Liquidity Hoarding and Investment under Uncertainty*

Patrick Bolton[†] Neng Wang[‡] Jinqiang Yang[§]

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Abstract

This paper develops a model of real option exercising for a financially constrained firm. We show that costly external financing induces the firm to hoard liquidity, e.g. cash. Importantly, liquidity hoarding has important effects, both conceptually and quantitatively, on the firm's real option exercising decisions and financing decisions. We find that the standard real options results do not hold when financial constraints are incorporated. For example, firm value may no longer increase with volatility due to the firm's precautionary motive to hoard cash. The marginal value of cash is not necessarily decreasing with liquidity due to the fact that the firm has embedded optionality. Importantly, our paper shows that volatility has both a positive and a negative effect on corporate investment and financing decisions. Our results suggest that we need to be

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[†]Columbia University, NBER and CEPR. Email: pb2208@columbia.edu. Tel. 212-854-9245.
[‡]Columbia Business School and NBER. Email: neng.wang@columbia.edu. Tel. 212-854-3869.
[§]The School of Finance, Shanghai University of Finance & Economics (SUFE). Email: yang.jinqiang@mail.sufe.edu.cn.

cautious in prescribing real option results based on standard complete-markets frictionless models to MBA students and practitioners. These prescriptions can be misleading and sometimes wrong.

1 Introduction

Much of the theory of investment under uncertainty (Dixit and Pindyck, 1994) is centered around a real options problem (McDonald and Siegel, 1986). The basic formulation of the real options problem takes the form of an investment timing option faced by a firm, who can at any time $t \geq 0$ acquire an asset of uncertain value $V(t)$ by incurring a fixed irreversible investment cost $I > 0$. Under the simplest formulation there is a constant interest rate $r > 0$ and $V(t)$ is assumed to follow a Geometric Brownian Motion, $dV(t) = V(t)(\mu dt + \sigma dW(t))$, where $W(t)$ is a standard Wiener process. The main result of the model is that it is not optimal for the firm to invest whenever the value of the asset exceeds the cost ($V(t) \geq I$). The firm can do better by optimally exercising a timing option and invest if only if $V(t) \geq \overline{V} > I$, where the optimal stopping point $\overline{V}$ can be easily characterized. A central comparative static prediction of the model is that the optimal stopping point $\overline{V}$ increases when the asset value is more uncertain (σ is higher) and that the value of the timing option is higher when σ is higher.

The real options formulation of investment under uncertainty provides a powerful foundation for corporations' widespread capital budgeting practice of pursuing only investments which have an *internal rate of return* $h(t)$ that exceeds a given *hurdle rate* $\overline{h}$ that is significantly higher than the firm's *cost of capital* r. Such a policy can be rationalized as equivalent to maximizing the value of a timing option, by investing if and only if the present value of the investment $V(t)$ exceeds the *'hurdle value'* $\overline{V} > I$.

The basic formulation of this real option problem has been extended in many different directions (to allow for time-varying investment costs, jumps in asset values, mean-reversion, time-varying volatility, etc.) and more and more elaborate formulations of corporate investment problems under uncertainty, which include staged investment options and abandonment options, have been considered. There is by now a vast theoretical and empirical literature on investment under uncertainty that builds on real options.

One aspect, however, that has remained virtually unexplored is how real options are affected by external financing costs and financial constraints. How is the timing of investment affected by the firm's retained earnings and cash holdings? And how is the value of real options affected by the availability of cash? These are the central questions addressed in this paper. When firms face external financing costs they are better off funding their capital expenditures via internal funds. However, internal can only be generated via operating profits, corporate saving, interest income, and external financing. Additionally, once the firm completes its investment, the value of the investment will be greater if the firm's costs can easily be covered with retained earnings. It is thus intuitively obvious that a firm's hoard of liquidity can be quite valuable and should affect both the optimal timing of investments and the values for both growth option and assets in place.

An important result from our analysis is that when firms face external financing costs the hurdle value depends on the firm's hoard of liquidity W in a highly non-linear and non-monotonic fashion. When the firm has almost no retained earnings (W is close to zero) it behaves like a financially unconstrained firm that faces an unusually high cost of capital (the firm's external financing costs). In other words, the firm then has a high hurdle rate mainly due to the fact that it faces high external funding costs. As retained earnings increase, however, the firm can fund a higher proportion of its investment with cheaper internal funds, so that one would expect the hurdle rate to decrease as W increases. This is true up to a point. Indeed, when the firm is close to being able to fully fund its capital expenditures from retained earnings it has a strong incentive to delay investment long enough to be able to fully cover its investment cost with internal funds. At that point, the hurdle rate is increasing in W, and increasingly so as W nears the total amount required for investment and for continuing operations after the investment is completed. In other words, the firm optimally becomes extremely conservative in its investment decisions around the point where it is about to accumulate sufficient internal funds to be able to cover investment outlays and future operating costs. This is a striking result that has important implications for corporate investment policy for financially constrained firms.

Related literature. Broadly, our work is related to (1) dynamic corporate financial models with financial constraints and (2) real options models. As we have noted earlier, vast majority of real option models assumes that the firm faces perfectly competitive capital markets.[1] We instead explicitly acknowledges the financial frictions and the implications of financing costs on investment. Liquidity hoarding is at the core of our model.

Our paper is most closely related to Boyle and Guthrie (2003) and Bolton, Chen, and Wang (2011), henceforth BCW. Boyle and Guthrie (2003) is among the first to

[1]Brennan and Schwartz (1985) use an option pricing approach to analyze investment in natural resources. McDonald and Siegel (1986) provide the standard continuous-time framework for analysis of a firms investment in a single project. Majd and Pindyck (1987) enrich the analysis with a time-to-build feature. Dixit (1989) uses the real option approach to examine entry and exit from a productive activity. Titman (1985) and Williams (1991) use the real options approach to analyze real estate development. Grenadier (1996, 2002) and Lambrecht and Perraudin (2003) extend real options to a game-theoretic environment.

study the effect of financial constraints and liquidity on the growth option exercising decisions. However, their model allows liquidity to be arbitrarily negative and hence banks may make losses. They do not explicitly model the external financing cost function. We also study the effects of operating leverage on financial constraints and financing decisions. BCW develop a q-theoretic model of investment for a financially constrained firm. As in our paper, BCW also model the liquidity accumulation for a financially constrained firm. However, BCW do not model firm's real option exercising decisions, which is the key of our paper. Whited (2006) studies the effects of external finance constraints on the timing of large investment projects by empirically estimate a structural model.

Our model also relates to the incomplete-markets real-options model of Miao and Wang (2007). In Miao and Wang (2007), the decision maker is risk averse and holds an illiquid option that is not fully spanned by the markets. The standard precautionary savings motive for a risk-averse consumer generates the negative effect of volatility on the option value. Therefore, In Miao and Wang (2007), volatility has both a positive and a negative effect on the entrepreneur's certainty equivalent wealth. In our paper, the firm is risk neutral but may behave in an endogenously risk averse manner as in BCW. This again is due to the financial constraints rather than the firm's owner being risk averse.

Financial frictions play a fundamental role in corporate investment. Fazzari, Hubbard, and Petersen (1988), Froot, Scharfstein, and Stein (1993), and Kaplan and Zingales (1997) are the first-generation static models. Hence, by construction, they're not suited to study the optionality involving investment and financing decisions. Recent dynamic models of investment with financial constraints include Gomes (2001), Hennessy and Whited (2005, 2007), Riddick and Whited (2009), and BCW, among others. However, these models do not focus on the *joint* real option exercising and liquidity hoarding decisions. We show that the interaction between real option exercising (irreversibility) and liquidity accumulation are critical in determining the firm's investment and financing decisions.

Our work is also related to two other sets of dynamic models of financing. First, DeMarzo, Fishman, He, and Wang (2012) develop a dynamic contracting model of corporate investment and financing with managerial agency, by building on Bolton and Scharfstein (1990) and using the dynamic contracting framework of DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007b).[2] These models derive optimal dynamic contracts and corporate investment with capital adjustment costs. Second, Rampini and Viswanathan (2010, 2011) develop dynamic models of collateralized financing, in which the firm has access to complete markets but is subject to endogenous collateral constraints induced by limited enforcement.

