Asset Pricing with Horizon-Dependent Risk Aversion

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Abstract

We study general equilibrium asset prices in a multi-period endowment economy when agents' risk aversion is allowed to depend on the horizon of the risk. In our pseudo-recursive preference framework, agents are time inconsistent for intra-temporal tradeoffs but time consistent for inter-temporal tradeoffs. Under standard log-normal consumption growth, we find that horizon-dependent risk aversion affects the pricing of risk if and only if volatility is stochastic. When risk aversion decreases with the horizon – as indicated by lab experiments – and the elasticity of intertemporal substitution is greater than 1, our model results in a downward sloping term structure of risk prices. The model can therefore explain the recent empirical results on the term structure of risky asset returns.

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1 Introduction

Most of the literature on general equilibrium asset pricing theory is premised on the assumption that risk aversion is constant across maturities. We investigate whether the standard tool box of asset pricing can be generalized to accommodate risk preferences that differ across temporal horizons, and whether such a generalization has the potential to address observed patterns in asset prices.

Inspired by ample experimental evidence that subjects are more risk averse to immediate than to delayed risks,[1] Eisenbach and Schmalz (2014) introduce a two-period model with horizon-dependent risk aversion and show it is conceptually orthogonal to other non-standard preferences such as non-exponential time discounting (Phelps and Pollak, 1968; Laibson, 1997), time-varying risk aversion (Constantinides, 1990; Campbell and Cochrane, 1999), and a preference for the timing of resolution of uncertainty (Kreps and Porteus, 1978; Epstein and Zin, 1989). In the present paper, we investigate the impact of horizon-dependent risk aversion preferences on asset prices in a dynamic framework. The conceptual difficulties of solving a multi-period model with dynamically inconsistent preferences are numerous. To start, the commonly used recursive techniques in finance and macroeconomics only apply to dynamically consistent preferences. At the same time, the only dynamically consistent time-separable expected utility preference is the special case of risk aversion constant across horizons.[2] In an effort to overcome these difficulties, we use techniques in the spirit of Strotz (1955) to solve the problem of a rational agent with horizon-dependent risk aversion preferences in a setting without time separability. Such an agent is dynamically consistent for deterministic payoffs, so that only uncertain payoffs induce time inconsistency. Unable to commit to future behavior but being aware of her preferences and perfectly rational, the agent optimizes today, taking into account reoptimization in future periods. Solving our model this way yields a stochastic discount factor that nests the standard Epstein and Zin (1989) case, with an new multiplicative term representing the discrepancy between the continuation value used for optimization at any period t versus the actual valuation at $t+1$.

We investigate the implications of horizon-dependent risk aversion on both the level

[1] See, e.g., Jones and Johnson (1973); Onculer (2000); Sagristano et al. (2002); Noussair and Wu (2006); Coble and Lusk (2010); Baucells and Heukamp (2010); Abdellaoui et al. (2011). See Eisenbach and Schmalz (2014) for a more thorough review.

[2] As a result, combining time-separability with horizon-dependent risk aversion in a dynamic model necessarily introduces inconsistent time preferences, which precludes isolating the effect on asset prices of horizon-dependent risk preferences.

and on the term structure of risk premia. We find the model can match risk prices in levels, very much in line with the long-run risk literature based on standard Epstein and Zin (1989) preferences (Bansal and Yaron, 2004; Bansal et al., 2014). Further, we find that the term structure of equity risk premia is non-trivial if and only if the economy features stochastic volatility. In such a setting, the horizon dependent risk aversion model can explain a downward-sloping term structure of equity risk premia, as documented empirically (see the literature review below). Interestingly, this effect is solely driven by a downward-sloping term structure of the price of volatility risk, which is a testable prediction.

Recent papers provide empirical support for our model's predictions, specifically for the term structure of volatility risk pricing. Dew-Becker et al. (2014) use data on variance swaps to show that investors only price volatility risk at the 1-month horizon and are essentially indifferent to news about future volatility at horizons ranging from 1 month to 14 years. Using different methodologies and standard index option data, Andries et al. (2015) find a negative price of volatility risk for maturities up to 4 months, but also a strongly nonlinear down-ward sloping term structure (in absolute value).

The paper proceeds as follows. Section 2 reviews the related literature. Section 3 presents a two-period model that illustrates the intuition of some of our result. Section 4 presents the dynamic model. Section 5 derives our formal results for the pricing of risk and its term-structure. Section 6 concludes.

2 Related Literature

This paper is the first to solve for equilibrium asset prices in an economy populated by agents with dynamically inconsistent *risk* preferences. It complements Luttmer and Mariotti (2003), who show that dynamically inconsistent *time* preferences of the kind also examined by Harris and Laibson (2001) have little power to explain cross-sectional variation in asset returns. Given that cross-sectional asset pricing involves intra-period risk-return tradeoffs, it is indeed quite intuitive that horizon-dependent *time* preferences are not suitable to address puzzles related to cross-sectional variation in returns.

Our formal results on the term structure of risk pricing are consistent with patterns uncovered by a recent empirical literature. Van Binsbergen et al. (2012) show the Sharpe ratios for short-term dividend strips are higher than for long-term dividend strips (see also van den Steen, 2004; van Binsbergen and Koijen, 2011; Boguth et al., 2012).[3] These empirical findings have led to a vigorous debate, because they appear to be inconsistent with traditional asset pricing models.

Our micro-founded model of preferences implies a downward sloping pricing of risk, in a simple endowment economy. By contrast, other approaches typically generate the desired implications by making structural assumptions about the economy or about the priced shocks driving the stochastic discount factor directly. For example, in a model with financial intermediaries, Muir (2013) uses time-variation in institutional frictions to explain why the term structure of risky asset returns changes over time. Ai et al. (2013) derive similar results in a production-based RBC model in which capital vintages face heterogeneous shocks to aggregate productivity; Zhang (2005) explains the value premium with costly reversibility and a countercyclical price of risk. Other production-based models with implications for the term structure of equity risk are, e.g. Kogan and Papanikolaou (2010, 2014), Gârleanu et al. (2012) and Favilukis and Lin (2013). Similarly, Belo et al. (2013) offer an explanation why risk levels and thus risk premia could be higher at short horizons; by contrast, our contribution is about risk prices. Croce et al. (2007) use informational frictions to generate a downward-sloping equity term structure.

The predictions of our model do not rely on the possibility of rare disasters, which is an assumption that some have argued may be more difficult to verify empirically. Further, our results are distinguishable from several alternative explanations for a downward-sloping term structure of equity risk premia: in our model, volatility risk is the only driver for a downward-sloping term structure of equity risk. Our predictions for the risk-pricing *levels* are also consistent with Campbell et al. (2012), who show that volatility risk is an important driver of asset returns in a CAPM framework and work who examining the relation between volatility risk and returns (Ang et al., 2006; Adrian and Rosenberg, 2008; Bollerslev and Todorov, 2011; Menkhoff et al., 2012; Boguth and Kuehn, 2013).

3 Static Model

Introducing horizon-dependent risk aversion into a time separable expected utility model with more than two periods necessarily introduces horizon-dependent inter-temporal trade-offs similar to quasi-hyperbolic discounting (for a detailed discussion, see Eisenbach and

[3] Giglio et al. (2013) show a similar pattern exists for discount rates over much longer horizons using real estate markets. Lustig et al. (2013) document a downward-sloping term structure of currency carry trade risk premia.

