale had a 35% allocation in 2013) are likely to be optimal to the extent that they risk tolerant and able to access or select better funds.

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2 A 10-20% allocation to fixed income and cash is typical for endowments and affiliated foundations, according to the 2014 NACUBO-Commonfund Study of Endowments. Results are similar using a benchmark 90/10 portfolio instead.
In contrast, breakeven returns are quickly increasing in the allocation to PE for more risk-averse LPs, as the special risks posed by PE rapidly outweigh its diversification benefits for these investors. Consequently, lower, if not zero, allocations are more prudent for these investors.

Finally, we consider the impact of fees. We find that moving from a hypothetical zero-fee fund to one with the typical 2% per year management fee plus 20% carried interest increases the breakeven PE asset alpha by about 50%-100%. Thus, in our model fees to the GP absorb one-third to one-half of the excess return (relative to the CAPM) generated by the fund.

This paper contributes to several strands of the private equity literature. Our model helps interpret empirical work on the performance of private equity funds. Robinson and Sensoy (2013), Harris, Jenkinson, and Kaplan (2014), Higson and Stucke (2014), and Phalippou (2013) all use recent data and provide similar estimates of the performance of PE as an asset class. While there is a general consensus on what performance has been, there is no clear agreement on whether this performance is sufficient to appropriately compensate LPs.

In an important paper, Sorensen, Wang, and Yang (2014) provide a different methodology to value private equity. Their approach uses European option-pricing techniques in continuous time, which require them to abstract from the secondary market and uncertainty about capital calls and distributions. These features are central to our approach and are naturally suited to the real-options framework adopted here. Consistent with the importance of these sources of liquidity risk to LPs, private equity is less attractive in our model than in Sorensen, Wang, and Yang (2014).

Our work is also related to recent theoretical work in asset pricing analyzing private equity performance measures, particularly the PME. Korteweg and Nagel (2013) and Sorensen and Jagannathan (2013) show that the PME is an appropriate risk-adjusted return for an investor with log utility in the case where the risks of private equity are spanned by the public market portfolio. In this case, the relevant PME benchmark is one. Our work shows that the liquidity risks of PE result in PME benchmarks that are often substantially greater than one, especially for more risk-averse LPs.

Finally, our work connects to the literature on the fees charged by private equity funds. A continuing debate is whether fees charged in practice are justified by the returns investors ultimately receive. Robinson and Sensoy (2013) partially address this question by showing that
higher-fee funds do not earn lower net-of-fee returns. Yet, their analysis speaks only to the cross-
section and not to whether the typical level of fees is excessive. In our model, the answer is
typically yes.

The rest of the paper is organized as follows. In Section I we provide some relevant
institutional detail of private equity funds. Section II explains our procedure for computing an
LP’s subjective valuation of private equity investment. Section III discusses base-case parameter
selection. Main results for portfolios of private equity and riskless bonds, as well as an allocation
across public equity, private equity, and riskless bonds, are presented in Section IV. We offer a
summary in Section V.

I. Investments in Private Equity Funds – The Institutional Detail

Private equity funds are generally organized as limited liability partnerships (LLPs) with
a contracted life of ten years. The fund is managed by a private equity firm such as Blackstone,
which serves as the general partner (GP) of the partnership. Investors in the fund, typically large
institutions, are the limited partners (LPs). LPs are passive investors in the fund. At fund
inception, LPs commit to provide capital for management fees and for the fund to make
investments in portfolio companies. These commitments are not transferred to the GP
immediately. Instead, the GP calls capital for investments at its discretion when it encounters
investment opportunities. Similarly, LPs receive cash distributions when the GP chooses to exit
portfolio companies through an IPO, acquisition, or liquidation.

GPs are compensated with a management fee and a share of the profits earned by the
fund, called carried interest. The most common management fee is 2% of committed capital per
year, so that an LP who committed $100 million would pay a total of $20 million in management
fees, leaving $80 million for investments in portfolio companies. The most common carried
interest arrangement is 20% of profits, with LPs receiving their committed capital back, plus a
hurdle rate of 8%, before carried interest is earned. Typically, a catch-up provision specifies that
GPs receive 100% of further distributions until they have received 20% of total profits. After that
point, additional proceeds are split between LPs and GPs according to the carried interest rate.
See Gompers and Lerner (1999), Metrick and Yasuda (2010), and Robinson and Sensoy (2013)
for descriptions of different management fee and carried interest arrangements.
There are two main types of private equity funds categorized by the nature of their portfolio companies. Venture capital funds invest in early-stage pre-IPO companies. Buyout funds, in contrast, invest in more-established companies, including public firms. Given this difference, systematic differences in risk and performance are typically observed, hence empirical analysis is often undertaken and results reported on venture capital funds separately from buyout funds. Henceforth when discussing issues that apply to both we use the general term private equity, as in Section II which presents our valuation model. When presenting numerical results, as in Section IV, we report two sets of analyses with parameters selected to represent either venture capital or buyout funds.

II. The Model

Consider a risk averse LP with initial wealth \( W_0 \) and a CRRA utility function for date \( T \) wealth \( W_T \) given by

\[
U(W_T) = \frac{W_T^{1-\gamma}}{(1-\gamma)}
\]

where \( \gamma \) measures the LP’s risk aversion.\(^3\) The LP can allocate wealth at date 0 across three assets: a risk-free bond, the public equity market, and a private equity fund managed by a GP. Let \( x \) denote the fraction of initial wealth \( W_0 \) invested in the private equity fund, \( y \) denote the fraction of initial wealth allocated to the public market, and \( z = (1-x-y) \) denote the remaining fraction of initial wealth invested in the risk-free bond, with \( 0 \leq x, y, z \leq 1 \). Denote corresponding dollar amounts as \( X_0, Y_0, \) and \( Z_0, \) respectively.

A. Preliminaries

The private equity fund has a maximum investment life of \( T \), though the GP may sell or “liquidate” fund assets and distribute proceeds earlier. The LP’s participation in the private equity fund involves a quantity of capital fixed at date 0, known as committed capital, which is itself split into two categories. The first is investment capital, denoted by \( I_0 \), which the LP pays to the GP once suitable assets are identified and the investment capital is called by the GP. The

\(^3\) An alternative specification is the utility of lifetime periodic consumption, for which an investment in private equity requires a temporary commitment of capital. See Sorensen et al. (2014).
second are management fees, which are a fixed percentage \( m \) of committed capital and are paid at the end of every period with the first payment on date 1 and the last payment on the liquidation date. To simplify analysis of fund performance we collapse this arrangement into a single cash outflow by the LP at date 0 and a single cash inflow on the liquidation date. In particular, we assume that both investment capital and the present value of management fees for the full horizon \( T \) are set aside at date 0 in an interest bearing account earning the risk-free rate \( r_f \). This avoids uncertainty in the ability of the LP to supply investment capital when called.\(^4\) Interest earned on the investment capital prior to the call date is retained by the LP in the risk-free account. Management fees are paid at the end of each period out of the account. On the liquidation date, the single cash inflow is defined as the distribution from the GP and the remaining balance in the risk-free account.

Private equity funds often use leverage to increase the scale of asset purchases. We assume the GP issues zero-coupon debt in the amount of \( D_0 \) when investment capital is called and assets are purchased so that fund assets are initially worth \( A_0 = I_0 + D_0 \). The leverage ratio is defined as \( D_0/I_0 \). We assume that debt investors are risk-neutral and hence the yield on fund debt is \( r_f \).

In practice the GP makes multiple capital calls, invests in a portfolio of assets, and distributes proceeds from asset sales as they occur. We assume for simplicity a single capital call and a single liquidation date, on which proceeds from asset sales are distributed among debt investors, LPs, and the GP taking into account the seniority of debt investor claims, a hurdle rate for fund performance, and the carried interest earned by the GP. See the Appendix for details. We denote the share of asset sales flowing to LPs and GPs as \( LP(A_t) \) and \( GP(A_t) \), respectively.

To account for the correlation between the returns of public and private equity, and to allow for other links between the two markets, we need to model their joint dynamics. We assume that the returns of public \((U)\) and private \((V)\) equity are governed by a bivariate normal distribution over each fraction \( \Delta t \) of a year:

\[
(1) \quad r_U, r_V \sim BVN(\mu_U \Delta t, \mu_V \Delta t, \sigma_U^2 \Delta t, \sigma_V^2 \Delta t, \rho),
\]

\(^4\) This uncertainty can be important in practice when, for example, and LP makes multiple commitments in anticipation of a sequence of calls, and the calls instead cluster.
where $\mu$ and $\sigma$ are the annual mean and standard deviation of returns, respectively, and $\rho$ is the correlation between public and private equity returns. In some parts of the analysis we consider stand-alone investments in public or private equity, and in those cases we use the corresponding marginal distribution. Our focus is on the LP’s ability to exit the private equity fund on a secondary market prior to the liquidation date, and for this reason our valuation framework employs a discrete-time numerical technique. Over each period, we allow the value of public and private equity to either increase, stay the same, or decrease, with probabilities and magnitudes chosen to match the distributional assumption in (1). In other words, we calibrate a trinomial distribution to match the assumed normal distribution.

Figure 1 illustrates the possible paths the value of private equity investment capital can take over the $T$ periods in the life of the fund. Capital committed at date 0 may be called immediately and invested by the GP, in which case its value will either increase, stay the same, or decrease each period. If the investment capital is not called its value remains unchanged until the beginning of the next period. Figure 2 illustrates the possible paths the investment in both public and private equity can take. Figure 2A describes the three dimensions: time and the values of public and private equity. We assume that the investment in public equity occurs immediately upon the allocation decision at date 0. After the capital committed to the private equity fund is called there are nine joint outcomes possible at each point as shown in Figure 2B. As described in the Appendix and corresponding Table A1, we determine the nine branch probabilities by separating the public and private equity value changes. The public equity evolves according to its marginal distribution, and then the private equity evolves according to its conditional distribution, where the conditioning information is whether the public equity increased, stayed the same or decreased. Figure 2C shows this decomposition.

We achieve changes in asset values via multiplicative factors. From one date to the next, the LP’s stake in the public market can increase by a factor $e^u$, decrease by a factor $e^{-u}$ or stay the same. Once private capital is called and invested by the GP, the value of the purchased assets will increase by a factor $e^v$, decrease by a factor $e^{-v}$ or stay the same. When analyzing changes in either public or private markets in isolation, we will need the trinomial probabilities for changes in the value of the relevant market according to its marginal distribution. For these, we denote the probabilities of an increase, no change, and a decrease as $p_1$, $p_0$, and $p_{-1}$, respectively. In
general, we will need the joint probabilities for changes in the values of public and private markets. For these we denote the probability of an increase in both markets as $p_{1,1}$, with other branches defined similarly with the first subscript indicating the change in the public market.

The relatively recent emergence of secondary markets for private equity investments is a response to their extreme illiquidity. Illiquidity is especially important when the LP is subject to liquidity shocks, such as the tightening of credit markets and emergence of redemption restrictions in many investment vehicles, even money market funds, which occurred during the recent financial crisis. To assess the impact of illiquidity on the value of private equity investments, we assume that there exists the probability $\omega$ that a liquidity shock occurs the following period. If a liquidity shock does occur, we assume that the LP sells all risky assets on the secondary market and invests fully in the risk-free bond, i.e., a flight to safety.