[2]DeMarzo and Fishman (2007a) study optimal investment dynamics with managerial agency in a discrete time setting.

2 Model

Operating revenues and profits. We consider a firm with an investment opportunity modeled as in McDonald and Siegel (1986). At any point in time $t \geq 0$ the firm can exercize an investment opportunity by paying a fixed investment outlay $I > 0$. It then obtains a stochastic revenue Y_t, which is assumed to follow the geometric Brownian motion (GBM) process:

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dB_t^Y, \tag{1}$$

where μ is the drift parameter, σ the volatility parameter, and B^Y is a standard Brownian motion. Once it has exercised the option, the firm also incurs a flow operating cost $Z > 0$ and receives operating profits

$$Y_t - Z \tag{2}$$

per unit of time for as long as it continues operating the project. After undertaking the investment, the firm can continue operating the project as long as it wants. Should it deem that it is no longer worth continuing it can stop operating the project. In other words, the firm also has a liquidation option.

Before undertaking the investment it does not incur any costs, and while the firm is waiting to invest, the revenue process Y_t continues to evolve according to (1). In sum, the simplest formulation of the life-cycle of the firm in our model allows for three phases: a start-up phase, a mature phase, and a scale-down phase. In an extension we also consider a richer life-cycle where the firm may have multiple investment opportunities.

We assume that investors are risk neutral, so that all cash flows are discounted at the risk-free rate, r. Equivalently, we may interpret that all cash flow generating processes are discounted under the risk-neutral measure. We pursue the risk-neutral measure interpretation when we analyze the firm's risk-return tradeoff. We will show how financial constraints and liquidity hoarding influence the firm's expected rate of return.

Cash and liquidity hoard. At the beginning of the start-up phase ($t = 0$) the firm is endowed with a stock of cash (or, more generally, a liquidity hoard comprising both cash and marketable securities) of W_0. As long as the firm does not spend this cash it simply accumulates liquid wealth at the risk-free rate r as follows:

$$dW_t = rW_t dt\,, \quad t \geq 0. \tag{3}$$

Note that since the firm earns the risk-free rate r on its cash it does not need to pay out any cash to its shareholders. In other words, as under the Modigliani and Miller irrelevance proposition, shareholders are indifferent to the firm's payout policy. If the firm were to earn less than the risk-free rate on its cash it would also face an optimal payout decision. For simplicity we do not consider this generalization of the model. As the firm incurs no cost in carrying cash, without loss of generality, the firm never pays out its cash as long as it operates the asset.

When $W < I$, the firm will have to raise external financing or continue to accumulate internal funds in order to finance the cost of exercising the growth option. The firm will also need funds to finance operations after the growth option is exercised.

We introduce external financing cost as follows: if the firm needs external funds F it incurs a cost of external capital of $\Phi(F)$, where $\Phi(F)$ is an increasing, convex function. In other words, to be able to deploy funds F raised externally, the firm must raise a total amount $F+\Phi(F)$, so that the *'gross spread'* is given by $\frac{\Phi(F)}{F+\Phi(F)}$, the ratio between the total financing cost $\Phi(F)$ and the total amount raised $F+\Phi(F)$.

Firm value: mature and start-up phases. During the mature phase (after exercising its growth option) the firm's liquidity W_t accumulates as follows:

$$dW_t = (rW_t + Y_t - Z)dt + dC_t\,, \quad W_t \geq 0\,. \tag{4}$$

The first term in (4) denotes the firm's internal funds W_t (which earn the interest rate r) plus the operating revenue Y_t minus the operating cost Z. The second term dC_t denotes the net external funds the firm chooses to raise through an external equity issue. Finally, after the firm has chosen to scale-down its investment it simply collects the remaining cash W_t and closes down.

First, we define the firm's value, $G(W, Y)$, after the firm's growth option is exercised, as follows,

$$G(W_t, Y_t) = \max_{\tau_2 \geq t,\, dC} \mathbb{E}_t \left[\int_t^{\tau_2 \wedge \tau_L} e^{-r(s-t)}(-\Phi(dC_s)) + e^{-r(\tau_2-t)} G^*(W_{\tau_2}, Y_{\tau_2}) \mathbf{1}_{\tau_2<\tau_L} \right], \tag{5}$$

where $G^*(W_t, Y_t)$ is the first-best firm value for a financially unconstrained firm given by (7). Note that we have two stopping times: τ_2 is the stopping time that the firm accumulates sufficient liquidity such that it will never become constrained, and τ_L is the stochastic liquidation time. The indicator function $\mathbf{1}_A$ takes the value of one if event A occurs and zero otherwise. If the firm strictly prefers issuing external equity rather liquidation, liquidation is never optimal and $\tau_L = \infty$.

If liquidation is suboptimal, the firm must raise costly external financing to be able to continue operating the project should it run out of cash. That is, when $W_s = 0$ at any time s where the firm also incurs operating losses, $Y_s < Z$, it has to raise funds dC_s that are at least sufficient to cover operating losses, to be able to continue operations. Therefore, at any such time s the change in the firm's stock of cash dW_s is given by $F_s = dC_s$ and the corresponding external financing cost is given by $\Phi(dC_s)$.

In the start-up phase, the firm's optimal investment timing problem is to choose the hurdle $\overline{Y}(W)$ so as to maximize the value of its timing option:

$$P(W_t, Y_t) = \max_{\tau_1, F} \mathbb{E}\left[e^{-r(\tau_1 - t)}\left[G(W_{\tau_1} + F - I, Y_{\tau_1}) - F - \Phi(F)\right]\right], \tag{6}$$

where τ_1 is the first time that Y_t crosses the *investment frontier* $\overline{Y}(W)$ for a liquidity hoard $W_{\tau_1} \geq W$, and where $G(W, Y)$ is the firm's value in the mature phase (after the firm has exercised its investment option). To be able to invest, the firm must have total available funds $W + F$ that cover at least the investment outlay I, so that F must satisfy the financing condition, $F + W \geq I$.

The firm's dynamic optimization problem thus involves a sequence of two optimal stopping decisions: an investment timing decision followed by an abandonment decision. We now proceed to a characterization of the firm's optimal policy. Proceeding by backwards induction we begin by characterizing the optimal abandonment policy before moving to the description of the optimal investment timing decision.

3 The First-Best Solution

We summarize the model solution in the MM world where there is no financial imperfection. First, consider the firm's value in the mature phase.

3.1 Firm value in the mature phase.

With perfect capital markets, we know that in the mature phase, the value for a firm with cash W and its asset in place has the following simple additive form:

$$G^*(W,Y) = V^*(Y) + W\,. \tag{7}$$

Here, $V^*(Y)$ is the present discounted value of future operating profits $Y - Z$ and accounts for the firm's optimal exercising of its abandonment option. For sufficiently low values of Y, i.e. $Y \leq Y_a^*$, where Y_a^* is to be determined soon, the asset is abandoned and hence $V^*(Y) = 0$. For the more interesting case where $Y \geq Y_a^*$, we may write $V^*(Y)$ as the solution of the following ODE:

$$rV(Y) = Y - Z + \mu Y V'(Y) + \frac{\sigma^2 Y^2}{2} V''(Y)\,,\ Y \geq Y_a^*\,, \tag{8}$$

with the standard value-matching and smooth-pasting conditions:

$$V(Y_a^*) \ = \ 0\,, \tag{9}$$

$$V'(Y_a^*) \ = \ 0\,. \tag{10}$$

Importantly, the abandonment threshold Y_a^* is endogenous and part of the model solution. The value $V^*(Y)$ admits the following unique closed-form solution:

$$V^*(Y) = \left(\frac{Y}{r-\mu} - \frac{Z}{r}\right) + \left(\frac{Y}{Y_a^*}\right)^{\gamma}\left(\frac{Z}{r} - \frac{Y_a^*}{r-\mu}\right), \quad Y \geq Y_a^*. \tag{11}$$

The first term in (11) is the present discounted value of its operating profits if the firm were to remain in operation forever (which would be suboptimal for sufficiently low Y). The second term gives the additional value created if the firm were to optimally exercise its abandonment option. Without capital market frictions, the firm optimally operates its physical asset if and only if $Y \geq Y_a^*$, where

$$Y_a^* = \frac{\gamma}{\gamma-1}\frac{r-\mu}{r}Z, \tag{12}$$

and the constant γ is given by

$$\gamma = \frac{1}{\sigma^2}\left[-\left(\mu - \frac{\sigma^2}{2}\right) - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] < 0. \tag{13}$$

Next, we turn to the firm's value in the start-up phase.