Schmalz, 2014). This is undesirable since we want to study the effects of horizon-dependent risk aversion in isolation. Our general model in Section 4 solves this problem by dropping time separability, though this comes at the cost of more analytical complexity and less intuitive clarity. Here we present a simple two-period model with time separability and uncertainty both in the immediate and proximate future to illustrate the effect of horizon-dependent risk aversion on risk pricing.

Consider a two-period model with uncertainty in both periods. The agent has time separable expected utility U_t in periods $t = 0, 1$ given by

$$U_0 = E\big[v(c_0) + \delta u(c_1)\big] \quad \text{and} \quad U_1 = E\big[v(c_1)\big],$$

where v and u are von Neumann-Morgenstern utility indexes and v is more risk averse than u. At the beginning of period 0 the agent forms a portfolio of two risky assets and a risk free bond. Asset 0 is a claim on consumption in period 0 while asset 1 is a claim on consumption in period 1. Consumption in the two periods is i.i.d. Denoting the prices of the two assets by p_0 and p_1, respectively, the first-order conditions for the agent's portfolio choice yield:

$$\begin{aligned} & E\big[v'(c_0)\,(c_0 - p_0)\big] = 0 \\ \text{and} \quad & E\big[\delta u'(c_1)\,(c_1 - (1+r)\,p_1)\big] = 0. \end{aligned}$$

Eisenbach and Schmalz (2014) show that the equilibrium prices p_0, p_1 and r satisfy:

$$p_0 < (1+r)\,p_1.$$

In this two-period setting, horizon-dependent risk aversion therefore leads to an equilibrium term-structure of risk premia that is downward sloping.

This simple example illustrates how horizon-dependent risk aversion can affect the pricing of risk at different horizons as intuition would suggest. There are, however, important limitations to this example. The setting is subtly different from standard asset pricing models, even with only two periods $t = 0, 1$: there is uncertainty in both periods and a period's decision is made before the period's uncertainty resolves. This allows for horizon-dependent risk aversion to have a term-structure effect without worrying about inconsistent inter-temporal tradeoffs, since only one such tradeoff arises. However, the period-0 portfolio choice problem above implicitly assumes that the agent has no opportunity to re-trade the claim to period-1 consumption at the beginning of period 1.

To generalize this setting, the next section presents our fully dynamic model, which allows for retrading every period.

4 Dynamic Model

Our approach is to generalize the model of Epstein and Zin (1989) (hereafter EZ) to allow for horizon-dependent risk aversion without affecting intertemporal substitution.

4.1 Preferences

Let $\{\gamma_h\}_{h\geq 0}$ be a decreasing sequence representing risk aversion at horizon h. In period t, the agent evaluates a consumption stream starting in period $t+h$ by:

$$V_{t,t+h} = \left((1-\beta)\, C_{t+h}^{1-\rho} + \beta E_{t+h}\big[V_{t,t+h+1}^{1-\gamma_h}\big]^{\frac{1-\rho}{1-\gamma_h}} \right)^{\frac{1}{1-\rho}} \quad \text{for all} \quad h \geq 0. \tag{1}$$

The agent's utility in period t is given by setting $h=0$ in (1) which we denote by $V_t \equiv V_{t,t}$ for all t:

$$V_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t\big[V_{t,t+1}^{1-\gamma_0}\big]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}}.$$

As in the EZ model, utility V_t depends on (deterministic) current consumption C_t and a certainty equivalent $E_t\big[V_{t,t+1}^{1-\gamma_0}\big]^{\frac{1}{1-\gamma_0}}$ of (uncertain) continuation values $V_{t,t+1}$, where the aggregation of the two periods occurs with constant elasticity of intertemporal substitution given by $1/\rho$. However, in contrast to the EZ model, the continuation value $V_{t,t+1}$ is *not* the same as the agent's utility V_{t+1} in period $t+1$:

$$\begin{aligned} V_{t,t+1} &= \left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1}\big[V_{t,t+2}^{1-\gamma_1}\big]^{\frac{1-\rho}{1-\gamma_1}} \right)^{\frac{1}{1-\rho}} \\ &\neq \left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1}\big[V_{t+1,t+2}^{1-\gamma_0}\big]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}} = V_{t+1} \end{aligned}$$

The key feature of the definition (1) is that certainty equivalents at different horizons h are formed with different levels of risk aversion γ_h. Imminent uncertainty is treated with risk aversion γ_0, uncertainty one period ahead is treated with γ_1 and so on.

In contrast to EZ, the preference of our model captured by $V_t \equiv V_{t,t}$ is *not* recursive since $V_{t+1} \equiv V_{t+1,t+1}$ does not recur in the definition of V_t.[4] This non-recursiveness is a direct implication of the horizon-dependent risk aversion, in which uncertain consumption streams starting in $t+1$ are evaluated differently by the agent's selves at t and $t+1$. Crucially, this disagreement arises only for *uncertain* consumption streams as for any *deterministic* consumption stream the horizon-dependence in (1) becomes irrelevant and we have $V_{t,t+1} = V_{t+1}$. Our model therefore implies dynamically inconsistent risk preferences while maintaining dynamically consistent time preferences.

An interesting question is the possibility to axiomatize the horizon-dependent risk aversion preferences we propose. The static model in Section 3 could be axiomatized as a special version of the temptation preferences of Gul and Pesendorfer (2001). Their preferences deal with general disagreements in preferences at a period 0 and a period 1. In our case, the disagreement is about the risk aversion so an axiomatization would require adding a corresponding axiom to the set of axioms in Gul and Pesendorfer (2001). Our dynamic model builds on the functional form of Epstein and Zin (1989) which capture non-time-separable preferences of the form axiomatized by Kreps and Porteus (1978). However, our generalization of Epstein and Zin (1989) explicitly violates Axiom 3.1 (temporal consistency) of Kreps and Porteus (1978) which is necessary for the recursive structure.

To solve our model, we follow the tradition of Strotz (1955), assuming that the agent is fully rational when making choices in period t to maximize V_t. This means self t realizes that its evaluation of future consumption given by $V_{t,t+1}$ differs from the objective function V_{t+1} which self $t+1$ will maximize. The solution then corresponds to the subgame-perfect equilibrium in the sequential game played among the agent's different selves (Luttmer and Mariotti, 2003).[5]

4.2 Stochastic Discount Factor

For asset pricing purposes, the object of interest is the stochastic discount factor (SDF) resulting from the preferences in equation (1). We can arrive at the SDF intuitively using a derivation based on the intertemporal marginal rate of substitution:[6]

$$\frac{\Pi_{t+1}}{\Pi_t} = \frac{dV_t/dW_{t+1}}{dV_t/dC_t}.$$

[4]Only the definition of $V_{t,t+h}$ for different h in (1) is recursive since the object for $h+1$ recurs in the definition of the object for h.

[5]See Appendix A for more details.

[6]See Appendix A for a more rigorous derivation.