Rather than require the LP to sell risky assets when a liquidity shock occurs, we could allow the LP to decide at each node whether to exercise his real option to transact on the secondary market or continue to hold the position in the private equity fund for an additional period. Early exercise may be optimal, for example, when liquidity shocks are somewhat predictable by the LP, and if haircuts widen substantially when liquidity shocks occur. In unreported results, we find that in our model LP’s do not generally exercise early unless the probability of a liquidity shock is quite large. We provide an illustration of the early exercise decision in Section IV.

The probability of a liquidity shock is likely related to the performance or risky assets. We set the probability of a liquidity shock at a given node equal to a base rate of 1% per quarter that increases by 1% for every net decrease in private equity asset value, when no investment in public equity occurs, or by 1% for every net decrease in public equity asset value otherwise. We assume that if the LP sells his stake in the private equity fund in a secondary market the sale will require the LP accept a discount or haircut. Empirically, average haircuts vary over time, reflecting the overall appetite for alternative assets and suggesting that the size of the haircut has a systematic component. One might imagine an idiosyncratic component as well, if for example counterparties can prey on LPs that have experienced a liquidity shock.\(^5\) We could make the size of the haircut a function of asset values but for now assume that the haircut can take one of two

\(^5\) Ramadorai (2012) documents wider haircuts for hedge funds in distress.
values. The discount then is a multiplicative factor that is either in a high or low state, 
\[ H^H = (1 - h^H) \quad \text{or} \quad H^L = (1 - h^L) \] where \( h \) is a haircut with \( 0 \leq h^L < h^H < 1 \). We denote the probability of each state occurring as \( p^H \) or \( p^L \), which could be a function of the performance of asset values but we set as 50% for both. To operationalize these haircuts, suppose that the LP approaches the secondary market at some date \( t \) after the call date \( k \), and fund asset levels are worth \( A_t \) at that time. We assume that the market price for the LP’s stake in the fund would be a fraction \( H \) of the payoff that would accrue to the LP if the GP were to immediately sell the fund assets, i.e., the LP’s payoff after debt holders and the GP’s carried interest were paid. Similarly, if the LP approaches the secondary market at some date \( t \) prior to or on the call date \( k \), the market price for the LP’s stake would be a fraction \( H \) of the investment capital. In all cases, the LP would retain the current value of interest earned on the deposit of investment capital. With regards to management fees, we assume that the investor who buys from the LP on the secondary market is willing to assume responsibility for paying them. However, the new LP will only pay a maximum of the same percentage management fee as the original LP. If the new LP buys at a price less than the original LP’s investment capital, the original LP must make a transfer payment to the new investor equal to the time \( t \) value of the difference between the original management fees (which the GP still expects to receive) and those consistent with the same percentage management fee applied to the lower purchase price.

B. Asset valuation in the lattice

Let \( P_{t,i,j,k} \) denote the LP’s subjective valuation of their investment at node \((t, i, j)\), where \( t \) denotes the date, ranging from 0 to \( T \), and \( i \) and \( j \) denote the number of net increases in public and private equity asset values, respectively. As will become clear, we also need to condition the value on the date on which private capital is called; let \( k \) denote a potential call date. At date 0, the probability of capital being called on a given date \( t \) is denoted by \( \pi_t \). To emphasize the uncertainty over capital calls we assume the earliest capital can be called is \( t = 1 \), reflecting also the time required for the GP to identify suitable assets. Further, let \( K < T – 1 \) denote the last date on which capital can be called, which is typically defined in a partnership agreement. Since public equity investment occurs at time 0, \( i \) ranges from \( t \) to \(-t\). However, on the private equity
In particular, nodes with \( |j| \leq t-k \) are reachable.

Denote the probability that the GP sells fund assets and distributes proceeds at node \((t, i, j)\), and conditional on call date \(k\), as \(\phi(t, i, j, k)\). Since the lattice uses a backward recursion, both investment values \(P_{t,i,j,k}\) and liquidation probabilities \(\phi(t, i, j, k)\) are conditional on reaching a given node without having yet sold the asset. The probability of an asset sale can be a function of the duration of the private equity investment defined by the call date \(k\) and the current date \(t\), as well as the cumulative performance of public or private equity assets defined by the number of net increases given by \(i\) or \(j\). Presumably, the GP would not sell fund assets prior to \(T\) if the GP’s payoff, after bondholders and the LP received their share, were zero, since the GP always has an incentive to delay liquidating assets since this allows him to continue to draw management fees each period. For this reason we set the probability of an asset sale at node \((t, i, j)\), and conditional on a particular call date \(k\), equal to a function of the GP’s payoff given an immediate sale as a percentage of the maximum payoff a GP could achieve which occurs when fund assets reach maximum value at date \(T\) conditional on the call date \(k\), i.e.

\[
\phi(t, i, j, k) = \sqrt{\frac{GP(A_0e^{ij})}{GP(A_0e^{v(T-k)})}}.
\]

We take the square root of the ratio in (2) because the probability of an asset sale prior to \(T\) would otherwise generally be extremely low given the size of the maximum GP payoff that occurs in the extreme upper node of the lattice.

Each value \(P\) is a certainty equivalent, that is, the value of a risk-free bond that provides the same utility for sure as the expected utility of the investment the following period. To determine the value of the LP’s investment at date 0, we consider the range of possible values at date \(T - 1\), which are based on the size of the payoffs to the LP at the final date \(T\).

The LP’s subjective valuation at node \((T - 1, i, j)\), conditional on call date \(k\), can be expressed as:

\[
\frac{1}{1-\gamma} \left( \left(1 + r_f \right) P_{T-1,i,j,k} \right)^{1-\gamma} = \frac{1}{1-\gamma} \left( \sum_{a=1}^{1} \sum_{b=1}^{1} p_{a,b} \left( LP(A_0e^{v(j+b)}) + Int_T + Y_0e^{u(j+b)} + Z_T \right)^{1-\gamma} \right)
\]
where the LHS is the certain utility of date \( T \) wealth generated by investing the date \( T – 1 \) investment value in a risk-free account and the RHS is the expected utility of random date \( T \) wealth. The components of the RHS include: the stake in the private equity fund which generates a payoff \( LP(\cdot) \) that is a function of private equity asset values at date \( T \), the time \( T \) value of interest income earned by the LP on the deposit of investment capital between date 0 and the capital call, denoted by \( \text{Int}_T \), the time \( T \) value of public equity denoted by \( Y_0e^{u[i+ii]} \), and the time \( T \) value of the riskless bond, \( Z_T \). Since by assumption the liquidation date has not occurred on or before \( T – 1 \), the LP knows the payoff will occur at date \( T \), hence the liquidation probability is irrelevant. Similarly, since by assumption the last possible call date \( K \) occurs prior to \( T – 1 \) the LP knows capital has been called and the probability \( \pi \) is irrelevant. The expectation is formed over the nine branches emanating from node \((T – 1, i, j)\) reflecting the possible changes in public and private equity asset values between date \( T – 1 \) and date \( T \). We can solve explicitly for the portfolio value at node \((T – 1, i, j)\) as:

\[
(4) \quad P_{T-1,i,j,k} = \left(1 + r_f\right)^{-1} \left\{ \sum_{i=1}^{1} \sum_{j=1}^{1} P_{i,j,i} \left[ LP\left( A_0e^{x_{j+j}} \right) + \text{Int}_T + Y_0e^{u[i+ii]} + Z_T \right] \right\}^{1/\gamma}.
\]

Now we begin a backward recursion to date 0, i.e., dates \( t \) ranging from \( T – 2 \) to 0. As we iterate backwards through time, we take into account uncertainty regarding liquidation and call dates. There are three possible cases involving the relation between \( t, k \), and \( K \) as described next.

First, for \( t \geq K \), the call date is known by the LP and it is necessary to compute conditional values for all possible call dates:

\[
(5) \quad P_{t,i,j,k} = \left(1 + r_j\right)^{-1} \sum_{i=1}^{1} \sum_{j=1}^{1} P_{i,j,i} \left\{ \phi \left[ LP\left( A_0e^{x_{j+j}} \right) + \text{Int}_{t+1} + \text{Fee}_{t+1} + Y_0e^{u[i+ii]} + Z_{t+1} \right] \right\}^{1/\gamma}.
\]
There are four possible outcomes at each future node. First, with probability $\phi$, the GP liquidates private equity fund assets, and the LP receives their share in the proceeds. In this instance we assume the LP marks their portfolio to market, hence we add to the payoff $LP(\cdot)$: $Int_{t+1}$ the time $t + 1$ value of interest earned on the deposit of investment capital between date $0$ and the call date, $Fee_{t+1}$, the time $t + 1$ value of capital set aside at time $0$ to pay future management fees, as these will no longer be paid and can be withdrawn, the value of public equity, and the value of the riskless bond. Second, with probability $(1 - \phi)\omega p^H$ the GP does not liquidate and a liquidity shock occurs with high haircuts. In this case the LP’s payoff from private equity is the assumed secondary market value involving the haircut $H^H$. The withdrawal of remaining fees is adjusted as necessary to account for a transfer payment to the new LP. Third, with probability $(1 - \phi)\omega p^L$, the liquidity shock occurs with low haircut. Fourth, with probability $(1 - \phi)(1 - \omega)$ neither a GP liquidation nor a liquidity shock occurs, and the payoff to the LP is the previously computed continuation value.

Second, for $t = K - 1$, the LP will have observed if a call has occurred on or before $t$. Thus, for $k \leq t$, the valuation is computed as in (5). If a call has not yet occurred, the LP knows capital will be called on date $K$ since $K$ is the last possible call date. In this situation, the value of the investment is:

$$P_{t,x,0,K} = (1 + r_j)^{-1} \left( \sum_{i=0}^{1} p_{ii} \left( \begin{array}{c}
\omega p^H (H^HI_0 + Int_{t+1} + Fee^*_t + Y_0e^{\nu_{(i)}(i+\omega)} + Z_{t+1})^{1-\gamma} + \\
(1 - \omega) P_{t+1,x+i}^{1-\gamma,0,K}
\end{array} \right) \right)^{\gamma}$$

where $p_{ii}$ are the branch probabilities following the public market’s marginal distribution. Note here the haircut is applied to the investment capital $I_0$ since it has not yet been deployed by the GP. Consequently, the liquidation probability is irrelevant.

Third, for $0 \leq t < K - 1$, the LP again will have observed if a call has occurred on or before $t$ and for $k \leq t$ investment values are given by (5). If a call has not yet occurred, then the LP can form a subjective valuation by considering two outcomes at each adjacent node on date $t + 1$: either capital will be called on that date or it won’t. So, for $k = t + 1$, 
where $\pi_{t+1}$ is the probability of a capital call on date $t + 1$ conditional on capital not yet called prior to that date. So, for dates $t$ prior to the penultimate call date, we compute $t + 1$ conditional values. The first $t$ are conditional on calls occurring on dates 1 through $t$. The last is the value conditional on the capital not yet called, and is denoted as a call date $t + 1$. As such, the value $P_{t+1, i, ii, 0, k+1}$ in the RHS of (7) is the value conditional on a call after date $t + 1$ and as the iteration through time continues it recursively incorporates all possible future call dates. At date 0 only one value is computed, $P_{0,0,0,1}$, which is the LP’s ex-ante valuation of the portfolio.