3.2 Firm value in the start-up phase.

We denote the *first-best* value of a financially unconstrained firm by $P^*(W,Y)$. As for the firm's value $G^*(W,Y)$ in the mature phase, the first-best value $P^*(W,Y)$ takes the following simple additive form:

$$P^*(W,Y) = H^*(Y) + W\,. \tag{14}$$

Intuitively, with perfect capital markets, again, firm value is given by the sum of its cash holding W and the the value of its growth option, $H^*(Y)$, can be valued independently from its liquidity holding.

Using the standard real options analysis, we know that $H(Y)$ is the solution the following ODE:

$$rH(Y) = \mu Y H'(Y) + \frac{\sigma^2 Y^2}{2} H''(Y), \tag{15}$$

subject to the value-matching and smooth-pasting boundary conditions:

$$H(Y_i^*) \quad = \quad V^*(Y_i^*) - I, \tag{16}$$

$$H'(Y_i^*) \quad = \quad V^{*\prime}(Y_i^*). \tag{17}$$

Note that $V^*(Y)$ given in (11) is the value of assets in place in the mature phase. Additionally, the growth option is worthless at $Y = 0$ as the origin is an absorbing state for a GBM process Y, i.e.

$$H(0) = 0. \tag{18}$$

The first-best option value $H^*(Y)$ has the following closed-form solution:

$$H^*(Y) = \left(\frac{Y}{Y_i^*}\right)^\beta \left[\frac{Y_i^*}{r-\mu} - \frac{Z}{r} + \left(\frac{Y_i^*}{Y_a^*}\right)^\gamma \left(\frac{Z}{r} - \frac{Y_a^*}{r-\mu}\right) - I\right], \tag{19}$$

where β is a constant given by:

$$\beta = \frac{1}{\sigma^2}\left[-\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 1. \tag{20}$$

Furthermore, the optimal investment threshold Y_i^* is the solution to the following equation:

$$(\beta-\gamma)\left(\frac{Y_i^*}{Y_a^*}\right)^{\gamma}\left(\frac{Z}{r}-\frac{Y_a^*}{r-\mu}\right)+(\beta-1)\frac{Y_i^*}{r-\mu}-\beta\left(\frac{Z+rI}{r}\right)=0\,. \tag{21}$$

Next, we turn to the analysis for a financially constrained firm.

4 Abandonment and Financing in the Mature Phase

Suppose that the firm is running the asset in place and that it has a liquidity hoard W. With external financing costs, liquidity hoard W will influence the firm's decision and its valuation. Denote the firm's value in the mature phase by $G(W, Y)$. The firm can be in three regions: (1) the "interior" region where the firm hoards cash and operates. In this region, the firm will not raise any external financing but simply continues its operation; (2) the financially "unconstrained" region where the firm behaves as if it is in the perfect capital market (MM) world; and (3) the issuance/liquidation region.

First, we consider the "interior" region where the firm hoards cash and operates its asset in place. In this region, the firm is financially constrained but does not raise any external financing.

4.1 The interior liquidity hoarding region.

Using the standard principle of optimality, we characterize the firm's value $G(W, Y)$ as the solution to the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rG(W,Y)=(rW+Y-Z)G_W(W,Y)+\mu Y G_Y(W,Y)+\frac{\sigma^2Y^2G_{YY}(W,Y)}{2}, \tag{22}$$

Intuitively, (22) states that the expected change of firm value $G(W, Y)$, given by the ride side of (22), equals $rG(W, Y)$. The first term (the G_W term) on the right side of (22) represents the effect of the firm's savings $(rW + Y - Z)$ on firm value. The second term (the G_Y term) represents the marginal effect of a change in expected earnings μY on firm value, and the last term (the G_{YY} term) encapsulates the effects of the volatility of changes in earnings Y on firm value.

With financial frictions, liquidity generally is more valuable than its pure monetary value. Typically, the firm's decision of whether to abandon the project or not is influenced by its financial considerations and the prospect of having to incur external financing costs. All else equal, the costs of external financing ought to be an additional inducement to abandon a project yielding low revenues. Therefore, one would expect that the prospect of having to incur external financing costs would lower the firm's valuation for its asset in place and result in a higher abandonment threshold, so that $\underline{Y}(W) \geq Y_a^*$.

The optimal abandonment threshold $\overline{Y}(W)$. Now consider the situation where a financially constrained firm is just indifferent between abandoning the firm or not. First, at the moment of indifference, firm value must be continuous, in that

$$G(W, \underline{Y}) = W, \tag{23}$$

which states the the firm's value equals to its liquidity hoard W at the moment of abandoning its asset. Equation(23) implicitly defines the abandonment threshold $\overline{Y}(W)$. Moreover, since the abandonment decision is optimally made, the marginal values along both Y and W margins shall also be matched before and after the abandonment of the asset in place, in that

$$G_W(W, \underline{Y}(W)) = 1, \tag{24}$$

$$G_Y(W, \underline{Y}(W)) = 0. \tag{25}$$

The smooth-pasting condition (24) states that the firm's marginal value of liquidity G_W at the moment of liquidation is one, and (25) implies that G_Y equals zero at the optimally chosen abandonment threshold $\overline{Y}(W)$.

To fully characterize the value for a financially constrained firm, $G(W, Y)$, we need to first characterize the behavior of financially unconstrained firms.

4.2 The financially unconstrained region.

There are two ways that the firm can be financially unconstrained with probability one at all times: (1) it internally generates sufficient liquidity at all times; or (2) the firm already has sufficient liquidity hoard W.

Type-1 financially unconstrained firm: $Y \to \infty$.

If the firm's internally generated cash flow Y is very high, the firm will be fully liquid even without any liquidity hoard W, as internally generated revenue Y is fully sufficient to cover its flow operate cost Z per period without ever having to raise external funds. In the limit where $Y \to \infty$, the following boundary value condition must hold:

$$\lim_{Y\to\infty} G(W, Y) = \lim_{Y\to\infty} G^*(W, Y), \tag{26}$$

where $G^*(W, Y)$ is the value for a financially unconstrained firm in the mature phase and is given by (7).

Type-2 financially unconstrained firm: $W \geq W^*$.

The firm is able to implement the first-best abandonment option policy and avoid costly external financing permanently with probability one provided that the firm has sufficiently high liquidity hoarding. The question is how high the firm's liquidity hoard W has to be for a firm to be permanently unconstrained. As long as the firm does not abandon its asset in place prematurely and involuntarily, that is, when $Y > Y_a^*$, the firm achieves its first-best policy and thus it is financially unconstrained. That is, as long as the corporate savings rate is weakly positive at Y_a^*, then the firm will never need to raise external financing and is not forced to liquidate the firm sub-optimally. That is, we need $rW + Y_a^* - Z \geq 0$, which implies

$$W \geq \frac{Z - Y_a^*}{r} = W^* \,. \tag{27}$$

Using the explicit formula (27) for the abandonment threshold Y_a^*, we write W^* as

$$W^* = \frac{(r - \gamma\mu)}{r^2(1-\gamma)} Z \,. \tag{28}$$

For a firm whose liquidity W is greater than W^*, its value $G(W, Y)$ equals the first-best firm's value:

$$G(W, Y) = G^*(W, Y), \quad \text{for} \quad W \geq W^* \,. \tag{29}$$

This condition states that when the firm has a liquidity hoard that is greater than or equal to W^* it is permanently financially unconstrained. To see that this condition must hold, note that for all $Y \geq Y_a^*$ the firm can finance its operations entirely out of its cash hoard. And for $Y < Y_a^*$ the firm voluntarily abandons its asset and distributes W to shareholders.

4.3 The equity issuance/liquidation region.

As BCW show, in a world with constant financing opportunities where financing terms do not change over time, the firm has no need to issue equity unless it absolutely has to. By delaying equity issuance, the firm saves the time value of money for the financing costs. In the mature phase, with positive liquidity hoarding W, the firm has sufficient slack to cover any operating losses over a given small time interval. Therefore, the firm never issues equity before it exhausts its cash.