The derivative of current utility V_t with respect to current consumption C_t is standard and given by:

$$\frac{dV_t}{dC_t} = V_t^{\rho}\left(1-\beta\right)C_t^{-\rho}. \tag{2}$$

The derivative of current utility V_t with respect to next-period wealth W_{t+1} is almost standard:

$$\frac{dV_t}{dW_{t+1}} = \frac{dV_t}{dV_{t,t+1}} \times \frac{dV_{t,t+1}}{dW_{t+1}} = V_t^{\rho}\beta E_t\big[V_{t,t+1}^{1-\gamma_0}\big]^{\frac{\gamma_0-\rho}{1-\gamma_0}} V_{t,t+1}^{-\gamma_0} \times \frac{dV_{t,t+1}}{dW_{t+1}}. \tag{3}$$

At this point, however, we cannot appeal to the envelope condition at $t+1$ to replace the term $dV_{t,t+1}/dW_{t+1}$ by $dV_{t,t+1}/dC_{t+1}$. This is because $V_{t,t+1}$ is the value self t attaches to all future consumption while the envelope condition at $t+1$ is in terms of the objective function of self $t+1$ which is given by V_{t+1}:

$$\frac{dV_{t+1}}{dW_{t+1}} = \frac{dV_{t+1}}{dC_{t+1}} = V_{t+1}^{\rho}\left(1-\beta\right)C_{t+1}^{-\rho}. \tag{4}$$

The disagreement between selves t and $t+1$ requires us to take an extra step. Due to the homotheticity of our preferences, we can rely on the fact that both $V_{t,t+1}$ and V_{t+1} are homogeneous of degree one which implies that

$$\frac{dV_{t,t+1}/dW_{t+1}}{dV_{t+1}/dW_{t+1}} = \frac{V_{t,t+1}}{V_{t+1}}. \tag{5}$$

This relationship captures a key element of our model: The marginal benefit of an extra unit of wealth in period $t+1$ differs whether evaluated by self t (the numerator on the left hand side) or by self $t+1$ (the denominator on the right hand side).

To arrive at the stochastic discount factor, we first combine the relationship (5) with the envelope condition (4) to eliminate the term $dV_{t,t+1}/dW_{t+1}$ in the derivative of current utility with respect to next-period wealth (3). Then we can combine (3) with (2) to form the marginal rate of substitution and arrive at:

$$\frac{\Pi_{t+1}}{\Pi_t} = \underbrace{\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}}_{\text{(I)}} \times \underbrace{\left(\frac{V_{t,t+1}}{E_t\big[V_{t,t+1}^{1-\gamma_0}\big]^{\frac{1}{1-\gamma_0}}}\right)^{\rho-\gamma_0}}_{\text{(II)}} \times \underbrace{\left(\frac{V_{t,t+1}}{V_{t+1}}\right)^{1-\rho}}_{\text{(III)}}.$$

The SDF consists of three multiplicative parts:

(I) The first term is that of the standard time-separable CRRA model with discount factor β and constant relative risk aversion ρ.

(II) The second part originates from the wedge between the risk aversion and the inverse of the elasticity of intertemporal substitution, i.e. from the non time separable framework. It is similar to the standard EZ model, taking risk aversion as the immediate one, γ_0.

(III) The third part is unique to our model and originates from the fact that, with horizon-dependent risk aversion, different selves disagree about the evaluation of a given consumption stream, depending on their relative horizon. Since the SDF Π_{t+1}/Π_t captures trade-offs between periods t and $t+1$, the key disagreement is how selves t and $t+1$ evaluate consumption starting in period $t+1$.

If we set $\gamma_h = \gamma$ for all horizons h, our SDF for horizon-dependent risk aversion preferences simplifies to the standard SDF for recursive preferences: it nests the model of EZ which, in turn, nests the standard time-separable model for $\gamma = \rho$.

5 Pricing of Risk and the Term Structure

To derive the pricing of risk under horizon-dependent risk aversion preferences, we consider a simplified version of the model where risk aversion for immediate risk is given by γ, and by $\tilde{\gamma}$ for all future risks. This framework, and our derivations for risk pricing, easily extends to a case where risk aversion is decreasing up to a given horizon, after which, for risks beyond, it remains constant ($\tilde{\gamma}$).

Our general model (1) thus becomes:

$$V_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}$$
$$\tilde{V}_t = \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\tilde{\gamma}} \right]^{\frac{1-\rho}{1-\tilde{\gamma}}} \right)^{\frac{1}{1-\rho}} .$$

The second equation is simply the standard EZ framework with risk aversion $\tilde{\gamma}$. If solutions for the recursion on the continuation value $\tilde{V}$ are derived, the value function V is automatically obtained from the first equation. The simplified version of the SDF is:

$$\frac{\Pi_{t+1}}{\Pi_t} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \times \left(\frac{\tilde{V}_{t+1}}{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma} \times \left(\frac{\tilde{V}_{t+1}}{V_{t+1}}\right)^{1-\rho} \tag{6}$$

As in the standard EZ framework, closed-form solutions for $\tilde{V}$ (and thus for V) obtain only for the knife-edge case of a unit elasticity of intertemporal substitution (EIS), $\rho = 1$. To aid in the comparison of the standard EZ framework with horizon-independent risk aversion and our generalization to horizon dependence, we therefore start by analyzing the case of unit-EIS in Section 5.1. We then go beyond the special case of $\rho = 1$ in Section 5.2 by studying solutions for V and $\tilde{V}$ under the approximation of a discount factor close to unity, $\beta \approx 1$.

5.1 Closed-Form Solutions under Unit EIS

To determine the pricing implications of our model, we analyze the wedge between the continuation value $\tilde{V}_{t+1}$ and the valuation V_{t+1}, which is the key difference between the SDF in our framework (6) and in the standard EZ framework. Denoting logs by lowercase letters, we consider a Lucas-tree economy with an exogenous endowment process given by

$$c_{t+1} - c_t = \mu + \phi_c x_t + \alpha_c \sigma_t W_{t+1},$$

where the time varying drift, x_t, and the time varying volatility, σ_t, have evolutions

$$x_{t+1} = \nu_x x_t + \alpha_x \sigma_t W_{t+1}$$
$$\sigma_{t+1}^2 - \sigma^2 = \nu_\sigma \left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma \sigma_t W_{t+1}.$$

Both state variables are stationary (ν_x and ν_σ are contracting), and for simplicity, we assume the three shocks are orthogonal.

Lemma 1. *Under these specifications for the endowment economy, and $\rho = 1$, we find:*

$$v_t - \tilde{v}_t = -\frac{1}{2}\beta\left(\gamma - \tilde{\gamma}\right)\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v^2\alpha_\sigma^2\right)\sigma_t^2, \tag{7}$$

where ϕ_v and ψ_v are constant functions of the model parameters such that:

$$\phi_v = \beta\phi_c \left(I - \nu_x\right)^{-1}$$

and

$$\psi_v = \frac{1}{2}\frac{\beta\left(1-\tilde{\gamma}\right)}{1-\beta\nu_\sigma}\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v^2\alpha_\sigma^2\right).$$

Observe from equation (7) that $v_t < \tilde{v}_t$ at all times, as should be expected. Indeed, $\tilde{v}_t$ is derived from the standard EZ model with risk-aversion $\tilde{\gamma}$, whereas v_t is derived from our horizon-dependent risk aversion model with a higher risk-aversion $\gamma > \tilde{\gamma}$ for immediate risk. More striking, however, is that the difference in the valuations under the two models is constant when volatility σ_t is constant.