### III. Calibration

Our base case assumes the private equity fund has a 12-year life which we model with 48 quarterly time steps. Figure 3 shows our assumptions about the distribution of the GP’s timing decisions with regards to asset purchases. Metrick and Yasuda (2010) report that investment periods are typically five years long and that the bulk of the investment occurs in the first three years. In our base case, as depicted in Figure 3, asset purchases can occur at the end of any quarter during the first five years of the fund’s life with time-varying probabilities $\pi$. With regards to the probability of asset sales, Metrick and Yasuda use 20% per year post investment, noting that the probability is likely a function of asset values. In our analysis, as described in Section II, we set the probability of an asset sale at a given node equal to the square root of the ratio of the GP’s share of the proceeds to the maximum share possible given the call date and the cumulative performance of the asset at the node.
We assume annual returns for public equity have a mean of 10% and standard deviation of 15%. For annual returns of private equity assets, we use a base-case mean of 20%. Harris et al. (2014) report average internal rates of return of 14.2% for buyout funds and 16.8% for venture capital funds. These are based on after-fee LP cash flows, hence corresponding asset returns, though unobservable, are certainly higher. Most of our empirical analysis is based on inferring the break-even level of private equity asset returns. For private equity asset return standard deviation, Metrick and Yasuda (2010) use a volatility 60% for individual buyout investments and 90% for individual firms in venture capital portfolios, based on results in Cochrane (2005). They report on average 24 investments per venture capital fund and 15 per buyout fund, and based on Campbell et al. (2001) assume that the pairwise correlation between these investments is 20%. With an equal-weight portfolio, these assumptions imply that the portfolio volatility of buyout funds is 30% and the volatility of venture capital funds is 43%. We then round and assume 30% for buyout funds and 45% for venture capital funds.

To illustrate the lattice topology with quarterly time steps, we use the base-case mean and standard deviation for annual public equity returns of 10% and 15%, respectively, and the mean and standard deviation of 20% and 30%, respectively, for buyout fund assets. These parameter correspond to quarterly means of 2.5% and 5.0%, respectively, and quarterly standard deviations of 7.5% and 15.0%, respectively. We consider two correlations between public equity and buyout fund assets of 0.3 and 0.6. Listed in Table A-1 in the Appendix are branch probabilities determined to generate a discrete trinomial distribution that matches the mean and standard deviation of a normal distribution. Our procedure described in Appendix 1 results in step sizes for public equity of 9.8% and either 19.6% or 21.0% for buyout funds, per quarter, depending on the correlation with public equity. Panels A and B show results with correlation of 0.3 whereas panels C and D show results with correlation of 0.6. In panels A and C, listed for public equity is the marginal probability of each of the three possible returns from the marginal probability distribution, and listed for the buyout fund assets is the probability of each of the three possible returns given one of the public equity returns from their conditional probability distribution. Note that the conditional standard deviation of buyout fund asset returns is equal in all cases, since the

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7 Naturally all parameters used should be forward-looking estimates but for our purpose we rely on historical summary statistics. These vary with the length of the estimation window. Over the 50-year period 1964 – 2013 the monthly return of the Fama-French market factor has an annualized mean of 10.9% and standard deviation of 15.6%.
bivariate normal distribution implies a conditional standard deviation equal to the unconditional standard deviation scaled by $\sqrt{1-\rho^2}$. The conditional mean of buyout fund assets is substantially affected by the return of public equity and the correlation between the two. The conditional mean following a positive public equity return, for example, is 9.4% on a quarterly basis when correlation is 0.3 versus 13.8% when correlation is 0.6. Panels B and D show the corresponding joint probabilities. Note that probabilities along the diagonal are higher in Panel D given the higher correlation between the two markets.

The match between the discrete distribution of the lattice and the assumed continuous distribution is illustrated in Figure 4, which shows the probability of landing on each node at time $T = 48$ as well as the corresponding probability from the bivariate normal distribution.

We assume that liquidity shocks are rare events with base-case probability $\omega$ equal to 1% per quarter, consistent with the estimate of a global consumption shock in Nakamura et al. (2013) of 3.7% per annum. We change the probability in the lattice at date $t$ by increasing $\omega$ by 1% for every net decrease in asset value between the date capital is called and $t$. For nodes with a net increase in asset value we leave $\omega$ at 1%. When analyzing stand-alone private equity investments private equity asset values determine $\omega$. When considering the joint evolution of public and private equity, however, the performance of public equity determines $\omega$. The intuition is that an LP is more likely to suffer a liquidity crisis when its relatively more liquid assets are declining in value.

For GP fees we assume a 2% management fee, a 20% performance fee, and an 8% hurdle rate, all of which are typical as found in Metrick and Yasuda (2010).

With regards to LP risk aversion, Mehra and Prescott (1985) review the use of $\gamma$ in prior literature, state that most studies use values between one and two, and argue that the parameter should be restricted to a maximum of ten. More recently, Campbell and Cochrane (1999) use $\gamma = 2$ in their seminal work on consumption-based asset pricing. As described in the following section, we find that for a 12-year investment horizon, and our assumptions about the risk-free rate and public equity’s expected return and standard deviation, values for $\gamma$ between 1.5 and 3.5 generate reasonable optimal asset allocations. Hence, these are the values we use throughout the paper.
Table 2 summarizes the base case assumptions for each of the parameters of the model.

IV. Results

A. Impact of Uncertainty of Timing of Calls and Distributions

As mentioned previously, prior academic work abstracts from the uncertainty of timing of LP cash flows in private equity funds. In this subsection we focus on uncertainty regarding the dates of capital calls and the return of equity capital by the GP. For this reason we assume here no secondary market for LP investments.

In much of our analysis it will be helpful to have a benchmark based on public equity for comparison. Consider an investor with initial wealth of $1,000,000 and a 12-year investment horizon. We assume public equity has expected annual return of 10%. The riskless rate is 4%. Using the same type of trinomial lattice as described in Section II, we determine the allocation weights across public equity and riskless bonds that maximizes the certainty equivalent of the portfolio, over a range of risk aversion levels and public equity volatility. We restrict allocation to public equity to be 100%. Figure 5 shows the results. For low levels of risk aversion and volatility the investor should invest all wealth in public equity. For volatility of 15%, which is typical of estimates of long-run volatility for the stock market, optimal allocation is 100% for $\gamma = 3.0$ and below. However, given short-term liquidity needs, all investors hold some fraction of wealth in the riskless asset. In the 2014 NACUBO study of university endowments, for example, the typical endowment allocates roughly 20% to cash and bonds, hence for our benchmark we assume an 80/20 portfolio.

Consider an LP who invests in a portfolio of 80% private equity, instead of public equity, and 20% riskless bonds, where the bond allocation can be interpreted as optimally trading off risk and return as well as accommodating short-term liquidity needs as discussed above. We assume a risk aversion parameter $\gamma$ of 2.5. We consider five different assumptions about the private equity fund cash flows. In the first three cases, we assume that the GP waits until the end of 12 years before distributing the LP’s claim on fund assets, and vary the assumption about

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8 The position in riskless bonds also avoids cases in which the LP receives a payoff of zero in a levered fund when terminal asset values are less than the amount owed to debt investors – given the assumed utility function these cases have a dramatic effect on valuation.
capital calls. In the first, consistent with the assumption in Sorensen et al. (2014), capital is called immediately. In the second we assume that the capital call occurs at the end of 2.5 years instead, reflecting the time spent by the GP in identifying suitable investments. In the third we assume that capital is called with equal probability each quarter over the first 4.75 years, resulting in an expected delay of 2.5 years. The fourth and fifth case are the same as the second and third, respectively, but in each we allow for the GP to liquidate assets prior to the end of 12 years. We assume that for all nodes after a given call date, and which correspond to a positive GP payoff if assets were liquidated, the probability of a disbursement is equal to the square root of the ratio of the GP payoff to the maximum possible GP payoff for that call date.

We apply these assumptions to three types of private equity funds: venture capital with no leverage and buyout funds with leverage of either 100% or 300% initial equity. Venture capital assets are assumed to have volatility of 45% whereas buyout fund assets have volatility of 30%. For all three types of funds we calibrate the expected asset return by numerically determining when the 80/20 portfolio with an immediate call and payoff at the end of year 12 has the same certainty equivalent as a 80/20 portfolio involving public equity instead. The public equity is assumed to have an expected return and volatility of 10% and 15%, respectively.

Table 2 shows the certainty equivalents and performance metrics of each type of fund and for each of the five assumptions about the timing of LP cash flows. To measure risk and return we focus on the total value to paid in capital (TVPI) defined simply as the ratio of the final LP payoff to the initial committed capital, in this illustration 80% of initial wealth. The advantage of TVPI over a return measure like IRR is that different assumptions about cash flow timing result in different holding periods, and TVPI reflects the length of the holding period along with asset performance in one metric. We report the quartiles of the TVPIs possible in the lattice taking into account the probability of each occurring. In other words, we record each possible outcome and its probability implied by all the paths that land on the same node, then sort by the magnitude of the outcomes. The quantiles are then defined by the cumulative probability of these outcomes.9

Panel A shows results for venture capital. The first column shows results for an immediate call with disbursement at the end of year 12. The certainty equivalent is 160.76% of initial wealth, corresponding to a 20.6% expected annual asset return necessary to match the

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9 Expected outcomes and the variance of outcomes are heavily influenced by extreme values occurring at the very top or bottom of the lattice.
certainty equivalent of a public equity/riskless bond portfolio. The above par certainty equivalent indicates that for this investor public equity is attractive. Here the expected investment horizon and commitment are both exactly 12 years. This first case generates a median TVPI of 8.6 with only a 6.8% probability of generating a TVPI less than one. The next two columns reduce the expected investment period to 9.5 years given their assumption about delayed calls. This shorter investment period naturally reduces their median TVPIs to 5.1, and increases the likelihood of a TVPI below one to 9%. Their certainty equivalents are 144.47 and 143.83, respectively, a drop of about 10% from the first case, reflecting the loss of earning power when capital remains uninvested by the GP.

The fourth and fifth columns reduce the expected investment horizon further to about 8.7 years by allowing the GP to disburse assets early. The expected disbursement date is roughly 10 months prior to the end of year 12. The impact of early disbursement is largest in the upper end of the distribution of TVPIs, with the 75th percentile dropping to about 8.8 from 11.5 with delayed calls but no early disbursement. The certainty equivalents are 140.31 and 139.22, respectively, 3% lower than the prior two cases. The lower value reflects the additional uncertainty over call dates, and the increase in the probability of a low TVPI that would accompany the most delayed calls, since the investment period is shortest. The combined impact of uncertainty in call and distribution dates is a decline in certainty equivalent of about 12.5%.

Panels B and C show results for buyout funds with 100% and 300% leverage, respectively. The results are not directly comparable to those in Panel A since both the expected return and volatility of fund assets are different. With the additional leverage, the first column shows that asset returns need only be 17.3% to match the certainty equivalent of the unlevered 80/20 public equity and riskless bond portfolio.10 The median TVPI increases from 8.6 in Panel A to 10.0 in Panel B with the levered position in the underlying asset. Despite the increase in median TVPI the certainty equivalents are the same in the first column across the panels by design, and reflect the additional variability in outcomes. In contrast, with uncertainty in the timing of cash flows, certainty equivalents and median TVPIs are higher in the case of no early disbursement but delayed calls. Here the increase in median TVPIs must offset the additional uncertainty generated by leverage. In contrast, for the case of early disbursement, Panel B shows

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10 The differences in the certainty equivalents in the first column across the panels is due to discreteness in the numerical procedure.
that certainty equivalents decrease with leverage and a lower asset return, a result of median TVPIs dropping from about 4.0 to 3.5. Note also that the expected investment horizon drops from about 8.7 years to 6.7 years, reflecting the impact of leverage on the assumed function for the probability of an early disbursement. Lastly, in Panel C, the higher leverage and resulting levered asset volatility requires a higher asset expected return to maintain the same certainty equivalent in the first column. In all other cases certainty equivalents increase.