When the firm runs out of its cash, the firm has an option to either issue equity or to liquidate its asset in place, whichever is in the interest of its shareholders. Intuitively, equity issuance or liquidation depends on how valuable the asset in place. This line of reasoning suggests that

$$G(0,Y) = \begin{cases} G(F,Y) - F - \Phi(F)\,, & \underline{Y}(0) < Y < Z\,, \\ 0\,, & Y \leq \underline{Y}(0)\,. \end{cases} \tag{30}$$

This condition states that firm value is continuous before and after raising external financing. When the firm runs out of cash it is instantly liquidated unless it raises new external financing. Following an operating loss ($Y < Z$) the firm will raise new funds should it run out of cash, provided that $Y > \underline{Y}(0)$, where $\underline{Y}(0)$ is endogenously determined liquidation threshold.

Should the firm seek to raise new funds, its optimal external financing amount F

is given by the first-order condition (FOC):

$$G_W(F, Y) = 1 + \Phi'(F). \tag{31}$$

Intuitively, the marginal value of cash $G_W(W, Y)$ equals the marginal cost of financing $1 + \Phi'(F)$. Equation (31) defines the optimal financing amount F as a function of Y, $F(Y)$. An immediate implication of this condition is that the firm's marginal value of cash G_W is greater than 1 as $\Phi'(F) > 0$. Naturally, the firm's marginal value of cash depends on the revenue Y the firm expects to obtain from the project.

Summary. In a dynamic environment, the condition for a firm to be financially unconstrained is much tighter than in a static setting. The reason is as follows. For a firm to be financially unconstrained in a dynamic setting, the firm cannot have demand for funds at any moment. That is, with probability one, the firm has no demand for funding. Only under this condition, can the firm be assured that its marginal value of cash is one.

In our model, the firm can be financially unconstrained in one of the two ways; it either internally generates enormous amount of funds or it has sufficient liquidity to cover all the potential needs. We obtain the the optimal abandonment option exercising and financing decisions for a financially constrained firm by (1) solving the PDE (22) subject to the value-matching condition (23), the smooth-pasting conditions (24) and (25), the limiting conditions (26)) and (29) where firm value approaches the first-best one, and financially the equity issuance/liquidation conditions (30) and (31).

Having characterized the firm's optimal financial policy $\{\underline{Y}(W), F(Y)\}$ and value $G(W, Y)$ in the mature phase, we now turn to the firm's problem in the start-up phase.

Obviously, a rational forward-looking firm fully anticipates its financial constraints in the mature phase and acts accordingly in the start-up phase.

5 Investment and Financing in the Start-up Phase

In the start-up phase, the firm faces both the one-time investment cost I and also future operating cost Z. Before analyzing the effect of financial constraints on the firm's investment option exercising and financing decisions, we first reason how much liquidity the firm needs in order to be financially unconstrained.

5.1 The financially unconstrained region: $W \geq I + W^*$.

For a firm to be dynamically financially unconstrained in its start-up phase, we need to require that its liquidity to be greater than $W^* + I$, the sum of I and W^* given in (28). Intuitively, if the firm has liquidity hoard greater than $W^* + I$, with probability one, the firm can cover both its investment cost I and its future liquidity shortfall to continue operating the asset. The firm distorts neither the investment option nor the abandonment option exercising decisions. Therefore, the firm in its start-up phase is financially unconstrained if and only if $W \geq I + W^*$.

Note that unlike in the mature phase, the firm will be constrained if $W < I$ no matter how large its earnings Y is. The intuition is as follows. The investment cost I is lumpy (stock) while the earnings Y is flow, the firm cannot possibly finance the cost I to exercise its growth option if $W < I$. Instead, it has to access the costly external capital market to cover its investment cost I, if it chooses to exercise its investment option immediately.

Next, we derive the optimal investment policy and also calculate the start-up value for a financially constrained firm. There are two sub-cases for a financially constrained firm: (1) the firm has a sufficient liquidity hoard to fund the investment outlay I entirely out of its internal funds, but may not have sufficient funds to be immune to a liquidity shortage (with probability one) and hence involuntary liquidation/equity issuance may occur with positive probability; (2) the firm cannot even cover its investment cost I and hence needs to raise funds to cover part of the investment costs and to also build the firm's future liquidity hoard.

5.2 The Medium Cash-holding Region: $I \leq W < I + W^*$

Consider now the situation of a firm with moderate financial slack. This firm has sufficient internal funds W to cover the investment cost I, but not quite enough cash to ensure that it will never run out of internal funds in the mature phase: $I \leq W < I + W^*$. In the forward-looking sense, the firm is still financially constrained and the marginal value of cash is greater than one. For such a firm it is optimal not to raise any external funds when it exercises its growth option, so that $F = 0$. Importantly, the firm realizes that exercising the investment option drains its cash holding by I and hence the firm may be led to raise external funds in the future to cover operating losses in the mature phase. Therefore, the value liquidity in the start-up phase, we need to incorporate the value of liquidity in the mature phase.

Intuitively, in the start-up phase, the firm's value $P(W, Y)$ is the solution to the following HJB equation:

$$rP(W,Y) = rWP_W(W,Y) + \mu Y P_Y(W,Y) + \frac{\sigma^2 Y^2}{2} P_{YY}(W,Y). \tag{32}$$

The first term on the right side of (32) reflects the firm's savings effect on firm value. The second and the third terms capture the effects of a change in expected earnings μY, and of the volatility of changes in earnings Y on firm value, respectively.

Next, we characterize the investment threshold $\overline{Y}(W)$. First, at the moment of exercising the investment option, firm value must be continuous, which implies the following value-matching condition:

$$P(W, \overline{Y}) \quad = \quad G(W - I, \overline{Y})\,. \tag{33}$$

Note that (33) defines $\overline{Y}(W)$, the investment threshold $\overline{Y}$ as an implicit function of liquidity W. Along the investment threshold $\overline{Y}(W)$, we have the following smooth-pasting conditions along both the liquidity W and the earnings Y dimensions,

$$P_W(W, \overline{Y}) \quad = \quad G_W(W - I, \overline{Y})\,, \tag{34}$$

$$P_Y(W, \overline{Y}) \quad = \quad G_Y(W - I, \overline{Y}). \tag{35}$$

Note that when the growth option is exercised it is entirely financed out of internal funds, and hence liquidity W decreases by I. For this case, the firm will only raise external financing when it runs out of cash and it needs to cover operating losses in the mature phase. Finally, we note the natural boundary condition,

$$P(W, 0) = W. \tag{36}$$

Condition (36) states that when the absorbing state $Y = 0$ is reached, there is no investment opportunity to be ever obtained so that the only valuable asset of the firm is its cash W.

5.3 The Low Cash-holding Region, $0 < W < I$.

If the firm chooses to raise external financing when its cash holding satisfies $W < I$, it may raise the amount more than needed to just cover its investment cost. Intuitively, the firm is forward looking and is aware that it also needs funds to cover operating losses in the mature phase. By issuing more funds than the amount needed to cover the investment needs may be economical due to the fixed cost, for example. We use $F(W)$ to denote the amount of external financing as a function of W. The gross financing amount is then $F(W) + \Phi(F(W))$. The value of the firm in the start-up phase $P(W,Y)$ is still given by the solution to the HJB equation (32) and the boundary condition (36) at $Y = 0$ is unchanged.

Importantly, we now have different boundary conditions to characterize the investment threshold $\overline{Y}(W)$. The value matching condition states that

$$P(W,\overline{Y}) \quad = \quad G(W + F - I, \overline{Y}) - F - \Phi(F)\,, \tag{37}$$

which defines the investment threshold $\overline{Y}(W)$ as a function of W in this region. We also require that the marginal value of cash to be equal before and after the investment option exercising. Similarly, the marginal value of earnings shall also equal before and after the investment option exercising. Therefore, we have the following two smooth-pasting conditions:

$$
\begin{aligned}
P_W(W, \overline{Y}(W)) &= G_W(W + F - I, \overline{Y}(W)), &(38)\\
P_Y(W, \overline{Y}(W)) &= G_Y(W + F - I, \overline{Y}(W)). &(39)
\end{aligned}
$$

How much does the firm issue? The following FOC for $F(W)$, obtained via differentiation with respect to W in equation (37), must also be satisfied:

$$
1 + \Phi'(F(W)) = P_W(W, \overline{Y}(W)) = G_W(W + F - I, \overline{Y}(W)). \tag{40}
$$

Intuitively, (40) states that the marginal value of cash equals the marginal cost of funding $1 + \Phi'(F(W))$. Note again that the marginal value of cash at the exercise boundary $\overline{Y}$ is greater than one, $P_W(W, \overline{Y}(W)) = 1 + \Phi'(F(W)) > 1$. Internal funds are valuable before and after the investment option is exercised.