Corollary 1. *Under constant volatility in the consumption process, the ratios $\tilde{V}/V$ are constant and therefore do not affect excess returns, both in levels and in the term-structure.*

This is one of the central results of our paper and, as shown below, is not limited to the special case of $\rho = 1$. The intuition is that when our time-inconsistent representative agent is aware that prices will be set, the following period, by her next-period self, then the term-structure of prices is affected by risk-horizon dependent risk-aversion only if unexpected shocks to volatility can occur.

This result makes clear that the intuition from the simple two-period horizon-dependent risk aversion model of Section 3 does not trivially extend to the dynamic model and that there is no tautological relationship between horizon-dependent preferences and horizon-dependent risk pricing. It makes also clear, however, why the generalized EZ preferences we employ in this paper are necessary to derive interesting predictions. Before we make use of that feature, we derive one more result under the $\rho = 1$ case.

Proposition 1. *In the knife-edge case $\rho = 1$, the stochastic discount factor satisfies:*

$$\frac{\Pi_{t+1}C_{t+1}}{\Pi_t C_t} = \beta \underbrace{\left(\frac{\tilde{V}_{t+1}^{1-\gamma}}{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]}\right)}_{multiplicative\ martingale}$$

Borovicka et al. (2011) show the pricing of consumption risk, at time t, and for horizon h is determined by $E_t[\Pi_{t+h}C_{t+h}]$. Under the $\rho = 1$ case, the evolution of the risk adjusted payoffs as multiplicative martingales, yields $E_t[\Pi_{t+h}C_{t+h}]$ independent of the horizon h and thus a flat term-structure of risk prices, even under stochastic consumption volatility.

In the following section, we relax the assumption $\rho = 1$, and analyze the term-structure impact of our horizon-dependent risk-aversion model, under stochastic volatility.

5.2 General Case and Role of Volatility Risk

We consider the general case $\rho > 0$, $\rho \neq 1$, and we approximate the two relations

$$V_t = \left((1-\beta) C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}},$$
$$\tilde{V}_t = \left((1-\beta) C_t^{1-\rho} + \beta E_t \left[\tilde{V}_{t+1}^{1-\tilde{\gamma}} \right]^{\frac{1-\rho}{1-\tilde{\gamma}}} \right)^{\frac{1}{1-\rho}},$$

under $\beta \approx 1$.

When the coefficient of time discounting β approaches 1, the recursion in $\tilde{V}$ can be re-written as

$$E_t \left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}} \right)^{1-\tilde{\gamma}} \left(\frac{C_{t+1}}{C_t} \right)^{1-\tilde{\gamma}} \right] \approx \beta^{-\frac{1-\rho}{1-\tilde{\gamma}}} \left(\frac{\tilde{V}_t}{C_t} \right)^{1-\tilde{\gamma}},$$

an eigenfunction problem, in which $\beta^{-\frac{1-\rho}{1-\tilde{\gamma}}}$ is an eigenvalue.

Lemma 2. *Under the Lucas-tree endowment process considered in the previous section, this eigenfunction problem admits a unique eigenvalue, and eigenfunction (up to a scalar multiplier):*

$$\tilde{v}_t - c_t = \tilde{\mu} + \phi_v x_t + \psi_v \sigma_t^2,$$

where

$$\phi_v = \phi_c (I - \nu_x)^{-1},$$
$$\psi_v = \frac{1}{2} \frac{1-\tilde{\gamma}}{1-\nu_\sigma} \left(\alpha_c^2 + \phi_v^2 \alpha_x^2 + \psi_v^2 \alpha_\sigma^2 \right) < 0,$$

and

$$\log \beta = -(1-\rho) \left(\mu + \psi_v \sigma^2 (1-\nu_\sigma) \right).$$

Note the eigenvalue solution for β yields $\beta < 1$, as desired, for $\rho < 1$.[7] It also makes valid the approximation around 1: Using the calibration of Bansal and Yaron (2004) for

[7] Even though $\psi_v < 0$, the term $\left(\mu + \psi_v \sigma^2 (1-\nu_\sigma) \right)$ remains positive for all reasonable parameter values.

the consumption process, we obtain solutions for β above 0.998, for any values of ρ between 0.1 and 1, and $\tilde{\gamma}$ between 1 and 10.

To derive the pricing equations, we use the approximation, valid for β close to 1:

$$\frac{V_t}{\tilde{V}_t} \approx \frac{E_t\left[\tilde{V}_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}{E_t\left[\tilde{V}_{t+1}^{1-\tilde{\gamma}}\right]^{\frac{1}{1-\tilde{\gamma}}}}.$$

Theorem 1. *Under the Lucas-tree endowment process and the $\beta \approx 1$ approximation we have*

$$v_t - \tilde{v}_t = -\left(\gamma - \tilde{\gamma}\right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma_t^2 < 0,$$

and the stochastic discount factor satisfies

$$\begin{aligned} \pi_{t,t+1} = {} & \bar{\pi}_t - \gamma \alpha_c \sigma_t W_{t+1} + \left(\rho - \gamma\right) \phi_v \alpha_x \sigma_t W_{t+1} \\ & + \left(\left(\rho - \gamma\right) + \left(1 - \rho\right)\left(\gamma - \tilde{\gamma}\right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right) \psi_v \alpha_\sigma \sigma_t W_{t+1}, \end{aligned}$$

where

$$\begin{aligned} \bar{\pi}_t = {} & -\mu - \rho \phi_c x_t - \left(1-\rho\right) \psi_v \sigma^2 \left(1 - \nu_\sigma\right) \left(1 - \left(\gamma - \tilde{\gamma}\right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right) \\ & - \Big(\left(\rho - \gamma\right)\left(1 - \gamma\right) - \left(1-\rho\right)\left(\gamma - \tilde{\gamma}\right) \nu_\sigma\Big) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma_t^2. \end{aligned}$$

Our model yields a negative price for volatility shocks, consistent with the existing long-run risk literature, and the observed data for one-period returns. Observe further that the SDF loading on the drift shocks $\alpha_x \sigma_t W_{t+1}$ is unaffected by the specificities of our horizon-dependent risk-aversion model – it is exactly the same as in the standard EZ model. Any novel pricing effects we obtain – both in level and in the term-structure – derive from the volatility shocks. For this reason, we shut down the drift shocks in the part that follows, and assume $x_t = 0$ and $\alpha_x = 0$ in the remainder of the paper.

We now analyze the pricing of volatility risk in the term-structure. Denote by $P_{t,h}$ the price at time t for a claim to the endowment consumption at horizon h, and let $P_{t,0} = C_t$ for all t. The one-period holding returns for such assets are determined by

$$R_{t \to t+1,h} = \frac{P_{t+1,h-1} - P_{t,h}}{P_{t,h}},$$

and we denote by $\mathrm{SR}_{t,h}$ the conditional sharpe ratio for the one-period holding return at time t for a claim to consumption in period $t+h$.