Another way to illustrate the impact of uncertainty of timing in cash flows is to vary the asset expected return for each type of fund and measure the gap between the certainty equivalent valuation in the case of fixed call and disbursement dates to the case of uncertainty in call and disbursement dates. Figure 6 shows the relation between asset expected returns and certainty equivalents in which the investment period of the fund with a call at time 0 and fixed disbursement date (“No Uncertainty” in the figures) is set equal to the expected investment period of the fund with a uniform distribution for call dates over the first 19 quarters and the same disbursement probability as used in Table 2 (“With Uncertainty”). In this way we control for the mechanical positive relation between holding period and valuation and focus instead on the variability in outcomes resulting from the uncertainty in timing of LP cash flows. For the three types of funds, naturally there is a positive relation between expected asset return and the certainty equivalent. More importantly, there are two types of differences between the No Uncertainty and With Uncertainty assumptions. For high levels of asset return, the No Uncertainty assumption results in dramatically higher valuations. With high returns, the longer the investment period the better. For extremely low returns, the No Uncertainty assumption results in lower valuations than the With Uncertainty case for the opposite reason. With the levered funds in Figures E and F there is an interesting middle region with relatively high asset expected returns but the With Uncertainty case still generates a higher valuation, resulting from the assumed level of LP risk aversion. As shown in Figures B and C, the additional variation in outcomes generated by the longer holding period is not as important for an LP with lower risk aversion of 1.5.

B. Impact of Liquidity Shock and Time-Varying Haircuts
In Figure 7 we show certainty equivalents of private equity investments when the LP is vulnerable to liquidity shocks and can access a secondary market. Liquidity shocks occur at a base rate of 1% per quarter, with rates increasing as asset values fall as described in Section II. To show the impact of haircuts we show results for Small Haircuts, with $H^L = 95\%$ and $H^H = 90\%$, and Large Haircuts, with $H^L = 90\%$ and $H^H = 50\%$. Certainty equivalents are computed over a range of expected asset returns. As before we compute certainty equivalents for portfolios of 80% private equity and 20% riskless bonds. We show results for venture capital funds, with asset volatility of 45% and no leverage, and buyout funds, with asset volatility of 30% and leverage of either 100% equity capital or 300% equity capital. Figures A through C show results for less risk averse investors with $\gamma = 1.5$ and Figures D through F show results for less more averse investors with $\gamma = 2.5$. Naturally in all cases certainty equivalents are increasing in asset expected return. More importantly, the gap between certainty equivalents for Small and Large Haircuts is substantial and widens as expected returns increase. For higher risk aversion, the difference reaches a maximum of roughly 25% for all three fund types. For lower risk aversion, the difference is not as large but still reaches a maximum of roughly 15% for all three fund types. These results indicate that the efficiency of the secondary market for private equity investments has a significant impact on their valuation.

Perhaps the most important question facing LPs is what rate of return is required to compensate for the risk of underlying assets, the uncertainty of the timing of cash flows in private equity funds, as well as the uncertainty of the size of the haircut required to transact in the secondary market. Our methodology takes as an input the expected return of private equity asset values, and by varying this input we can quickly determine using numerical methods the return required to generate a break-even valuation. For this analysis we use as a benchmark the certainty equivalent of a portfolio of 80% public equity, with expected return of 10% and volatility of 15%, and 20% in riskless bonds with a return of 4%. The private equity investments are also portfolios with a 20% allocation to riskless bonds.

Table 3 reports the break-even expected returns for assets in venture capital funds in Panel A, as a function of LP risk aversion as well as the size of the haircut parameters $H^H$ and $H^L$. In all cases the break-even return is increasing in risk aversion and decreasing in the sale prices available in the secondary market as modeled by the haircut parameters. For the least risk
averse LP, the breakeven asset returns range from 22.9% at the most efficient secondary market ($H^k = 95\%$ and $H^H = 90\%$) to 25.5% for the least efficient secondary market ($H^k = 90\%$ and $H^H = 50\%$). The difference of 2.6% annually is one measure of the cost of inefficient secondary markets for VC funds. For the most risk averse LP, the breakeven asset returns range from 37.5% to 57.4%. Here the cost of inefficient secondary markets is 19.9% annually. Clearly risk aversion has a first-order impact on break-even returns suggesting that venture capital funds are appropriate as a stand-alone investment for only a subset of potential investors. Of course investors hold private equity funds in a well-diversified portfolio, and we explore in the next subsection conditions under which more the more risk averse investor would hold a position in private equity.

Another metric for break-even performance is the alpha of fund assets. These are straightforward to compute by subtracting from break-even asset returns their expected return from a single factor model. We use public equity as a proxy for the single factor. For venture capital, the asset beta is the product of the assumed 0.6 correlation and the ratio of venture capital asset standard deviation to that of public equity, 45%/15% or 3, yielding a beta of 1.8. This implies an expected asset return of 14.6% assuming a 4% risk-free rate and 10% expected return for public equity. Similarly, for buyout assets, the beta is the correlation of 0.3 times the ratio 30%/15%, yielding an asset beta of 0.6 and an expected return of 7.6%. Panel B lists the break-even alphas for venture capital assets. The impact of the secondary market efficiency is modest for the lowest risk aversion, but extreme for the most risk-averse, suggesting that for many investors it is unlikely that venture capital funds are appropriate when secondary markets are inefficient.

Panel C shows the realized LP returns that correspond to the break-even asset returns in Panel A. The break-even LP returns are more salient since they can be compared to historical LP returns to assess whether private equity funds have in the past generated sufficient returns to compensate investors for their risk and various costs. Given the uncertain call and distribution dates in private equity funds, there are a number of different ways that realized returns can be computed. For example, for a given investment horizon $T$, the LP’s capital is only deployed by the manager for the time between call and distribution dates, and assumptions are required to account for the periods outside this window. Since different paths through the lattice can result in dramatically different exit dates from the fund, we first compute the IRR on a quarterly basis that sets the present value of the payoff equal to the initial committed capital. Payoffs can occur from
either a distribution of asset proceeds by the GP or a secondary market transaction. Next, we compute a median IRR taking into account the likelihood of each possible payoff occurring in the lattice.\textsuperscript{11}

As seen in Panel C, the relation between the LP’s break-even return, risk aversion, and the haircut parameters is qualitatively the same as in Panel A. One other result stands out: the break-even LP return is roughly 50-60\% of gross return of fund assets. This result is consistent with anecdotal evidence of a 50-50 split between LPs and the GP. For comparison, Harris et al. (2014) report an average IRR for LPs in venture capital funds of 16.8\%, indicating that the average venture capital fund in their sample generated returns that exceed the break-even returns in Table 3 from the perspective of the least risk-averse investor but generally fell short otherwise, even with highly efficient secondary markets. Common performance metrics for private equity funds are the TVPI, discussed previously, and the Public Market Equivalent (PME). In this exercise we compute a median TVPI using the distribution of gross returns from the set of possible payoffs in the lattice. We compute PMEs by scaling each possible payoff at date \( t \) by the future value at \( t \) of the committed capital grossed up by the assumed public equity return of 10\%. Panels D and E show the median TVPI and PME, respectively, which correspond to the break-even asset returns in Panel A. As before, break-even TVPI and PME are substantially affected by the efficiency of the secondary market, especially for the more risk-averse investor. The break-even PME, for example, falls from 2.87 to 1.86 for the medium risk-averse investor as markets are made more efficient. For comparison, Harris et al. (2014) report average PMEs of 1.22 for buyout funds and 1.36 for venture capital funds. Here again the average fund outperforms the break-even benchmark for less risk-averse investors but underperforms those of more risk-averse investors.

Table 4 shows break-even asset returns and LP returns for a buyout fund with 100\% leverage. Assets have annual volatility of 30\%. Break-even asset returns are slightly lower than the venture capital fund results in Table 3. In contrast, break-even fund returns are in almost all cases somewhat higher, indicating that for the utility function considered, the levered investment in the lower volatility assets of the buyout fund is riskier for the LP than the unlevered investment in the higher volatility assets of the venture capital fund. For the least risk averse

\textsuperscript{11} We focus on median performance metrics due to the presence of influential outliers at the extremes of the lattice.
investor and the most efficient secondary market, for example, the break-even venture capital fund returns is 11.7% versus 14.7% for the buyout fund. Table 5 shows break-even returns for a fund with a leverage ratio of 3. In all cases the break-even fund returns are higher than in Table 4, as one would expect for the additional risk generated by leverage. The difference is much larger for the highest level of risk aversion. For the most efficient secondary market, for example, the break-even fund return is 24.0% for lower leverage and 38.2% for higher leverage. The average IRR of buyout funds in Harris et al. (2014) is 14.2%. Regardless of the haircuts, risk aversion, and leverage, this indicates that the average buyout fund in their sample underperformed a public equity benchmark.

One feature of private equity funds that drives a wedge between the returns of underlying portfolio companies and the return experienced by an LP is the fee structure charged by GPs. The most common is a 2% annual management fee and 20% carried interest, the base case assumptions in this paper. To illustrate the impact of fees on the returns earned by LPs, we repeat the break-even analysis described above for the LP with $\gamma = 1.5$. We determine the break-even asset return that sets the certainty equivalent of an 80/20 portfolio of private equity and riskless bonds to the certainty equivalent of an 80/20 portfolio involving public equity instead. Then we convert this to an asset alpha by a single-factor model. As described above, for the base-case assumptions, venture capital assets have a factor loading of 1.8 whereas buyout fund assets have a factor loading of 0.6, where public equity is used as the risk factor.

Table 6 shows results for each fund type across a range of management fees and levels of carried interest. In Panels A and B, corresponding to venture capital and 1X buyout funds, the maximum fees increase the break-even asset alpha by roughly 6% per annum. In Panel C, for 3X buyout funds, the asset alpha increases by about 4%.

C. Early Exercise

We illustrate in this subsection how the lattice can be used to study an LP’s real option to transact on the secondary market, which we call a discretionary sale. For this purpose we change our assumptions about the size of haircuts, the probability of a liquidity shock, and the relation between the two. The effect of these changes is to increase the benefit of exercising the option to sell on the secondary market prior to $T$ and prior to a liquidity shock.
Suppose that LPs who undertake discretionary sales do so at the lower haircut $H^L$ but those who are forced to sell due to a liquidity shock suffer a larger haircut of 50%. Furthermore, suppose that the probability of a liquidity shock at node $(t, j)$ equals a base rate of 1% per quarter that increases by 0.5% times $-t \times j$ for $j < 0$, i.e. the probability of a liquidity shock increases when asset values drop and this relationship strengthens over time. We cap the probability of a liquidity shock at 90%.

We report in Table 7 certainty equivalents of portfolios consisting of 20% riskless bonds earning 4% per year and 80% in a venture capital fund which uses no leverage and has a portfolio of assets with volatility of 45%. The LP has risk aversion of 2.5. Results are listed as a function of the expected return of fund assets and the efficiency of the secondary market during normal times. The first column, as a benchmark, shows values when the LP opts never to use the secondary market unless a liquidity shock occurs. Naturally the certainty equivalent increases with the asset expected return. The other columns show values when the LP optimally decides to execute discretionary sales when the payoff from doing so exceeds the certainty equivalent of holding the portfolio. As the haircut shrinks the certainty equivalents increase. Note also that the difference between the no-exercise values and the others shrinks as expected asset return increases since this reduces the probability of a liquidity shock.