6 Analysis

We now turn to a quantitative analysis of the model. We assume that the cost function for external financing takes the following affine form:

$$
\Phi(F) = \phi_0 + \phi_1 F,
$$

where ϕ_0 is the fixed cost component, and $\phi_1 > 0$ is the marginal cost of financing. We set $\phi_0 = 0.389$ and $\phi_1 = 0.053$ as in Altinkilic and Hansen (2000). The other baseline (annualized) parameter values that we take for our numerical solution are as follows: the risk-free interest rate is $r = 4\%$, the expected growth rate of revenue is $\mu = 2\%$, and the volatility is $\sigma = 20\%$. The investment cost is set at $I = 10$ and the operating cost is $Z = 1$.

We numerically solve for $G(W,Y)$ and $P(W,Y)$ as follows. We divide the parameter set into in two regions separated by the break-even line $rW + Y - Z = 0$. In the positive earnings region, with $rW + Y - Z > 0$, we solve for $G(W,Y)$ and $P(W,Y)$ via an initial value problem by starting from $W = W^*$ and gradually decreasing W. In the loss-making region, i.e. $rW + Y - Z < 0$, we solve for $G(W,Y)$ and $P(W,Y)$ by starting W from its origin $W = 0$. Along the $Y-$dimension we have a free-boundary problem.

6.1 The Mature Phase

The Liquidation decision. Figure 2 plots the firm's optimal liquidation threshold $\underline{Y}(W)$ for a financially constrained firm. First, the firm is unconstrained when its cash holding $W \geq W^* = 17.6$. The first-best liquidation threshold equals $Y_a^* = 0.293$, which by definition is independent of cash holdings W.

Second, when $W = 0$, without external funds, the firm would have been liquidated at the first time τ where $Y_\tau = Z = 1$. However, that would be suboptimal for the firm. Indeed, as long as $Y \geq \underline{Y}(0) = 0.358$, the firm will issue equity and keep the firm alive. We can see that there is a significant range of earnings $0.358 \leq Y \leq 1$ that the firm will find it optimal to issue equity in order to keep the firm alive.

More generally, the liquidation threshold $\underline{Y}(W)$ decreases in liquid wealth W from $\underline{Y}(0) = 0.358$ to $Y_a^* = 0.293$ as we increase W from the origin to $W^* = 17.6$. Intuitively, the more liquid wealth, the less distorted the liquidation exercising decision.

For $W > W^* = 17.6$, the firm does not need any external financing and will exercise its abandonment option if and only if $Y \le Y_a^* = 0.293$. Quantitatively, the effect of financial constraints effectively disappears if $W \ge 8$.

The Value of the Mature Firm $G(W, Y)$. Figure 3 plots the firm's value $G(W, Y)$ and the marginal value of liquid wealth, $G_W(W, Y)$, for different levels of earnings Y for three levels of the fixed cost, $\phi = 0$, $\phi_0 = 0.389$ and $\phi_1 = 0.053$. We note that $G_W(W, Y)$ is highly nonlinear and non-monotonic with respect to W. Panel B plots the marginal value of cash, $G_W(W, Y)$, holding earnings $Y = 0.5$. We note that G is convex in W for low values of W and concave in W for high values of W.

For sufficiently high values of earnings Y, the marginal value of liquidity $G_W(W, Y)$ is decreasing in W. Intuitively, financing constraints become less severe as the firm's liquidity W increases.

The Financing Decision. The firm will only issue equity when it runs out of cash in order to save the time value of financing costs. Figure 1 plots the optimal financing amount at $W = 0$. First, Figure 1 shows that the external financing ought to be zero even at $W = 0$, i.e. $F = 0$, when $Y \ge Z = 1$. In this region, the firm's operating cost Z can be fully covered by its revenue Y_t with probability one. Second, the firm raises no external financing for sufficiently low revenue, i.e. $Y \le \underline{Y} = 0.358$ because the firm's abandonment option is sufficiently in the money (note that external financing costs further erodes the option value of abandonment). Third, the (net) financing amount F is at first increasing with Y in the region $Y \le 0.5$ and then decreasing with Y in the region $0.5 < Y < Z = 1$. For our numerical example, the maximal net amount of financing is $F(Y) = 5.2$, which is around $Y = 0.5$. When

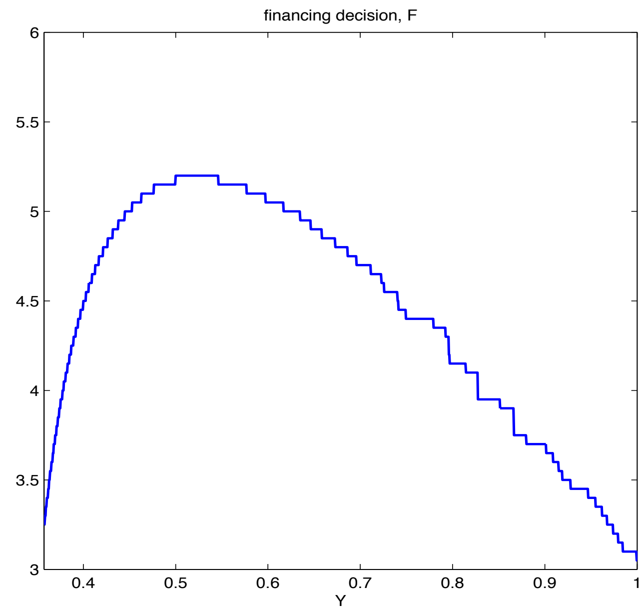

Figure 1: **The financing decision at $W = 0$ with $\phi_0 = 0.389$, $\phi_1 = 0.053$.**

Y is low (i.e. $Y \leq 0.5$), the firm does not want to hoard too much liquidity as the firm may be liquidated in the near term. As Y increases, the marginal value of liquidity increases and the firm issues more equity. When Y is sufficiently high, the firm relies less on external financing as internally generated cash flows in the future, appropriately discounted to the present value, become an important source of financing. As the firm's cash-flow generating ability increases (a higher Y), the firm relies less on external financing, which is costly, and hence lowers its demand.

Let $V(W, Y) = G(W, Y) - W$ as the firm's enterprise value, i.e. firm value in excess of its cash holding. Note that $V^*(Y) = G^*(W, Y) - W$. Intuitively, $1 - V(W, Y)/V^*(Y)$ measures the percentage value loss due to financial constraints. Figure 8 plots the percentage value losses due to financial constraints for different levels of revenue, $Y = 0.5, 1, 2$, where $Z = 1$. Notice that that the value losses are decreasing in cash, W, because the liquid wealth could mitigate the financial friction. For given liquid wealth, the lower the level of revenue implies more likelihood of external refinancing or distort liquidation. Naturally, the corresponding value losses of assets in place are lower for lower level of revenue. Especially, the losses will be 1 when Y is lower than $\underline{Y}$ because the firm is liquidated, for example the losses are always $1-V(W,Y)/V^*(Y) = 1$ when $W \leq 2.3$ for $Y = 0.3$ as presented by the blue line in Figure 8.