Theorem 2. *Pricing in the term-structure is given by:*

$$\frac{P_{t,h}}{C_t} = \exp\left(a_h + A_h \sigma_t^2\right),$$

and the conditional Sharpe ratios are given by:

$$\mathrm{SR}_{t,h} = \frac{1 - \exp\left[-\left(\bar{r} + A\sigma_t^2 - \left(\rho - \gamma + (1-\rho)(\gamma - \tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2 \sigma_t^2\right)\right]}{\sqrt{\exp\left(\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\sigma_t^2\right) - 1}},$$

where $\bar{r}$ and A are constant (independent of t and h) and A_h is determined by the initial condition $A_0 = 0$ and the recursion:

$$A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right) = A - \left(\rho - \gamma + (1-\rho)(\gamma - \tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2.$$

From Theorem 1 and Theorem 2, observe that both the pricing of volatility risk and the term-structure of Sharpe ratios for one-period returns on the consumption claims at various horizons depend mostly on a term of the model parameters:

$$\left(\rho - \gamma + (1-\rho)(\gamma - \tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v \tag{8}$$

The first term in this expression, $\rho - \gamma$, is a standard EZ term while the second term, $(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}$, is new and originates in the horizon-dependent risk aversion. If the novel term is dominated by the standard EZ term then our model of preferences has no significant impact on the pricing of volatility risk in either its level or its term-structure.

Corollary 2. *Horizon-dependent risk aversion affects risk prices and their term-structure if and only if $\tilde{\gamma}$ is close to 1. The effect is stronger, the more persistent the volatility risk.*

In Figure 1, we plot the term-structure of the Sharpe ratios of one-period holding returns on horizon-dependent consumption claims, for various values of the ratio $\frac{1-\nu_\sigma}{1-\tilde{\gamma}}$, which determines how much impact the variations in risk-aversion across horizons have. For each value of $\frac{1-\nu_\sigma}{1-\tilde{\gamma}}$, the immediate risk aversion γ is chosen such that the pricing of volatility risk, given by the whole term in (8), always remains the same as in the

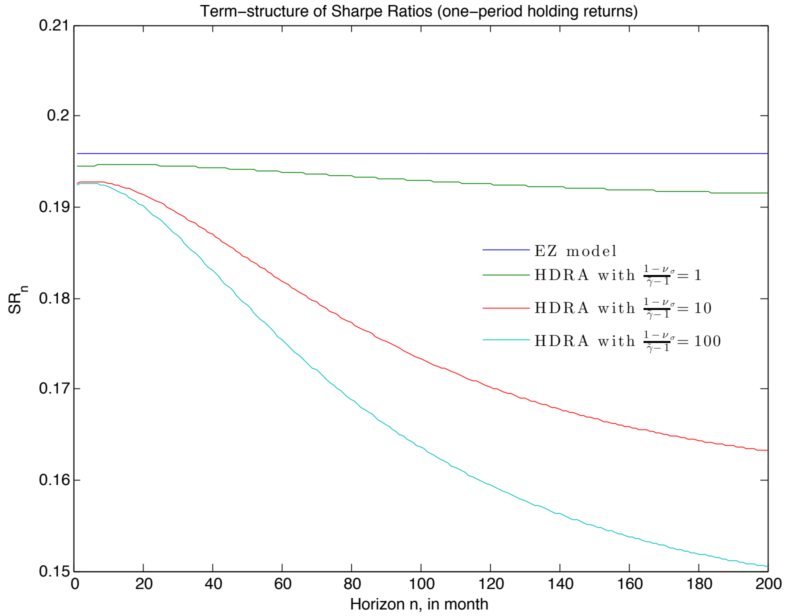

Figure 1: Calibrated term-structure. We use the parameters from Bansal and Yaron (2004) for μ, ν_σ, α_c and α_σ and $\rho = 1/1.5$. HDRA stands for "horizon-dependent risk aversion."

standard EZ model with risk aversion $\gamma = 10$. Figure 1 shows clearly that our horizon-dependent risk-aversion model can generate a downward sloping term-structure for the Sharpe ratios of one-period holding returns of consumption claims. This is in contrast to the standard EZ model which generates a flat term-structure – a result that has been highlighted in the literature starting with van Binsbergen et al. (2012), most recently by Dew-Becker et al. (2014). Observe, however, that for such a term-structure effect to be notable quantitatively, the long-horizon risk aversion $\tilde{\gamma}$ must be very close to one, i.e. approximate log utility. To match the pricing of volatility risk of the standard EZ model, the difference in risk aversion between the short horizon and the long horizon must become very large, unrealistically so under the very persistent volatility calibration of Bansal and Yaron (2004). However, this calibration problem can be largely avoided by making the time-varying volatility less persistent (without changing the volatility's stationary distribution).

To summarize, in a simple endowment economy, our model with horizon-dependent risk aversion has very specific implications for the level and the term-structure of the pricing of risk. If volatility is constant over time, our model does not affect the pricing of risk relative to the standard EZ model. Even with time-varying volatility, our model affects solely the loading of the stochastic discount factor on the shocks to volatility. The pricing of shocks to immediate consumption and of shocks to the consumption drift are unchanged from the standard EZ model. On the other hand, the pricing of the shocks to volatility presents a clear downward sloping term-structure (in absolute value), in contrast to the standard EZ model.

Recent papers provide empirical support for the central model prediction that volatility risk prices are higher for shorter-horizons. Dew-Becker et al. (2014) use variance swap prices with horizons up to 14 years and find that investors are essentially indifferent to news about future variance at horizons beyond 30 days ranging from 1 month to 14 years. Andries et al. (2015) use standard index option data with maturities between 30 and 360 days to calibrate a Heston (1993) model at different horizons. They find a significantly negative price of volatility risk for maturities up to 120 days, but also a strongly nonlinear down-ward sloping term structure (in absolute value). These results supplement earlier studies of volatility risk *premia*, such as those by Amengual (2008) and Ait-Sahalia et al. (2012).[8]

6 Conclusion

We solve for general equilibrium asset prices in an endowment economy in which assets are priced by an agent who can have different levels of risk aversion for risks at different maturities. Such preferences are dynamically inconsistent with respect to risk-return tradeoffs. We find horizon-dependent risk aversion preferences have a meaningful impact on asset prices, and have the ability to address recent puzzles in general equilibrium asset pricing unaccounted for by the standard models. In particular, we show that the price of risk depends on the horizon, but only if volatility is stochastic. This insight leads to several testable predictions.

We are not aware of competing mainstream endowment-economy models that can predict a downward-sloping term structure and that make similarly detailed and empirically valid predictions for its driver. Relaxing the common assumption that risk preferences are constant across maturities – and specifically, replacing it with the no more flexible assumption that short-horizon risk aversion is higher than long-horizon risk aversion – may thus be a useful tool in different subfields of asset pricing research.

[8]There is a large literature on variance risk premia more generally, including the seminal works by Coval and Shumway (2001); Carr and Wu (2009), and the link to political uncertainty (Amengual and Xiu, 2013; Kelly et al., 2014).