To illustrate the pattern of exercise decisions we plot in Figure 8 the early exercise region for the case where the expected asset return is 40% and the haircut is 10% for discretionary sales. Given our assumption about the probability of a liquidity shock the exercise region is in the lower half of the lattice, as depicted, and rises slightly over time. For the assumptions about liquidity shocks and haircuts made in the rest of the paper, early exercise is generally not optimal, hence in the next subsection we return to the case of no early exercise as we explore the allocation decision across public and private equity.

D. Allocation across Public and Private Equity

The valuation and break-even analyses in the prior subsections involve portfolios of private equity and a riskless bond. While this setup is useful for isolating the impact of the efficiency of the secondary market, it ignores possible interrelations between public and private equity markets. These naturally include a diversification benefit, but may also involve a
correlation between public markets and the probability of a capital call, private equity asset sale by the GP, or liquidity crisis, all of which feature prominently in our valuation model. In this section we examine a portfolio of riskless bonds, public equity, and either buyout or venture capital funds to provide additional insight in a context resembling the endowment of an institutional investor.

Following the throes of the Great Recession, public equity has generated high returns from a historical perspective, resulting in some investors reducing their allocation to other asset classes.\(^{12}\) One way to frame the analysis is to determine the minimum expected return of a private equity fund such that an investor is indifferent between a portfolio with no private equity and one with a given allocation to private equity. We assume a 20% allocation to riskless bonds earning 4% in all cases to accommodate an institution’s immediate liquidity needs. For a given level of risk aversion, we then compute the certainty equivalent of a portfolio that is 80% allocated to public equity with expected return of 10% and volatility of 15%. Next, we numerically determine the expected asset return for private equity such that a portfolio that is either 20%, 40%, or 60% allocated to private equity (with corresponding allocation of 60%, 40%, and 20% to public equity) generates the same certainty equivalent as the 80% allocation to public equity.

Results are displayed in Table 8 across sets of parameters used previously to match venture capital funds with no leverage and assets with volatility of 45%. In Panel A, increasing the allocation to private equity can generate a larger diversification benefit, but can increase the risk of large losses from forced sales at a discount to NAV, so that the lowest break-even return is at 40% allocation to private equity for \(\gamma = 1.5\) as opposed to one of the extremes. Perhaps most salient for our study, comparing break-even fund returns across secondary market parameters gives a measure of the cost of illiquidity in a portfolio context. The difference is as narrow as 0% per annum (\(\gamma = 1.5\) and \(x = 40\%\)) to as high as 6.9% per annum (\(\gamma = 3.5\) and \(x = 60\%\)).

Comparing these results to those in Table 3 shows that break-even venture capital returns are substantially lower in a portfolio context. For investors with \(\gamma = 2.5\), for example, and when secondary markets are least efficient, the break-even fund return in Panel C of Table 3 is 21.1%, and this drops to just 15.4% when allocation to venture capital is 20% in Panel C of Table 8.

\(^{12}\) For example, see “Calpers to Exit Hedge Funds” Wall Street Journal 9/15/14.
Viewed in this way, the typical fund in empirical studies such as Harris et al. (2014) is closer to this benchmark. One caveat is that our procedure allows an individual LP to determine his own benchmark given an estimate of risk aversion, the LP’s chosen allocation to venture capital, and the other parameters of our model.

Tables 9 and 10 show corresponding results for buyout funds with leverage of 100% and 300% of equity capital, respectively, and asset volatility of 30%. The additional leverage has little impact on break-even fund returns for investors with $\gamma = 1.5$. For more risk-averse investors, additional leverage raises the break-even fund return substantially for larger allocations, with a difference of 7.1% per annum the largest allocation and a more efficient secondary market. A similar pattern exists for TVPI and PME in Panels D and E, respectively, in the two tables.

Our last analysis fixes the bond allocation to 20% and then finds the utility-maximizing split between public and private equity. We do this for each type of private equity fund at the most efficient and least efficient secondary market parameters. Results are displayed in Table 11. In Panel A, for the least risk-averse investor, the optimal allocation to venture capital increases from 53% to 71% when secondary markets are made more efficient. For the most risk-averse investor the increase is smaller in absolute terms but larger in percentage terms, rising from 15% to 25%. This result indicates that the efficiency of the secondary market has a substantial effect on the optimal portfolio. Also, the optimal allocation to venture capital is widely varying across risk-aversion levels. Results in Panels B and C are qualitatively similar. Interestingly, as reported in the 2015 NACUBO study, the allocations in the less efficient secondary market case correspond to the average allocation to alternative investments in 2014 for university endowments as a function of the size of the endowment.

VI. Conclusion

This paper presents a valuation framework for private equity investment from the perspective of a risk-averse LP, incorporating explicitly institutional features that differentiate private equity from other types of vehicles. In particular, we allow for a distribution of call dates thereby reflecting the uncertainty of how long capital is actually deployed in private equity assets. We also allow for uncertainty surrounding when assets are liquidated by the GP. Most
importantly, we model LP exits on a secondary market in response to liquidity shocks. Secondary markets are quickly developing in practice to remedy the severe illiquidity in private equity investment. Up until now, there has been no research that studies the impact of the efficiency of secondary markets on the valuation of private equity stakes.

We find that cash flow uncertainty does matter, both by reducing the average time that capital is deployed by GPs as well as creating additional uncertainty in the distribution of LP payoffs. In our model, we assume that LPs set aside all committed capital at date 0 in an interest-bearing account, hence a distribution of call dates naturally leads to periods of low earning power for LPs. We also find that the efficiency of the secondary market can have a dramatic impact on the attractiveness of private equity investment, in some cases reducing the break-even fund return by 10% per year.

Our model can be used to guide LPs in their ex-ante allocation decision, as well as ex-post analysis of performance. In both cases our procedure requires as an input the LP’s assessment of the distribution of call dates as well as the subsequent likelihood of a GP’s distribution of fund proceeds. These parameters can be studied using historical data of LP cash flows.

We make a number of simplifications to our representation of an LP’s decision problem to gain tractability, and at least two of these we plan on relaxing in future work.

First, for a given private equity investment, we assume that all committed capital is called on a single date, and all fund assets are liquidated on a single subsequent date. This assumption likely overstates the cost of illiquidity, since in practice LPs hold a portfolio of private equity investments, with partial calls and liquidations for each, so that the allocation to private equity in whole is somewhat self-financing. That said, the popular press has documented cases in which the systematic component of calls and distributions has left LPs with significant liquidity crises. In future work we plan on incorporating multiple calls and distributions.

Second, we assume the LP has an investment horizon equal to the contracted life of a single private equity investment. Our valuation model therefore focuses on this relatively short time period. In practice, private equity investors often have a very long investment horizon. For University endowments, for example, one could argue the relevant horizon is infinite given the permanent role that endowments can play in a University’s budget. In subsequent work, we plan
on fixing the investment horizon at some arbitrarily long horizon and allowing LPs to roll over
distributions from private equity investments into new commitments.

In both cases, relaxing these assumptions will likely increase the dimensionality of the
valuation, resulting in the need for large-scale parallel processing to generate numerical
solutions.
Appendix

A.1. Waterfall

The computation of claimant shares follows closely that in Sorensen et al. (2014). Suppose that fund assets are worth $A_t$ on the liquidation date $t$. Distributions to the three types of claimants are defined by three discrete asset levels denoted $A_1$, $A_2$, and $A_3$. The first, $A_1$, is the amount due debt investors, i.e.

(A1) $A_1 = D_0 \left(1 + r_f \right)^{(t-k)}$.

Note that the value of the debt reflects the time from issuance, which is the call date $k$. We assume that debt investors receive 100% of asset proceeds until they receive $A_1$, the full amount they are due. The second, $A_2$, is defined by the LP’s hurdle rate $h$ applied to the committed capital and all management fees paid to the GP, i.e.

(A2) $A_2 = A_1 + I_0 \left(1 + h \right)^{(t-k)} + \left( m Y_0 / r_f \right) \left[ 1 - \left(1 + r_f \right)^{-1} \right] \left(1 + h \right)^{t}$.

Note that the hurdle rate is applied to the investment capital only from the call date $k$, whereas the hurdle rate is applied to all management fees paid beginning on date 1. We assume that, once the debt investors are fully repaid, the LP receives 100% of additional asset proceeds until they receive $A_2 - A_1$.

The third, $A_3$, is defined by the GP’s carried interest, $c$, which is a percentage of fund profits, typically 20%, as follows:

(A3) $c \left( A_1 - (Y_0 + A_1) \right) = A_3 - A_2$.

The left-hand side of (A3) is the GP’s contractual carried interest payment and the right-hand side of (A4) is the “catch-up” region of asset sales, assuming that the GP receives 100% of proceeds beyond $A_2$ until their carried interest is paid. The solution to (A3) is $A_3 = \left( A_2 - c \left( Y_0 + A_1 \right) \right) / (1 - c)$. Beyond $A_3$, asset sale proceeds are split between the LP and GP with the GP receiving a fraction $c$ of the incremental proceeds.

A.2. Branch Probabilities

When considering private equity ($V$) investment as a stand-alone investment we assume returns $r_v$ over each period of length $\Delta t$ are normally distributed

(A4) $r_v \sim N\left( \mu_v \Delta t, \sigma_v^2 \Delta t \right)$.

where $\mu$ and $\sigma$ are annual mean and standard deviation, respectively. Following Kamrad and Ritchken (1991) we can approximate the continuous normal distribution of $r_v$ using a discrete random variable $r_v^*$ governed by the trinomial distribution used in the lattice. From any node in the lattice, the discrete variable can take on one of three values

(A5) $r_v^* = \begin{cases} v = \lambda_v \sigma_v \sqrt{\Delta t} & \text{with probability } p_1 \\ 0 & \text{with probability } p_0 \\ -v = -\lambda_v \sigma_v \sqrt{\Delta t} & \text{with probability } p_{-1} \end{cases}$

where $\lambda_v \geq 1$ scales the familiar Cox, Ross, and Rubenstein (1979) binomial lattice step size. The free parameter $\lambda_v$ is selected numerically to both ensure all probabilities between 0 and 1 and to keep the three branch probabilities as close to 1/3 as possible to aid the ability of the discrete
distribution to approximate the assumed continuous normal distribution.\textsuperscript{13} Probabilities are selected so that the mean and variance of the approximating distribution match those of the true distribution for portfolio returns:

\[
E\left(r^p\right) = (p_1 - p_3)v = \mu_p \Delta t \quad (A6)
\]

\[
\text{Var}\left(r^p\right) = (p_1 + p_3)v^2 - (p_1 - p_3)^2v^2 = \sigma_p^2 \Delta t.
\]

Using (A6) we can solve for the three branch probabilities as:

\[
p_1 = \frac{1}{2} \left[\frac{\sigma_p^2 \Delta t + \mu_p \Delta t^2}{v^2} + \frac{\mu_p \Delta t}{v}\right]
\]

\[
p_0 = 1 - \frac{\sigma_p^2 \Delta t + \mu_p \Delta t^2}{v^2}
\]

\[
p_{-1} = \frac{1}{2} \left[\frac{\sigma_p^2 \Delta t + \mu_p \Delta t^2}{v^2} - \frac{\mu_p \Delta t}{v}\right].
\]

When considering a portfolio of both private equity and investment in public markets, we need to model the marginal distribution of public equity (\(U\)) as well as the joint distribution of public equity and private equity, consistent with the bivariate normal assumption:

\[
r_U, r_v \sim BVN\left(\mu_U \Delta t, \mu_U \Delta t, \sigma_U^2 \Delta t, \sigma_U^2 \Delta t, \rho\right).
\]

where \(\rho\) is the correlation between public equity and private equity returns.