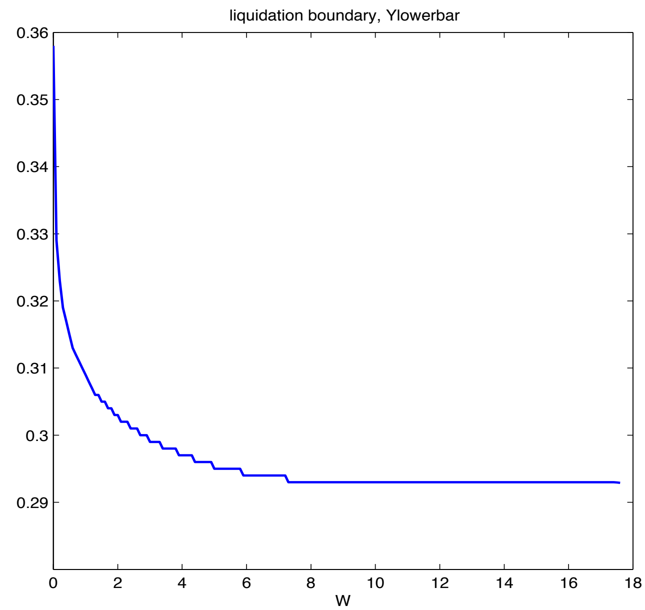

Figure 2: **The liquidation threshold $\underline{Y}(W)$.** The financing cost parameters are $\phi_0 = 0.389$ and $\phi_1 = 0.053$. The first-best liquidation (abandonment option exercising) threshold is $Y_a^* = 0.293$. The liquidation boundary at $W = 0$ is $\underline{Y}(0) = 0.358$. The liquidation boundary is monotonically decreasing with liquidity holding W.

The Investment Decision.

The Value of the Start-up Firm

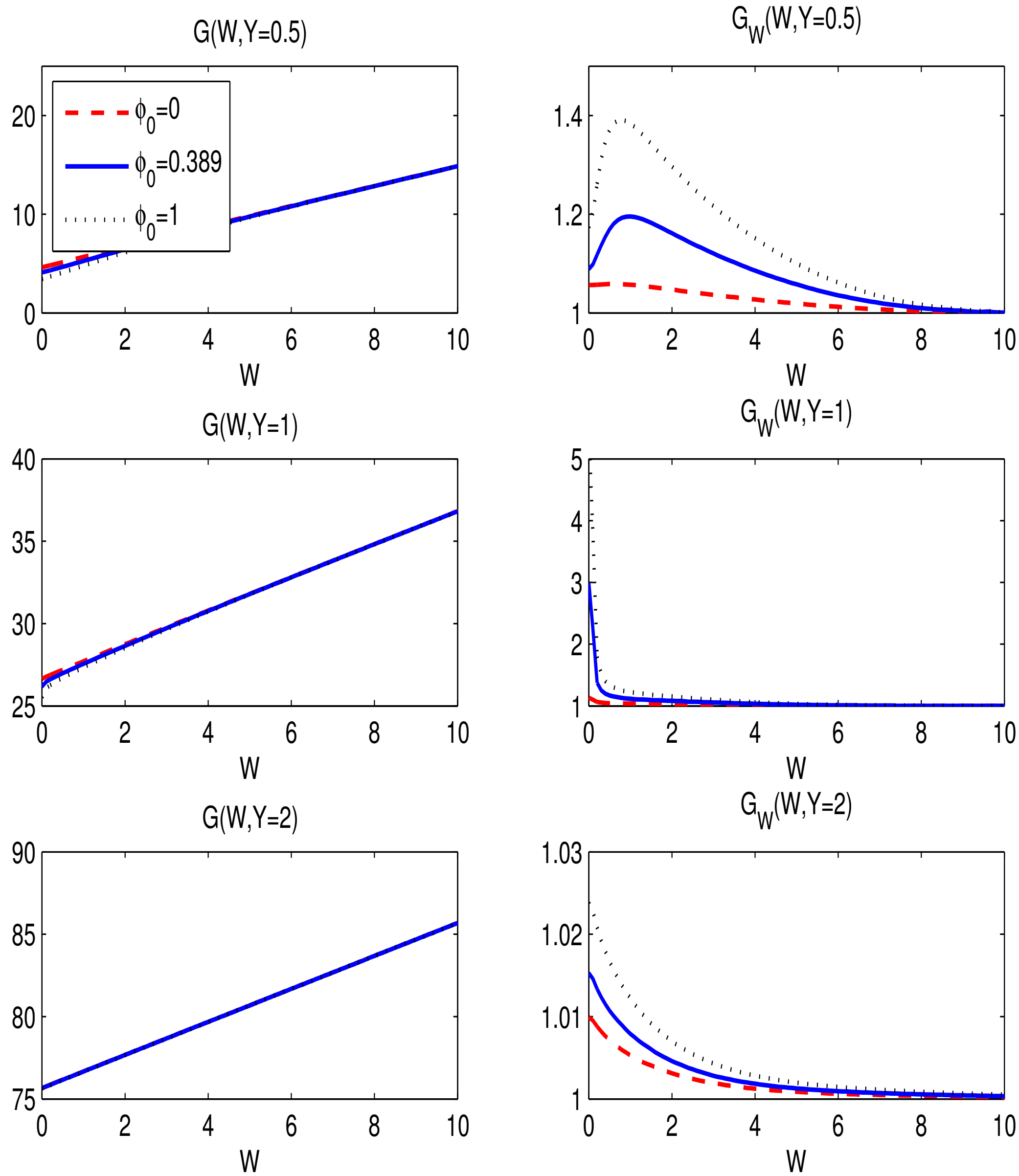

Figure 3: **Firm value after the growth option exercise, $G(W, Y)$, and the marginal value of liquidity $G_W(W, Y)$ for three levels of the fixed cost $\phi_0 = 0, 0.389, 1$.**

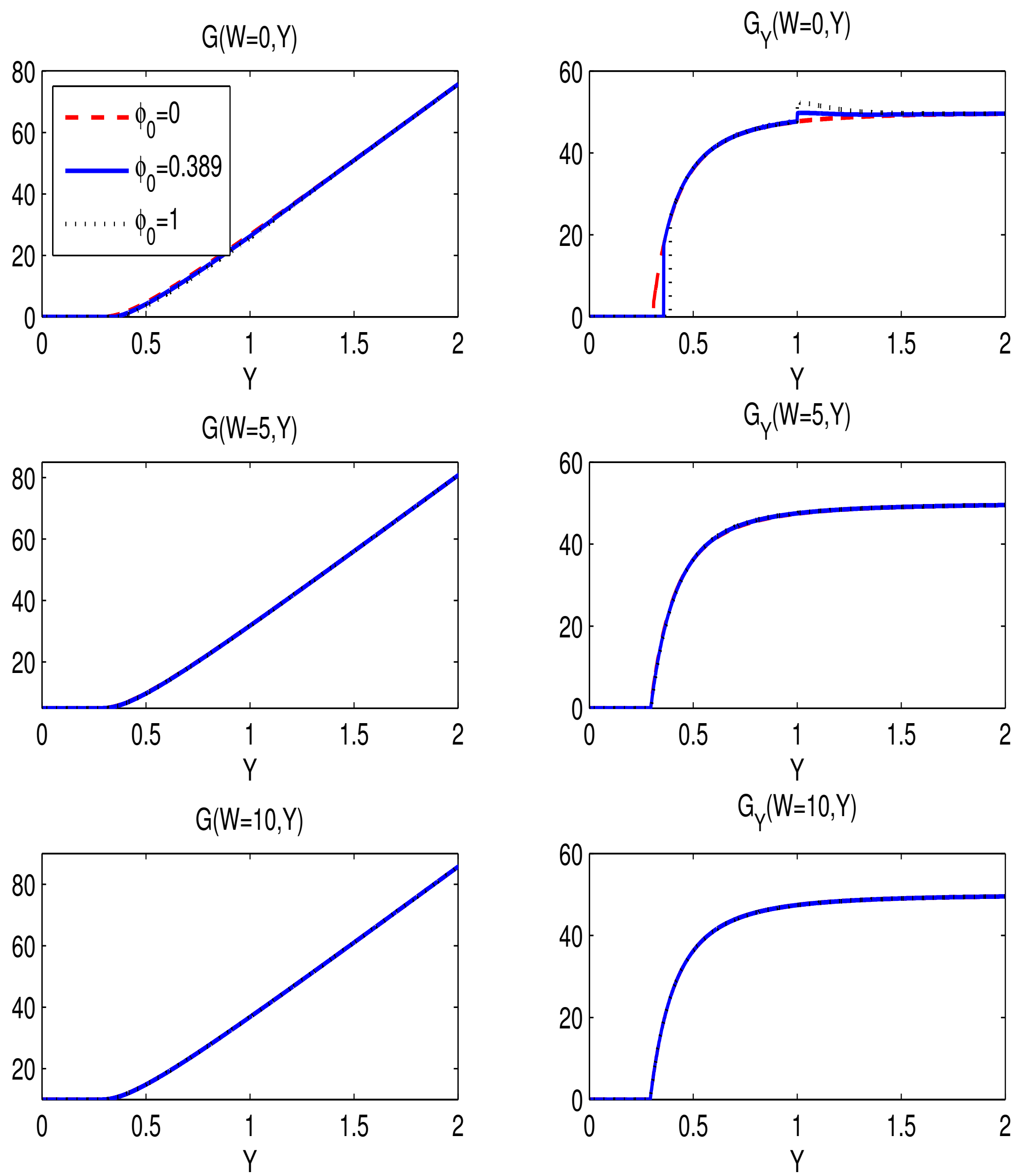

Figure 4: **Firm value after the growth option exercise, $G(W,Y)$, and the marginal value of revenue G_Y for three levels of the fixed cost $\phi_0 = 0, 0.389, 1$.**