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Appendix

A Derivation of the Stochastic Discount Factor

This appendix derives the stochastic discount factor of our dynamic model using an approach similar to the one used by Luttmer and Mariotti (2003) for dynamic inconsistency due to non-geometric discounting. In every period t the agent chooses consumption C_t for the current period and state-contingent levels of wealth $\{W_{t+1,s}\}$ for the next period to maximize current utility V_t subject to a budget constraint *and* anticipating optimal choice C^*_{t+h} in all following periods ($h \geq 1$):

$$\max_{C_t, \{W_{t+1}\}} \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[\left(V^*_{t,t+1} \right)^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}}$$

$$\text{s.t.} \quad \Pi_t C_t + E_t[\Pi_{t+1} W_{t+1}] \leq \Pi_t W_t$$

$$V^*_{t,t+h} = \left((1-\beta) \left(C^*_{t+h} \right)^{1-\rho} + \beta E_{t+h} \left[\left(V^*_{t,t+h+1} \right)^{1-\gamma_h} \right]^{\frac{1-\rho}{1-\gamma_h}} \right)^{\frac{1}{1-\rho}} \quad \text{for all} \quad h \geq 1.$$

Denoting by λ_t the Lagrange multiplier on the budget constraint for the period-t problem, the first order conditions are:[9]

- For C_t:

$$\left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[V_{t,t+1}^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}-1} (1-\beta)\, C_t^{-\rho} = \lambda_t.$$

- For each $W_{t+1,s}$:

$$\frac{1}{1-\rho} \left((1-\beta)\, C_t^{1-\rho} + \beta E_t \left[V_{t,t+1}^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}} \right)^{\frac{1}{1-\rho}-1} \beta \frac{d}{dW_{t+1,s}} \beta E_t \left[V_{t,t+1}^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}}$$
$$= \Pr[t+1, s]\, \frac{\Pi_{t+1,s}}{\Pi_t} \lambda_t.$$

Combining the two, we get an initial equation for the SDF:

$$\frac{\Pi_{t+1,s}}{\Pi_t} = \beta \, \frac{\frac{1}{1-\rho} \frac{1}{\Pr[t+1,s]} \frac{d}{dW_{t+1,s}} E_t \left[V_{t,t+1}^{1-\gamma_0} \right]^{\frac{1-\rho}{1-\gamma_0}}}{1} \frac{1}{(1-\beta)\, C_t^{-\rho}}. \tag{9}$$

[9] For notational ease we drop the star from all Cs and Vs in the following optimality conditions but it should be kept in mind that all consumption values are the ones optimally chosen by the corresponding self.

The agent in state s at $t+1$ maximizes

$$\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\left[\left(V_{t+1,s,t+2}^{*}\right)^{1-\gamma_0}\right]^{\frac{1-\rho}{1-\gamma_0}}\right)^{\frac{1}{1-\rho}}$$

and has the analogous first order condition for $C_{t+1,s}$:

$$\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\left[V_{t+1,s,t+2}^{1-\gamma_0}\right]^{\frac{1-\rho}{1-\gamma_0}}\right)^{\frac{1}{1-\rho}-1} (1-\beta)\, C_{t+1,s}^{-\rho} = \lambda_{t+1,s}.$$

The Lagrange multiplier $\lambda_{t+1,s}$ is equal to the marginal utility of an extra unit of wealth in state $t+1, s$:

$$\begin{aligned}\lambda_{t+1,s} = \frac{1}{1-\rho} &\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\left[V_{t+1,s,t+2}^{1-\gamma_0}\right]^{\frac{1-\rho}{1-\gamma_0}}\right)^{\frac{1}{1-\rho}-1} \\ &\times \frac{d}{dW_{t+1,s}}\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\left[V_{t+1,s,t+2}^{1-\gamma_0}\right]^{\frac{1-\rho}{1-\gamma_0}}\right).\end{aligned}$$

Eliminating the Lagrange multiplier $\lambda_{t+1,s}$ and combining with the initial equation (9) for the SDF, we get:

$$\frac{\Pi_{t+1,s}}{\Pi_t} = \beta\, \frac{\frac{1}{\Pr[t+1,s]}\frac{d}{dW_{t+1,s}} E_t\left[V_{t,t+1}^{1-\gamma_0}\right]^{\frac{1-\rho}{1-\gamma_0}}}{\frac{d}{dW_{t+1,s}}\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}\left[V_{t+1,s,t+2}^{1-\gamma_0}\right]^{\frac{1-\rho}{1-\gamma_0}}\right)} \left(\frac{C_{t+1,s}}{C_t}\right)^{-\rho}$$

Expanding the V expressions, we can proceed with the differentiation in the numerator:

$$\begin{aligned}\frac{\Pi_{t+1,s}}{\Pi_t} = E_t&\left[\left((1-\beta)\, C_{t+1}^{1-\rho} + \beta E_{t+1}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_0}-1} \\ &\times \left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}-1} \\ &\quad \times \beta\, \frac{\frac{d}{dW_{t+1,s}}\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)}{\frac{d}{dW_{t+1,s}}\left((1-\beta)\, C_{t+1,s}^{1-\rho} + \beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_0}}\right)} \left(\frac{C_{t+1,s}}{C_t}\right)^{-\rho}.\end{aligned} \tag{10}$$

For Markov consumption $C = \phi W$, we can divide by $C_{t+1,s}$ and solve both differentiations:

- For the numerator:

$$\frac{d}{dW_{t+1,s}}\left((1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_2}}\right)^{\frac{1-\gamma_1}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)$$

$$=\left((1-\beta)\,1+\beta E_{t+1,s}\left[\left((1-\beta)\left(\frac{C_{t+2}}{C_{t+1,s}}\right)^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_2}}\right)^{\frac{1-\gamma_1}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_1}}\right)$$

$$\times\,\phi_{t+1,s}^{1-\rho}W_{t+1,s}^{-\rho}.$$

- For the denominator:

$$\frac{d}{dW_{t+1,s}}\left((1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_0}}\right)$$

$$=\left((1-\beta)\,1+\beta E_{t+1,s}\left[\left((1-\beta)\left(\frac{C_{t+2}}{C_{t+1,s}}\right)^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_0}}\right)$$

$$\times\,\phi_{t+1,s}^{1-\rho}W_{t+1,s}^{-\rho}.$$

Substituting these into equation (10) and canceling we get:

$$\frac{\Pi_{t+1,s}}{\Pi_t}=\frac{(1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_2}}\right)^{\frac{1-\gamma_1}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_1}}}{(1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}\left[\left((1-\beta)\,C_{t+2}^{1-\rho}+\beta E_{t+2}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]^{\frac{1-\rho}{1-\gamma_0}}}$$

$$\times\,\beta\left(\frac{C_{t+1,s}}{C_t}\right)^{-\rho}\left(\frac{(1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1,s}[\dots]^{\frac{1-\rho}{1-\gamma_1}}}{E_t\left[\left((1-\beta)\,C_{t+1,s}^{1-\rho}+\beta E_{t+1}[\dots]^{\frac{1-\rho}{1-\gamma_1}}\right)^{\frac{1-\gamma_0}{1-\rho}}\right]}\right)^{\rho-\gamma_0}.$$

Simplifying and cleaning up notation, we arrive at the same SDF as in the main text:

$$\frac{\Pi_{t+1}}{\Pi_t}=\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{V_{t,t+1}}{E_t\left[V_{t,t+1}^{1-\gamma_0}\right]^{\frac{1}{1-\gamma_0}}}\right)^{\rho-\gamma_0}\left(\frac{V_{t,t+1}}{V_{t+1}}\right)^{1-\rho}.$$

B Exact solutions for $\rho = 1$

Suppose the risk aversion parameter differs only for immediate risk shocks: between t and $t+1$, risk aversion is γ, for all shocks further down, risk aversion is $\tilde{\gamma}$. The model simplifies to:

$$V_t = \left[(1-\beta)C_t^{1-\rho} + \beta\left(\mathcal{R}_{t,\gamma}\left(\widetilde{V}_{t+1}\right)\right)^{1-\rho}\right]^{\frac{1}{1-\rho}},$$
$$\tilde{V}_t = \left[(1-\beta)C_t^{1-\rho} + \beta\left(\mathcal{R}_{t,\tilde{\gamma}}\left(\widetilde{V}_{t+1}\right)\right)^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where

$$\mathcal{R}_{t,\lambda}(X) = \left(E_t\left(X^{1-\lambda}\right)\right)^{\frac{1}{1-\lambda}}.$$

Take the evolutions

$$c_{t+1} - c_t = \mu + \phi_c x_t + \alpha_c \sigma_t W_{t+1},$$
$$x_{t+1} = \nu_x x_t + \alpha_x \sigma_t W_{t+1},$$
$$\sigma_{t+1}^2 - \sigma^2 = \nu_\sigma\left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma \sigma_t W_{t+1},$$

and suppose the three shocks are independent. (We can relax this assumption.)