Boyle (1988) and Kamrad and Ritchken (1991) also make the assumption of bivariate normality in their analysis of options on two variables. In both papers, the authors allow variables to either increase, decrease, or stay the same over any increment of time. However, they restrict the process by assuming that if one variable stays the same, both do. In other words, there are five possible outcomes for the discrete joint distribution. In our analysis of public and private equity, we need to allow public equity values to change while private equity remains unchanged to model the situation between capital commitment and a capital call. Hence we develop our own procedure for establishing the topology of the lattice and the branch probabilities.

Boyle (1988) and Kamrad and Ritchken (1991) solve for all branch probabilities simultaneously by matching the mean and variance of each variable as well as the correlation between the two. In contrast, we employ a much simpler procedure that exploits the conditional distribution of one of the assets.\textsuperscript{14}

The procedure has two steps. In the first step, trinomial branch probabilities established in (A7) above are selected to match the marginal distribution of public equity. The marginal distribution of public equity is:

\[
r_U \sim N\left(\mu_U \Delta t, \sigma_U^2 \Delta t\right).
\]

The discrete trinomial distribution that best approximates (A9) is:

\textsuperscript{13} See Kamrad and Ritchken (1991) for a discussion.

\textsuperscript{14} Ho, Stapleton, and Subrahmanyam (1995) also exploit conditional probabilities when valuating options on multiple assets.
(A10) \[ r^*_U = \begin{cases} 
  u = \lambda_U \sigma_U \sqrt{\Delta t} & \text{with probability } p_1 \\
  0 & \text{with probability } p_0 \\
  -u = -\lambda_U \sigma_U \sqrt{\Delta t} & \text{with probability } p_{-1} 
\end{cases} \]

where the parameter \( \lambda_U \) is determined numerically as above and the probabilities are defined by (A7) using public equity’s mean and standard deviation.

In the second step, three sets of trinomial branch probabilities are computed to match the conditional distribution of private equity, where the conditioning information is the return experienced by public equity in the first step. For a bivariate normal distribution, the distribution of the return of private equity conditional on the return of public equity is:

\[
(A11) \quad r_{UV} \sim N \left( \mu_U \Delta t + \rho \sigma_U / \sigma_V \left( r_U - \mu_U \Delta t \right), \sigma_U^2 \left( 1 - \rho^2 \right) \Delta t \right)
\]

Naturally the joint probabilities can then be computed as the product of the marginal and conditional branch probabilities. In our discrete approximation, there are three possible returns for public equity, and hence three possible conditional distributions for private equity, which differ only by their mean. In each of the three cases the conditional variance is \( \sigma_U^2 \left( 1 - \rho^2 \right) \Delta t \) and this value is used in place of \( \sigma_U^2 \Delta t \) in (A7). Furthermore, the step size in (A5) \( \nu \) now equals \( \lambda_U \sigma_U \sqrt{\left( 1 - \rho^2 \right) \Delta t} \) with the free parameter \( \lambda_U \) selected via numerical methods to ensure all conditional probabilities lie between 0 and 1 and are as close to 1/3 as possible to aid convergence properties. We use a single step size for all three conditional distributions to maintain efficient lattice construction.

Case 1: Public equity increases with return \( r^*_U = u \). This results in a conditional mean for private equity equal to \( \mu_V \Delta t + \rho \sigma_V / \sigma_U \left( u - \mu_U \Delta t \right) \), which is used in place of \( \mu_U \Delta t \) in (A7).

Case 2: Public equity has no change with return \( r^*_U = 0 \). This results in a conditional mean for private equity equal to \( \mu_V \Delta t - \rho \sigma_V / \sigma_U \left( \mu_U \Delta t \right) \), which is used in place of \( \mu_U \Delta t \) in (A7).

Case 3: Public equity decreases with return \( r^*_U = -u \). This results in a conditional mean for private equity equal to \( \mu_V \Delta t - \rho \sigma_V / \sigma_U \left( u + \mu_U \Delta t \right) \), which is used in place of \( \mu_U \Delta t \) in (A7).

To illustrate, suppose that public equity has annual mean and standard deviation of 10% and 15%, respectively, whereas private equity has annual mean and standard deviation of 20% and 30%, and that the two are jointly normal with correlation of either 0.3 or 0.6. These values correspond to quarterly means of 2.5% and 5.0% for public and private equity, respectively, and quarterly standard deviations of 7.5% and 15.0%. Table A1 shows that the trinomial distribution for public equity’s marginal distribution has a step size \( \nu \) of 9.8% per quarter, with probabilities roughly 45%, 35%, and 20% for an increase, no change, and a decrease. In Panel A, with correlation .3, the conditional probability of private equity has a step size \( \nu \) of 19.6% per quarter. Conditional standard deviation is 14.3% per quarter, regardless of public equity’s change, whereas the conditional mean is 9.4% following an increase in public equity, and lower values otherwise. Panel B shows the joint probabilities for each combination of moves. Panels C and D repeat the calculations for a higher correlation of .6. Note that the joint probabilities in Panel D are higher along the diagonal than in Panel B.
Table A1. Branch Probabilities.

Listed are branch probabilities determined to generate a discrete trinomial distribution that matches the mean and standard deviation of a normal distribution. Public and private equity returns are governed by a bivariate normal distribution, with quarterly means of 2.5% and 5.0%, respectively, and quarterly standard deviations of 7.5% and 15.0%, respectively. Panels A and B show results with correlation of 0.3 whereas panels C and D show results with correlation of 0.6. In panels A and C, listed for public equity is the marginal probability of each of the three possible returns from the marginal probability distribution, and listed for private equity is the conditional probability of each of the three possible returns given one of the public equity returns. Panels B and D show the corresponding joint probabilities. Panels A and C also shows the step sizes in the lattice: 9.8% for public equity and either 19.6% or 21.0% for private equity.

<table>
<thead>
<tr>
<th>Panel A. Correlation = 0.3</th>
<th>Panel B. Joint Probabilities with Correlation = 0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Public Equity Return</td>
<td>Marginal Probability</td>
</tr>
<tr>
<td>Public Equity Return</td>
<td>Marginal Probability</td>
</tr>
<tr>
<td>$u = 9.8%$</td>
<td>0.4523</td>
</tr>
<tr>
<td>0</td>
<td>0.3502</td>
</tr>
<tr>
<td>-$u$</td>
<td>0.1974</td>
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</table>

<table>
<thead>
<tr>
<th>Panel C. Correlation = 0.6</th>
<th>Panel D. Joint Probabilities with Correlation = 0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Public Equity Return</td>
<td>Marginal Probability</td>
</tr>
<tr>
<td>Public Equity Return</td>
<td>Marginal Probability</td>
</tr>
<tr>
<td>$u = 9.8%$</td>
<td>0.4523</td>
</tr>
<tr>
<td>0</td>
<td>0.3502</td>
</tr>
<tr>
<td>-$u$</td>
<td>0.1974</td>
</tr>
</tbody>
</table>
References


NACUBO, 2015, NACUBO-Commonfund Study of Endowments.


Figure 1. Trinomial lattice.

The lattice shows possible changes in the value of capital committed by a limited partner (LP) at date 0. The capital may remain uninvested, in which case its value stays unchanged, or it may be called and invested, in which case the value can increase multiplicatively by one unit, stay the same, or decrease by one unit over each time period. The fund has finite life of $T$ periods, hence $T$ periods of change are possible if the capital is committed at date 0.
Figure 2. A bivariate distribution for public and private equity.

Figure 2A shows the three dimensions in the lattice used to model the bivariate distribution for changes in public and private equity. Figure 2B shows the nine possible joint outcomes over each time increment allowing for both public and private equity to increase, decrease, or stay the same. Figure 2C shows the same nine outcomes generated in two stages. In the first, the change in public equity is determined following its marginal distribution. In the second, the change in private equity is determined following its conditional distribution, in which the conditioning information is the change in public equity.

![Figure 2A. The three dimensions in the lattice](image1)

![Figure 2B. Possible outcomes generated by the bivariate distribution of public and private equity](image2)

![Figure 2C. Possible outcomes generated by public equity’s marginal distribution followed by private equity’s conditional distribution](image3)
**Figure 3. Asset purchases.**

Figure shows the assumed distribution of dates on which the GP of a 12-year private equity fund makes an acquisition.
Figure 4. Distribution comparison.
Figure 4A shows the joint probabilities of annualized private equity and public equity returns implied by a 48-step lattice approximating a 12-year investment horizon. Figure 4B shows the analytic probabilities from the underlying bivariate normal distribution.

Figure 4A. Probabilities implied by the lattice

Figure 4B. Probabilities implied by the bivariate normal distribution
Figure 5. Optimal allocation between public equity and riskless bonds.

Depicted is the optimal allocation to public equity with annual expected return of 10%, and volatility levels as listed, for a risk-averse investor, with the remainder of wealth invested in riskless bonds with annual return of 4%. The investor has $1,000,000 of initial wealth and risk aversion levels as displayed on the horizontal axis.
Figure 6. Impact of Uncertainty in Timing of Calls and Distributions.

Depicted are certainty equivalents of portfolios of 80% private equity and 20% riskless bonds as a function of the annual expected return of private equity assets from the perspective of a risk-averse investor with a 12-year horizon and a risk aversion level (γ) of 1.5 (Figures A through C) or 2.5 (Figures D through F). The risk-free rate is 4%. Figures A and D assume no leverage and 45% asset volatility. All others assume 30% asset volatility. Figures B and E assume leverage of 1X equity, whereas C and F assume leverage of 3X equity. The investor holds private equity investments until capital is distributed. Within each figure the two functions correspond to different assumptions about the timing of cash flows in the private equity fund.
Figure 7. Impact of Size of Haircut on Private Equity Values.

Depicted are certainty equivalents of portfolios of 80% private equity and 20% riskless bonds as a function of the annual expected return of private equity assets from the perspective of a risk-averse investor with a 12-year horizon and a risk aversion level ($\gamma$) of 1.5 (Figures A through C) or 2.5 (Figures D through F). The risk-free rate is 4%. Figures A and D assume no leverage and 45% asset volatility. All others assume 30% asset volatility. Figures B and E assume leverage of 1X equity, whereas C and F assume leverage of 3X equity. The investor faces the probability of a liquidity shock which forces an exit on the secondary market. Within each figure the two functions correspond to Large Haircuts (50% or 90%) and Small Haircuts (90% or 95%).
Figure 8. Early Exercise.

The shaded region as measured in quarter-years (horizontal axis) and asset levels (vertical axis) depicts cases in which a limited partner (LP) would optimally access the secondary market in order to exit a venture capital investment prior to fund liquidation. The LP has risk aversion level ($\gamma$) of 2.5. The fund has a maximum 12-year life. Asset returns are normally distributed with annual mean of 40% and volatility of 45%. Discretionary secondary market transactions involve a haircut of 5%. Forced secondary market transactions due to liquidity shocks involve a haircut of 50%. Liquidity shocks occur at a base rate of 1% (once every 100 quarters) that increases by 0.5% times the product of $t$ and $-j$ where $t$ is the number of periods since capital commitment and $j$ is the cumulative number of decreases in asset value.
Table 1. Base-case parameter values.