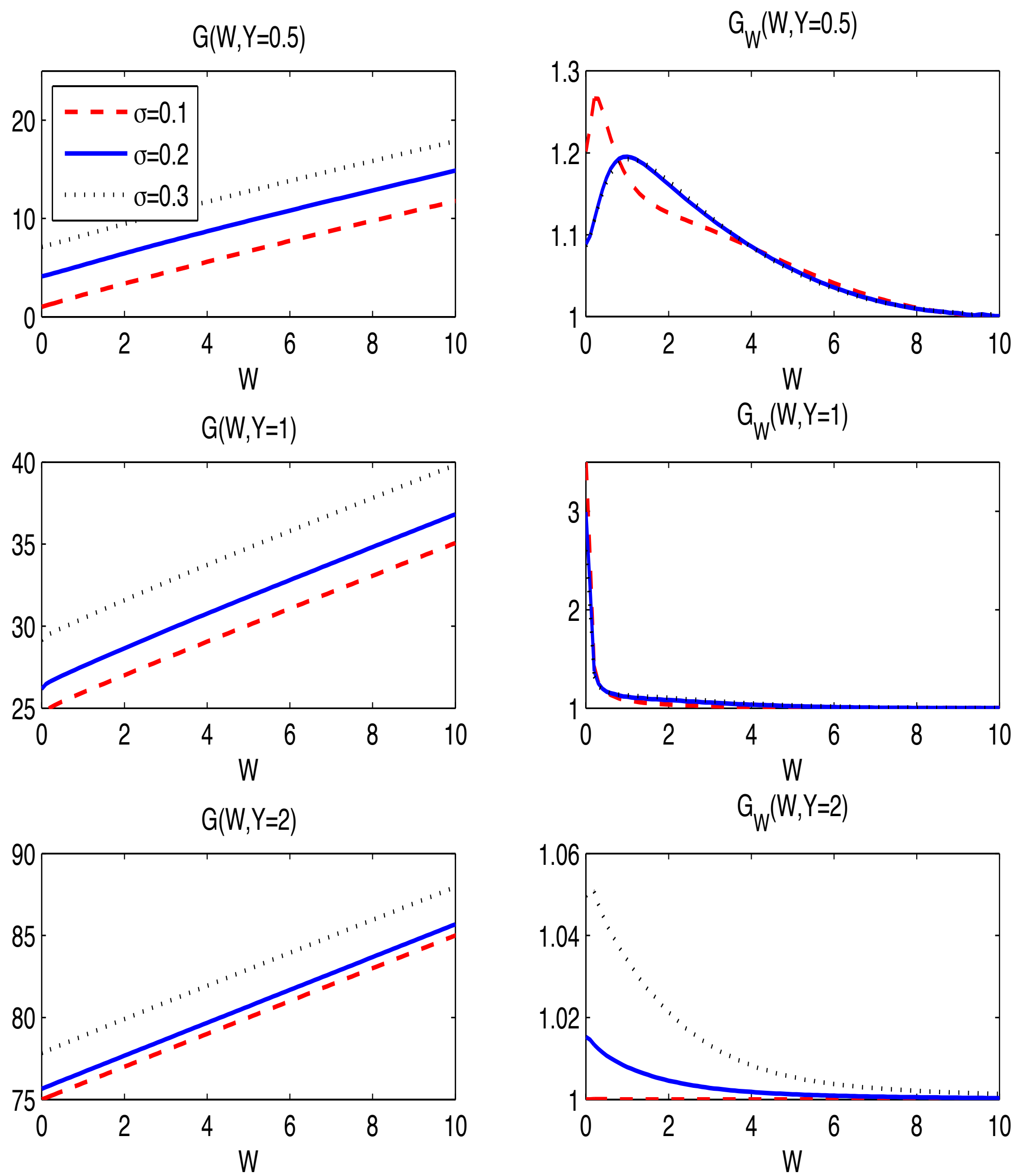

Figure 5: **Firm value after the real option exercise, $G(W, Y)$, and the marginal value of liquidity G_W for three levels of volatility $\sigma = 0.1, 0.2, 0.3$.**

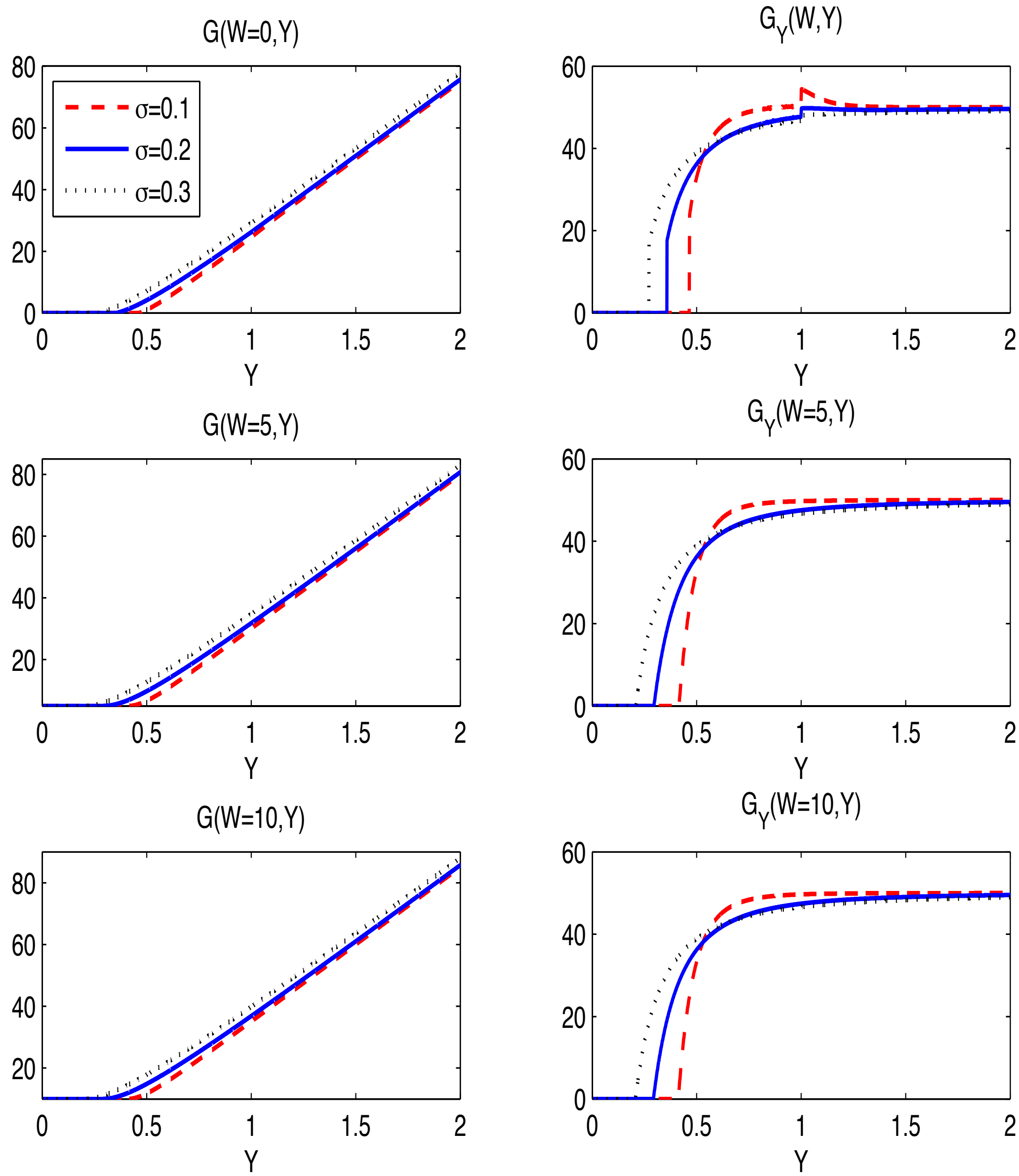

Figure 6: **Firm value after the real option exercise, $G(W,Y)$, and the marginal value of revenue G_Y for three different levels of volatility $\sigma = 0.1, 0.2, 0.3$.**