If $\rho = 1$, then the recursion for $\tilde{V}$ becomes

$$\frac{\widetilde{V}_t}{C_t} = \left(\mathcal{R}_{t,\tilde{\gamma}}\left(\frac{\widetilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)\right)^{\beta}.$$

Assume that

$$\tilde{v}_t - c_t = \tilde{\mu_v} + \phi_v x_t + \psi_v \sigma_t^2.$$

Then the solution to the recursion yields

$$\phi_v = \beta\phi_c\left(I - \nu_x\right)^{-1},$$

and

$$\psi_v = \frac{1}{2}\frac{\beta\left(1-\tilde{\gamma}\right)}{1-\beta\nu_\sigma}\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v^2\alpha_\sigma^2\right) < 0.$$

Because

$$\frac{V_t}{C_t} = \left(\mathcal{R}_{t,\gamma}\left(\frac{\widetilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)\right)^{\beta},$$

we have

$$\frac{V_t}{\tilde{V}_t} = \left[\frac{\mathcal{R}_{t,\gamma}\left(\widetilde{V}_{t+1}\right)}{\mathcal{R}_{t,\tilde{\gamma}}\left(\widetilde{V}_{t+1}\right)}\right]^{\beta},$$

which yields

$$\begin{aligned} & v_t - \tilde{v}_t = -\frac{1}{2}\beta\left(\gamma-\tilde{\gamma}\right)\left(\alpha_c^2 + \phi_v^2\alpha_x^2 + \psi_v^2\alpha_\sigma^2\right)\sigma_t^2, \\ \Rightarrow \quad & v_t - \tilde{v}_t = -\left(\gamma-\tilde{\gamma}\right)\frac{1-\beta\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2 < 0. \end{aligned}$$

C Approximation for $\beta \approx 1$

As in Appendix B, consider the simplified model with only two levels of risk aversion:

$$\begin{aligned} V_t &= \left[(1-\beta)C_t^{1-\rho} + \beta\left(\mathcal{R}_{t,\gamma}\left(\widetilde{V}_{t+1}\right)\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}, \\ \tilde{V}_t &= \left[(1-\beta)C_t^{1-\rho} + \beta\left(\mathcal{R}_{t,\tilde{\gamma}}\left(\widetilde{V}_{t+1}\right)\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}, \end{aligned}$$

where

$$\mathcal{R}_{t,\lambda}\left(X\right) = \left(E_t\left(X^{1-\lambda}\right)\right)^{\frac{1}{1-\lambda}}.$$

Also, as in Appendix B, take the evolutions:

$$\begin{aligned} c_{t+1} - c_t &= \mu + \phi_c x_t + \alpha_c\sigma_t W_{t+1}, \\ x_{t+1} &= \nu_x x_t + \alpha_x\sigma_t W_{t+1}, \\ \sigma_{t+1}^2 - \sigma^2 &= \nu_\sigma\left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma\sigma_t W_{t+1}, \end{aligned}$$

and suppose the three shocks are independent. (We can relax this assumption.)

Now, for β close to 1, we have:

$$\left(\frac{\tilde{V}_t}{C_t}\right)^{1-\tilde{\gamma}} = \beta^{\frac{1-\tilde{\gamma}}{1-\rho}}E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\tilde{\gamma}}\right].$$

This is an eigenfunction problem with eigenvalue $\beta^{-\frac{1-\tilde{\gamma}}{1-\rho}}$ and eigenfunction $\left(\tilde{V}/C\right)^{1-\tilde{\gamma}}$ known up to a multiplier. Let's assume:

$$\tilde{v}_t - c_t = \tilde{\mu} + \phi_v x_t + \psi_v \sigma_t^2.$$

Then we have:

- Terms in x_t (standard formula with $\beta = 1$):

$$\phi_v = \phi_c \left(I - \nu_x\right)^{-1}$$

- Terms in σ_t^2:

$$\psi_v = \frac{1}{2} \frac{(1-\tilde{\gamma})}{1-\nu_\sigma} \left(\alpha_c^2 + \phi_v^2 \alpha_x^2 + \psi_v^2 \alpha_\sigma^2\right) < 0$$

- Constant terms:

$$\log \beta = -\left(1-\rho\right)\left(\mu + \psi_v \sigma^2 \left(1-\nu_\sigma\right)\right)$$

For β close to 1, we have:

$$\frac{V_t}{\tilde{V}_t} \approx \frac{\mathcal{R}_{t,\gamma}\left(\widetilde{V}_{t+1}\right)}{\mathcal{R}_{t,\tilde{\gamma}}\left(\widetilde{V}_{t+1}\right)} = \frac{\left(E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}}{\left(E_t\left[\left(\frac{\tilde{V}_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\tilde{\gamma}}\right]\right)^{\frac{1}{1-\tilde{\gamma}}}},$$

and therefore:

$$\begin{aligned} & v_t - \tilde{v}_t = -\frac{1}{2}\left(\gamma - \tilde{\gamma}\right)\left(\alpha_c^2 + \phi_v^2 \alpha_x^2 + \psi_v^2 \alpha_\sigma^2\right)\sigma_t^2, \\ \Rightarrow \quad & v_t - \tilde{v}_t = -\left(\gamma - \tilde{\gamma}\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v \sigma_t^2 < 0. \end{aligned}$$

The stochastic discount factor becomes:

$$\begin{aligned} \pi_{t,t+1} = \bar{\pi}_t & - \gamma \alpha_c \sigma_t W_{t+1} + \left(\rho - \gamma\right)\phi_v \alpha_x \sigma_t W_{t+1} \\ & + \left(\left(\rho - \gamma\right) + \left(1-\rho\right)\left(\gamma - \tilde{\gamma}\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v \alpha_\sigma \sigma_t W_{t+1}, \end{aligned}$$

where

$$\bar{\pi}_t = -\mu - \rho\phi_c x_t - (1-\rho)\,\psi_v\sigma^2\,(1-\nu_\sigma)\left(1-(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)$$
$$-\left((\rho-\gamma)\,(1-\gamma)-(1-\rho)\,(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2.$$

Observe that in all the analysis the impact and the pricing of the state variable x_t is unaffected by the horizon dependent model. We can therefore simplify the analysis by setting $x_t = 0$ for all t. Going forward, take the evolutions:

$$c_{t+1} - c_t = \mu + \alpha_c\sigma_t W_{t+1},$$
$$\sigma_{t+1}^2 - \sigma^2 = \nu_\sigma\left(\sigma_t^2 - \sigma^2\right) + \alpha_\sigma\sigma_t W_{t+1},$$

and suppose the two shocks are independent.