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>2.5</td>
<td>LP risk aversion</td>
</tr>
<tr>
<td>$T$</td>
<td>48</td>
<td>Life of private equity fund in quarters</td>
</tr>
<tr>
<td>$r_f$</td>
<td>2.5%</td>
<td>Risk-free rate</td>
</tr>
<tr>
<td>$m$</td>
<td>2%</td>
<td>Annual management fee as a percentage of committed capital</td>
</tr>
<tr>
<td>$h$</td>
<td>8%</td>
<td>Hurdle rate expressed as a periodic return on all investment capital and management fees paid</td>
</tr>
<tr>
<td>$c$</td>
<td>20%</td>
<td>Carried interest paid to general partner as a percentage of fund profits after all debt is repaid and hurdle rate is achieved</td>
</tr>
<tr>
<td>$\mu_U, \sigma_U$</td>
<td>10%, 15%</td>
<td>Mean and standard deviation of public equity returns</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>45%</td>
<td>Standard deviation of returns for venture capital funds</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>30%</td>
<td>Standard deviation of returns for buyout funds</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.30</td>
<td>Correlation between public and private equity returns</td>
</tr>
</tbody>
</table>

Other parameters

$W_0$ | Initial LP wealth |
$D_0$ | Dollar amount of zero-coupon debt issued by GP to increase asset purchase |
$x, y, z$ | Fractions of LP wealth initially allocated to public equity, private equity and risk-free bond |
$X_0, Y_0, Z_0$ | Dollar amount of LP wealth initially allocated to public equity, private equity, and risk-free bond |
$\pi$ | Probability of capital calls |
$\phi$ | Probability of asset sales |
$H^N, H^S$ | Multiplicative haircut during normal times and liquidity shock, respectively for LP exit on secondary market |
$k$ | Date of capital call |
$\omega$ | Probability of a liquidity shock |

Outputs

$u, v$ | Return increments for public and private equity possible at each node in the lattice |
$p_1, p_0, p_{-1}$ | Trinomial branch to approximate a normal distribution |
$p_{i,j}$ | Branch probabilities to approximate a bivariate normal distribution |
$I_0$ | Dollar amount of committed capital invested by GP |
$A_0$ | Sum of investment capital and debt used for asset purchase |
$A_1, A_2, A_3$ | Asset values at disbursement date defining payoffs to debtholders, LP, and GP |
$LP(A_i)$ | LP’s payoff |
Table 2. Impact of Uncertainty in Timing of Calls and Distributions.

Listed are summary statistics of private equity funds from the perspective of a risk-averse investor with a 12-year horizon and a risk aversion level (γ) of 2.5. The five columns correspond to five assumptions about the timing of cash flows in the fund. The first three assume a fixed distribution date; the first assumes an immediate call with distribution in 12 years, the second assumes a fixed call in 2.5 years and distribution in 12 years, and the third assumes a random call with equal probability each quarter for the first 4.75 years and distribution in 12 years. The fourth and fifth follow the same call patterns as the second and third but allow for early distributions by the GP. Within each panel asset expected returns are set such that the LP’s subjective valuation of a portfolio of 20% riskless bonds and 80% private equity in the first column equals the subjective valuation of a portfolio of 20% riskless bonds and 80% public equity. Public equity has 10% expected return and 15% volatility. The risk-free rate is 4%. Panel A assumes no leverage and 45% asset volatility. Panels B and C assume 30% asset volatility with leverage of 1X and 3X equity, respectively. Listed is the certainty equivalent of the portfolios, the expected horizon of capital deployed by the GP, the expected horizon of the commitment between time 0 and disbursement, and quantiles of the distribution of total value to paid in capital (TVPI) of the private equity fund.

<table>
<thead>
<tr>
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<th>No Early Distribution</th>
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<tr>
<td></td>
<td>Call at 0</td>
<td>Fixed Call</td>
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<tr>
<td>Value</td>
<td>160.76</td>
<td>144.47</td>
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<tr>
<td>E[Horizon]</td>
<td>12.00</td>
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<tr>
<td>E[Commitment]</td>
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<td>12.00</td>
</tr>
<tr>
<td>25% TVPI</td>
<td>2.9</td>
<td>2.3</td>
</tr>
<tr>
<td>Median TVPI</td>
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<td>5.1</td>
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<tr>
<td>75% TVPI</td>
<td>19.9</td>
<td>11.5</td>
</tr>
<tr>
<td>Pr[TVPI&lt;1]</td>
<td>6.8%</td>
<td>9.6%</td>
</tr>
</tbody>
</table>

Panel A. Venture Capital (μ = 20.6%)

<table>
<thead>
<tr>
<th></th>
<th>Panel B. 1X Buyout (μ = 17.3%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>160.77</td>
</tr>
<tr>
<td>E[Horizon]</td>
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<tr>
<td>E[Commitment]</td>
<td>12.00</td>
</tr>
<tr>
<td>25% TVPI</td>
<td>4.2</td>
</tr>
<tr>
<td>Median TVPI</td>
<td>10.0</td>
</tr>
<tr>
<td>75% TVPI</td>
<td>18.5</td>
</tr>
<tr>
<td>Pr[TVPI&lt;1]</td>
<td>4.5%</td>
</tr>
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</table>

Panel B. 1X Buyout (μ = 17.3%)

<table>
<thead>
<tr>
<th></th>
<th>Panel C. 3X Buyout (μ = 18.4%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>160.76</td>
</tr>
<tr>
<td>E[Horizon]</td>
<td>12.00</td>
</tr>
<tr>
<td>E[Commitment]</td>
<td>12.00</td>
</tr>
<tr>
<td>25% TVPI</td>
<td>9.3</td>
</tr>
<tr>
<td>Median TVPI</td>
<td>19.0</td>
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<tr>
<td>75% TVPI</td>
<td>44.9</td>
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<tr>
<td>Pr[TVPI&lt;1]</td>
<td>5.1%</td>
</tr>
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</table>

Panel C. 3X Buyout (μ = 18.4%)
Table 3. Break-even Performance of Venture Capital Assets and Funds.

Listed are the annual break-even expected returns and alphas of venture capital fund assets, in Panels A and B, the corresponding returns for a risk-averse LP in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the LP’s subjective valuation of a portfolio of 20% riskless bonds and 80% venture capital equal to the subjective valuation of a portfolio of 20% riskless bonds and 80% public equity. Public equity has returns with annual mean of 10% and volatility of 15%. The venture capital fund has a maximum 12-year life, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns are normally distributed with annual volatility of 45%. The risk-free rate is 4%. The LP is forced to sell the investment on a secondary market with a multiplicative haircut applied to fair value during liquidity shocks occurring at a base rate of 1% per quarter. Results vary within the panels with the LP’s risk aversion ($\gamma$), as well as the multiplicative haircut levied on secondary market transactions which is either low ($H^l$) or high ($H^h$), each of which occurs with probability 50%.

$$
\begin{array}{c|c|c|c|c|c|c}
\gamma & H^l = 90% & H^l = 50% & H^l = 60% & H^l = 70% & H^l = 95% & H^l = 70% & H^l = 80% & H^l = 90% \\
\hline
1.5 & 25.5% & 24.7% & 24.1% & 23.9% & 23.4% & 22.9% \\
2.5 & 37.2% & 34.8% & 33.1% & 32.7% & 31.4% & 30.4% \\
3.5 & 57.4% & 48.1% & 43.3% & 42.4% & 39.5% & 37.5% \\
\end{array}
$$

$$
\begin{array}{c|c|c|c|c|c|c|c}
\gamma & \text{Panel A. Break-even VC Asset Returns} \\
\hline
1.5 & 10.7% & 9.9% & 9.3% & 9.1% & 8.6% & 8.1% \\
2.5 & 22.4% & 20.0% & 18.3% & 17.9% & 16.6% & 15.6% \\
3.5 & 42.6% & 33.3% & 28.5% & 27.6% & 24.7% & 22.7% \\
\end{array}
$$

$$
\begin{array}{c|c|c|c|c|c|c}
\gamma & \text{Panel B. Break-even VC Asset Alphas} \\
\hline
1.5 & 12.5% & 12.0% & 11.8% & 11.8% & 11.8% & 11.7% \\
2.5 & 21.1% & 19.7% & 19.2% & 18.8% & 17.4% & 16.8% \\
3.5 & 37.7% & 29.7% & 26.1% & 25.9% & 23.3% & 22.5% \\
\end{array}
$$

$$
\begin{array}{c|c|c|c|c|c|c|c}
\gamma & \text{Panel C. Break-even VC Fund Returns} \\
\hline
1.5 & 3.15 & 3.12 & 3.07 & 3.05 & 3.03 & 3.00 \\
2.5 & 7.69 & 6.70 & 5.59 & 5.55 & 5.43 & 5.22 \\
3.5 & 38.31 & 16.59 & 11.29 & 11.10 & 10.50 & 7.78 \\
\end{array}
$$

$$
\begin{array}{c|c|c|c|c|c|c|c}
\gamma & \text{Panel D. Break-even VC TVPI} \\
\hline
1.5 & 1.23 & 1.22 & 1.21 & 1.21 & 1.20 & 1.19 \\
2.5 & 2.87 & 2.31 & 2.23 & 2.22 & 2.03 & 1.86 \\
3.5 & 13.59 & 6.89 & 4.63 & 4.55 & 3.29 & 3.15 \\
\end{array}
$$

45
Table 4. Break-even Performance of 1X Buyout Assets and Funds.

Listed are the annual break-even expected returns and alphas of buyout fund assets, in Panels A and B, the corresponding returns for a risk-averse LP in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the LP’s subjective valuation of a portfolio of 20% riskless bonds and 80% buyout fund equal to the subjective valuation of a portfolio of 20% riskless bonds and 80% public equity. Public equity has returns with annual mean of 10% and volatility of 15%. The buyout fund has a maximum 12-year life, 1X leverage, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns are normally distributed with annual volatility of 30%. The risk-free rate is 4%. The LP is forced to sell the investment on a secondary market with a multiplicative haircut applied to fair value during liquidity shocks occurring at a base rate of 1% per quarter. Results vary within the panels with the LP’s risk aversion ($\gamma$), as well as the multiplicative haircut levied on secondary market transactions which is either low ($H^{ll}$) or high ($H^{hh}$), each of which occurs with probability 50%.

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Table 5. Break-even Performance of 3X Buyout Assets and Funds.

Listed are the annual break-even expected returns and alphas of buyout fund assets, in Panels A and B, the corresponding returns for a risk-averse LP in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the LP’s subjective valuation of a portfolio of 20% riskless bonds and 80% buyout fund equal to the subjective valuation of a portfolio of 20% riskless bonds and 80% public equity. Public equity has returns with annual mean of 10% and volatility of 15%. The buyout fund has a maximum 12-year life, 3X leverage, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns are normally distributed with annual volatility of 30%. The risk-free rate is 4%. The LP is forced to sell the investment on a secondary market with a multiplicative haircut applied to fair value during liquidity shocks occurring at a base rate of 1% per quarter. Results vary within the panels with the LP’s risk aversion ($\gamma$), as well as the multiplicative haircut levied on secondary market transactions which is either low ($H^L$) or high ($H^H$), each of which occurs with probability 50%.

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Table 6. Impact of Fees.