We have

$$\tilde{v}_t - c_t = \tilde{\mu} + \psi_v\sigma_t^2,$$

where

$$\psi_v = \frac{1}{2}\frac{(1-\tilde{\gamma})}{1-\nu_\sigma}\left(\alpha_c^2 + \psi_v^2\alpha_\sigma^2\right) < 0,$$

and

$$\log\beta = -(1-\rho)\left(\mu + \psi_v\sigma^2\,(1-\nu_\sigma)\right)$$

$$v_t - \tilde{v}_t = -(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma_t^2 < 0$$
$$v_t - c_t = \tilde{\mu} + \psi_v\sigma_t^2\left(1-(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)$$

The stochastic discount factor becomes:

$$\pi_{t,t+1} = \bar{\pi}_t - \gamma\alpha_c\sigma_t W_{t+1} + \left((\rho-\gamma)+(1-\rho)\,(\gamma-\tilde{\gamma})\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v\alpha_\sigma\sigma_t W_{t+1}$$

where

$$\bar{\pi}_t = -\mu - (1-\gamma)^2\,\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma^2$$
$$-\left((1-\gamma)^2 - (1-\rho)\,(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma)\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\left(\sigma_t^2-\sigma^2\right).$$

Term structure: Now let's look a the term structure for endowment consumption strips. Let the period-t price for the endowment consumption in h periods be $P_{t,h}$. For $h = 0$, we have $P_{t,0} = C_t$. For $h \geq 1$ we have:

$$\frac{P_{t,h}}{C_t} = E_t \left(\Pi_{t,t+1} \frac{C_{t+1}}{C_t} \frac{P_{t+1,h-1}}{C_{t+1}} \right).$$

We can guess that

$$\frac{P_{t,h}}{C_t} = \exp \left(a_h + A_h \sigma_t^2 \right),$$

with $a_0 = 0$ and $A_0 = 0$. Suppose $h \geq 1$, then:

$$\begin{aligned}
\log \Pi_{t,t+1} \frac{C_{t+1}}{C_t} \frac{P_{t+1,h-1}}{C_{t+1}} = & - (1-\rho) \left(1 - \gamma + (\gamma - \tilde{\gamma}) \nu_\sigma \right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma^2 \\
& - \left((1-\gamma)^2 - (1-\rho) \left(1 - \gamma + (\gamma - \tilde{\gamma}) \nu_\sigma \right) \right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma_t^2 \\
& + a_{h-1} + A_{h-1} \sigma^2 (1 - \nu_\sigma) + A_{h-1} \nu_\sigma \sigma_t^2 \\
& + (1-\gamma) \alpha_c \sigma_t W_{t+1} \\
& + \left(\left((\rho - \gamma) + (1-\rho)(\gamma - \tilde{\gamma}) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \right) \psi_v + A_{h-1} \right) \alpha_\sigma \sigma_t W_{t+1}.
\end{aligned}$$

We find the recursion

$$\begin{aligned}
A_h = & - \left((1-\gamma)^2 - (1-\rho) \left(1 - \gamma + (\gamma - \tilde{\gamma}) \nu_\sigma \right) \right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v + A_{h-1} \nu_\sigma \\
& + \frac{1}{2} \left(\left(\rho - \gamma + (1-\rho)(\gamma - \tilde{\gamma}) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \right) \psi_v + A_{h-1} \right)^2 \alpha_\sigma^2 + \frac{1}{2} (1-\gamma)^2 \alpha_c^2,
\end{aligned}$$

and

$$a_h = - (1-\rho) \left(1 - \gamma + (\gamma - \tilde{\gamma}) \nu_\sigma \right) \frac{1-\nu_\sigma}{1-\tilde{\gamma}} \psi_v \sigma^2 + a_{h-1} + A_{h-1} \sigma^2 (1 - \nu_\sigma).$$

The one-period excess returns on the dividend strips are given by:

$$R_{t+1}^h = \frac{P_{t+1,h-1} - P_{t,h}}{P_{t,h}} = \frac{\frac{P_{t+1,h-1}}{C_{t+1}} \frac{C_{t+1}}{C_t}}{\frac{P_{t,h}}{C_t}} - 1.$$

We have:

$$\log\left(R^h_{t+1}+1\right) = \overbrace{\mu + (1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\psi_v\sigma^2}^{\equiv\bar{r}}$$
$$+\left(A_{h-1}\nu_\sigma - A_h\right)\sigma_t^2 + \left(\alpha_c + A_{h-1}\alpha_\sigma\right)\sigma_t.$$

So the conditional Sharpe ratio term structure is given by:

$$\mathrm{SR}_t\left(R^h_{t+1}\right) = \frac{\exp\left(\bar{r}+\left(A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)-1}{\sqrt{\begin{cases}\exp\left(2\bar{r}+2\left(A_{h-1}\nu_\sigma - A_h + \left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)\\ -\left[\exp\left(\bar{r}+\left(A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)\right]^2\end{cases}}}$$
$$= \frac{1-\exp\left[-\left(\bar{r}+\left(A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\right)\sigma_t^2\right)\right]}{\sqrt{\exp\left(\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right)\sigma_t^2\right)-1}}$$

Observe that:

$$A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}\alpha_\sigma^2\right)$$
$$= \frac{(1-\rho)}{1-\tilde{\gamma}}\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\left(\rho-\gamma+(\gamma-\tilde{\gamma})\left((1-\rho)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}-1\right)\right)\frac{1}{2}\psi_v^2\alpha_\sigma^2$$
$$+\left(1-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\right)\frac{1}{2}\alpha_c^2$$
$$-\left(\rho-\gamma+(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2,$$

which we can re-write as:

$$A_{h-1}\nu_\sigma - A_h + \frac{1}{2}\left(\alpha_c^2 + A_{h-1}^2\alpha_\sigma^2\right) = A - \left(\rho-\gamma+(1-\rho)(\gamma-\tilde{\gamma})\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2,$$

where

$$A = \frac{(1-\rho)}{1-\tilde{\gamma}}\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\left(\rho-\gamma+(\gamma-\tilde{\gamma})\left((1-\rho)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}-1\right)\right)\frac{1}{2}\psi_v^2\alpha_\sigma^2$$
$$+\left(1-(1-\rho)\left(1-\gamma+(\gamma-\tilde{\gamma})\,\nu_\sigma\right)\right)\frac{1}{2}\alpha_c^2.$$

We therefore have:

$$\mathrm{SR}_t\left(R_{t+1}^h\right)=\frac{1-\exp\left[-\left(\bar{r}+A\sigma_t^2-\left(\rho-\gamma+(1-\rho)\left(\gamma-\tilde{\gamma}\right)\frac{1-\nu_\sigma}{1-\tilde{\gamma}}\right)\psi_v A_{h-1}\alpha_\sigma^2\sigma_t^2\right)\right]}{\sqrt{\exp\left(\left(\alpha_c^2+A_{h-1}^2\alpha_\sigma^2\right)\sigma_t^2\right)-1}}.$$