Listed are break-even private equity asset alphas as a function of the annual management fee and carried interest on private equity fund profits paid above a hurdle rate of 8%. The alphas are computed from break-even asset returns in a single-factor market model with public equity used as a proxy for the market. The break-even returns are determined by setting an LP’s subjective valuation of a portfolio of 20% riskless bonds earning 4% and 80% public equity equal to the valuation of a portfolio of 20% riskless bonds and 80% private equity instead. Public equity has expected return of 10% and volatility of 15%. The LP has risk aversion of 1.5 and $1 million of initial wealth. In Panel A, the venture capital fund has a maximum 12-year life, annual asset return volatility of 45%, asset correlation of 0.60 with public equity, and no leverage. In Panel B, the 1X buyout fund has a maximum 12-year life, annual asset return volatility of 30%, asset correlation of 0.30 with public equity, and 100% leverage. In Panel C, the 3X buyout fund is identical to that in Panel B save for 300% leverage. The LP is forced to sell all private equity investment on a secondary market with a multiplicative haircut applied to fair value during liquidity shocks occurring at a base rate of 1% per quarter. The haircut is either 5% of 10%, each of which occurs with probability 50%.

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Table 7. Early Exercise.

Listed are an LP’s subjective values of a portfolio of 20% riskless bonds earning 4% and 80% venture capital fund as a percentage of $1 million of initial wealth. The LP has risk aversion of 2.5. The venture capital fund has a maximum 12-year life, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns are normally distributed with annual volatility of 45% and annual expected return $\mu$ as listed. The fund uses no leverage. The LP can sell their stake on a secondary market with a haircut during normal times as listed. If a liquidity shock occurs the LP is forced to sell on a secondary market with a 50% haircut. Liquidity shocks occur at a base rate of 1% (once every 100 quarters) that increases by 0.5% times the product of $t$ and $-j$ where $t$ is the number of periods since capital commitment and $j$ is the cumulative number of decreases in asset value.

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Table 8. Break-even VC Fund Performance in a Portfolio Context.

Listed are the annual break-even expected returns and alphas of venture capital fund assets, in Panels A and B, the corresponding returns for a risk-averse LP in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the LP’s subjective valuation of a portfolio of 20% riskless bonds, $x\%$ venture capital fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% riskless bonds and the remainder in public equity. The allocation $x$ is varied in the table. Public equity is assumed to have returns with annual mean of 10% and volatility of 15%. The venture capital fund has a maximum 12-year life, no leverage, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns are normally distributed with annual volatility of 45% and correlation with public equity of 0.60. The risk-free rate is 4%. The LP is forced to sell the investment on a secondary market with a multiplicative haircut applied to fair value during liquidity shocks occurring at a base rate of 1% per quarter. Results vary within the panels with the LP’s risk aversion ($\gamma$), as well as the multiplicative haircut levied on secondary market transactions which is either low ($HL$) or high ($HH$), each of which occurs with probability 50%.

$H^L = 90\%$ and $H^H = 50\%

$H^L = 95\%$ and $H^H = 90\%

| $\gamma$ | Panel A. Break-even VC Asset Returns | | Panel B. Break-even VC Asset Alphas | | Panel C. Break-even VC Fund Returns | | Panel D. Break-even VC TVPI | | Panel E. Break-even VC PME |
|---|---|---|---|---|---|---|---|
| 1.5 | 24.1% | 22.7% | 23.2% | 22.8% | 20.8% | 21.0% | 1.20 | 1.12 |
| 2.5 | 29.5% | 29.3% | 31.7% | 27.1% | 25.8% | 26.8% | 1.70 | 1.64 |
| 3.5 | 34.1% | 36.1% | 42.5% | 30.1% | 29.5% | 31.6% | 2.27 | 2.68 |

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$

$\gamma$
Table 9. Break-even 1X Buyout Fund Performance in a Portfolio Context.

Listed are the annual break-even expected returns and alphas of buyout fund assets, in Panels A and B, the corresponding returns for a risk-averse LP in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the LP’s subjective valuation of a portfolio of 20% riskless bonds, $x \%$ buyout fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% riskless bonds and the remainder in public equity. The allocation $x$ is varied in the table. Public equity is assumed to have returns with annual mean of 10% and volatility of 15%. The buyout fund has a maximum 12-year life, 100% leverage, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns are normally distributed with annual volatility of 30% and correlation with public equity of 0.30. The risk-free rate is 4%. The LP is forced to sell the investment on a secondary market with a multiplicative haircut applied to fair value during liquidity shocks occurring at a base rate of 1% per quarter. Results vary within the panels with the LP’s risk aversion ($\gamma$), as well as the multiplicative haircut levied on secondary market transactions which is either low ($HL$) or high ($HH$), each of which occurs with probability 50%.

$$H^L = 90\% \text{ and } H^H = 50\% \quad \text{and} \quad H^L = 95\% \text{ and } H^H = 90\%$$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Panel A. Break-even 1X Buyout Asset Returns</th>
<th>Panel B. Break-even 1X Buyout Asset Alphas</th>
<th>Panel C. Break-even 1X Buyout Fund Returns</th>
<th>Panel D. Break-even 1X Buyout TVPI</th>
<th>Panel E. Break-even 1X Buyout PME</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>24.4% 20.8% 20.3% 23.2% 19.2% 18.5%</td>
<td>16.8% 13.2% 12.7% 15.6% 11.6% 10.9%</td>
<td>17.0% 14.6% 14.3% 17.0% 14.4% 13.8%</td>
<td>3.75 3.04 2.97 3.69 2.94 2.83</td>
<td>1.66 1.37 1.33 1.65 1.34 1.29</td>
</tr>
<tr>
<td>2.5</td>
<td>26.4% 24.0% 25.2% 24.2% 21.0% 21.5%</td>
<td>18.8% 16.4% 17.6% 16.6% 13.4% 13.9%</td>
<td>18.8% 16.8% 17.6% 17.9% 15.5% 16.0%</td>
<td>4.04 3.66 3.89 3.85 3.15 3.21</td>
<td>1.85 1.62 1.73 1.74 1.47 1.51</td>
</tr>
<tr>
<td>3.5</td>
<td>27.4% 27.0% 31.6% 24.0% 22.2% 24.2%</td>
<td>19.8% 19.4% 24.0% 16.4% 14.6% 16.6%</td>
<td>19.2% 19.1% 22.3% 17.7% 16.5% 17.9%</td>
<td>4.15 4.10 5.38 3.82 3.49 3.85</td>
<td>1.95 1.91 2.46 1.72 1.57 1.74</td>
</tr>
</tbody>
</table>
Table 10. Break-even 3X Buyout Fund Performance in a Portfolio Context.

Listed are the annual break-even expected returns and alphas of buyout fund assets, in Panels A and B, the corresponding returns for a risk-averse LP in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the LP’s subjective valuation of a portfolio of 20% riskless bonds, \( x \)% buyout fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% riskless bonds and the remainder in public equity. The allocation \( x \) is varied in the table. Public equity is assumed to have returns with annual mean of 10% and volatility of 15%. The buyout fund has a maximum 12-year life, 300% leverage, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns are normally distributed with annual volatility of 30% and correlation with public equity of 0.30. The risk-free rate is 4%. The LP is forced to sell the investment on a secondary market with a multiplicative haircut applied to fair value during liquidity shocks occurring at a base rate of 1% per quarter. Results vary within the panels with the LP’s risk aversion (\( \gamma \)), as well as the multiplicative haircut levied on secondary market transactions which is either low (\( H^L \)) or high (\( H^H \)), each of which occurs with probability 50%.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>Panel A. Break-even 3X Buyout Asset Returns</th>
<th>Panel B. Break-even 3X Buyout Asset Alphas</th>
<th>Panel C. Break-even 3X Buyout Fund Returns</th>
<th>Panel D. Break-even 3X Buyout TVPI</th>
<th>Panel E. Break-even 3X Buyout PME</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>17.2% 15.1% 15.5%</td>
<td>9.6% 7.5% 7.9%</td>
<td>17.2% 15.0% 15.5%</td>
<td>3.78 3.14 3.26</td>
<td>1.67 1.43 1.47</td>
</tr>
<tr>
<td>2.5</td>
<td>19.5% 19.0% 21.5%</td>
<td>11.9% 11.4% 13.9%</td>
<td>19.2% 18.1% 20.0%</td>
<td>4.36 4.18 5.05</td>
<td>1.96 1.89 2.24</td>
</tr>
<tr>
<td>3.5</td>
<td>21.2% 23.1% 29.5%</td>
<td>13.6% 15.5% 21.9%</td>
<td>21.6% 28.3% 25.0%</td>
<td>4.97 5.39 8.73</td>
<td>2.20 2.49 3.77</td>
</tr>
</tbody>
</table>

\( H^L = 90\% \) and \( H^H = 50\% \)

\( x = 20\% \) \( x = 40\% \) \( x = 60\% \)

\( H^L = 95\% \) and \( H^H = 90\% \)

\( x = 20\% \) \( x = 40\% \) \( x = 60\% \)
Table 11. Optimal Portfolio Weights.

Listed are utility-maximizing portfolio weights for public equity and private equity from the perspective of a risk-averse limited partner with a 12-year horizon, $1 million of initial wealth, and risk aversion as listed in the table. In all cases allocation to bonds is fixed at 20% and bonds have returns of 4%. The remaining 80% is split across public and private equity. Public equity has expected return of 10% and standard deviation of 15%. In Panel A, venture capital assets have expected return of 25.8%, standard deviation of 45%, and correlation of 0.60 with public equity. In Panel B, 1X buyout fund assets have expected return of 21.0%, standard deviation of 30%, and correlation of 0.30 with public equity. In Panel C, 3X Buyout fund assets have expected return of 17.0%, standard deviation of 30%, and correlation of 0.30 with public equity. Venture capital funds have no leverage, whereas 1X and 3X Buyout funds have leverage of 100% and 300% of initial equity. The LP is forced to sell private equity investment on a secondary market with a multiplicative haircut applied to fair value during liquidity shocks occurring at a base rate of 1% per quarter. Results vary within the panels with the multiplicative haircut levied on secondary market transactions which is either low \((H_L)\) or high \((H_H)\), each of which occurs with probability 50%.

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>Panel A. Venture Capital</th>
<th>Panel B. 1X Buyout</th>
<th>Panel C. 3X Buyout</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(H^L = 90%) and (H^H = 50%)</td>
<td>(H^L = 95%) and (H^H = 90%)</td>
<td></td>
</tr>
<tr>
<td>Public</td>
<td>VC</td>
<td>Public</td>
<td>Buyout</td>
</tr>
<tr>
<td>1.5</td>
<td>0.27</td>
<td>0.53</td>
<td>0.09</td>
</tr>
<tr>
<td>2.5</td>
<td>0.55</td>
<td>0.25</td>
<td>0.43</td>
</tr>
<tr>
<td>3.5</td>
<td>0.65</td>
<td>0.15</td>
<td>0.55</td>
</tr>
</tbody>
</table>

\(H^N = 90\%\) and \(H^S = 50\%\) | \(H^N = 95\%\) and \(H^S = 90\%\) |

\(\gamma\) | Public | Buyout | Public | Buyout | Public | Buyout | Public | Buyout |
| 1.5 | 0.29 | 0.51 | 0.23 | 0.57 | 0.29 | 0.51 | 0.23 | 0.57 |
| 2.5 | 0.52 | 0.28 | 0.47 | 0.33 | 0.52 | 0.28 | 0.47 | 0.33 |
| 3.5 | 0.62 | 0.18 | 0.58 | 0.22 | 0.62 | 0.18 | 0.58 | 0.22 |