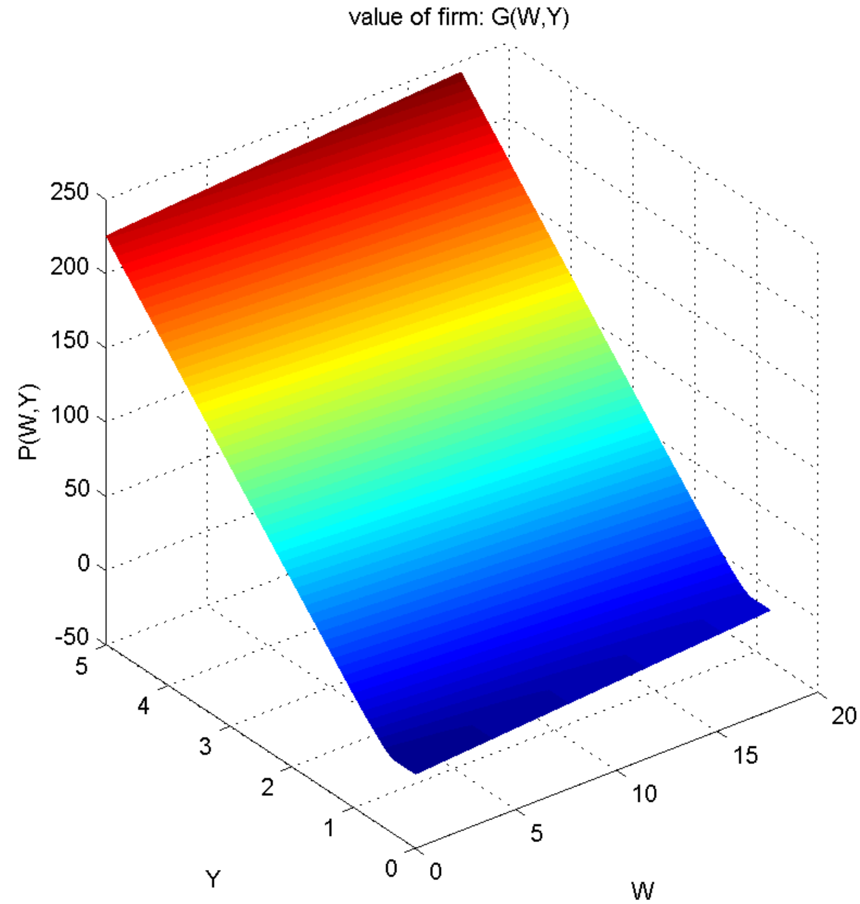

Figure 7: **Firm value after the real option exercise, $G(W, Y)$, The parameter values are $\phi_0 = 0.389$, and $\phi_1 = 0.053$.**

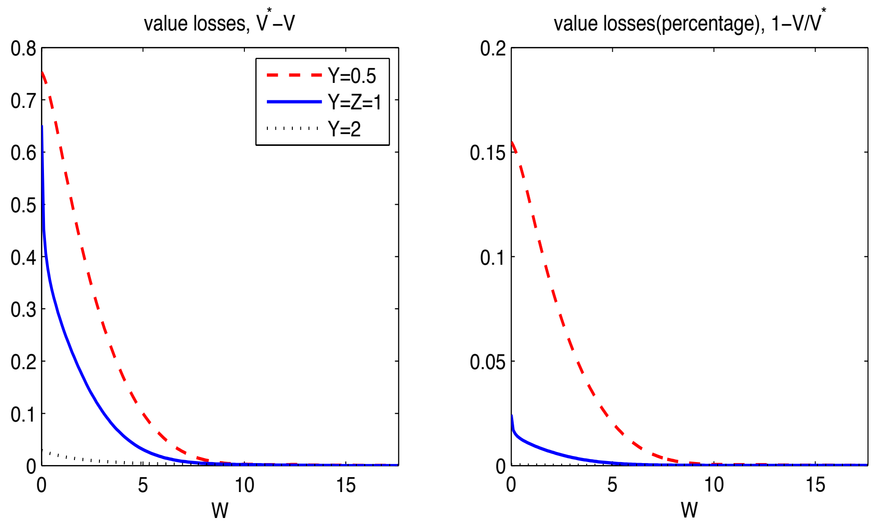

Figure 8: **The value losses (in level and percentage terms) due to financial constraints, $V^*(Y) - V(Y)$ and $1 - V(W,Y)/V^*(Y)$, for three different levels of revenue $Y = 0.5, 1, 2$. The financing cost parameters are $\phi_0 = 0.389$ and** $\phi_1 = 0.053$.

The Financing Decision.

7 Two growth options

To be completed.

8 Conclusion

We proposed a real option model for a financially constrained firm. Unlike in the standard real option models, the firm is financially constrained and faces costly external financing. Rationally, the firm hoards liquidity that will be useful for its investment (real option exercising) and also operations. We find that the firm's investment and financing decisions are highly interconnected. Liquidity is important for the firm to continue operating its asset in place. While the standard real options theory assumes that the firm owner/shareholder has an infinitely deep pocket and can inject funds whenever it is in the firm's interest. In reality, this is far from being true. Indeed, the recent financial crisis clearly indicates the importance of liquidity for financially constrained firms especially those with many growth options.

Unlike in the standard real option models, our framework predicts that the firm's option value may decrease in volatility. Intuitively, there are two opposing effects of volatility on firm value. On the one hand, due to the option's asymmetric payoff, firm value may increase with volatility as in standard complete-markets-based real option models. On the other hand, volatility can be costly as the firm may behave in a risk averse manner. We also develop a quantitative framework to value the marginal value of liquidity for a firm facing financial constraints and have investment options. We find that investment threshold is not monotonic in its liquidity holding. We also find that the amount of external financing is not monotonic in liquidity W. Our work shows the conceptual and quantitative importance of liquidity hoarding on real option exercising and financing decisions.

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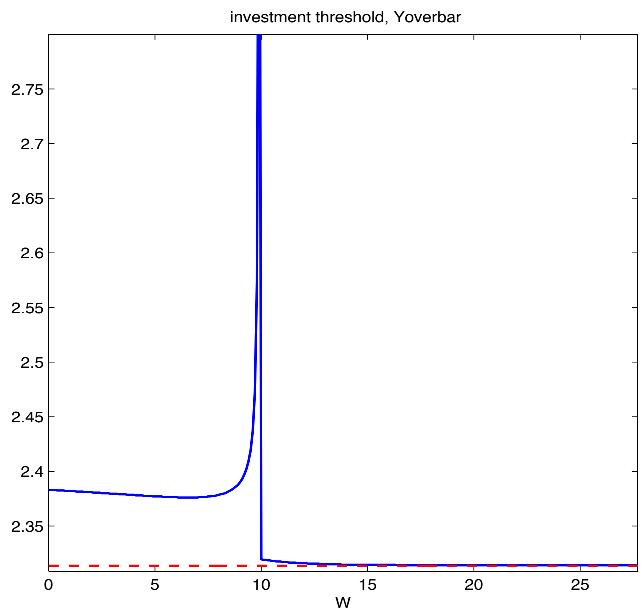

Figure 9: **The investment threshold $\overline{Y}(W)$ as a function of liquidity W. The financing cost parameters $\phi_0 = 0.389$ and $\phi_1 = 0.053$.**

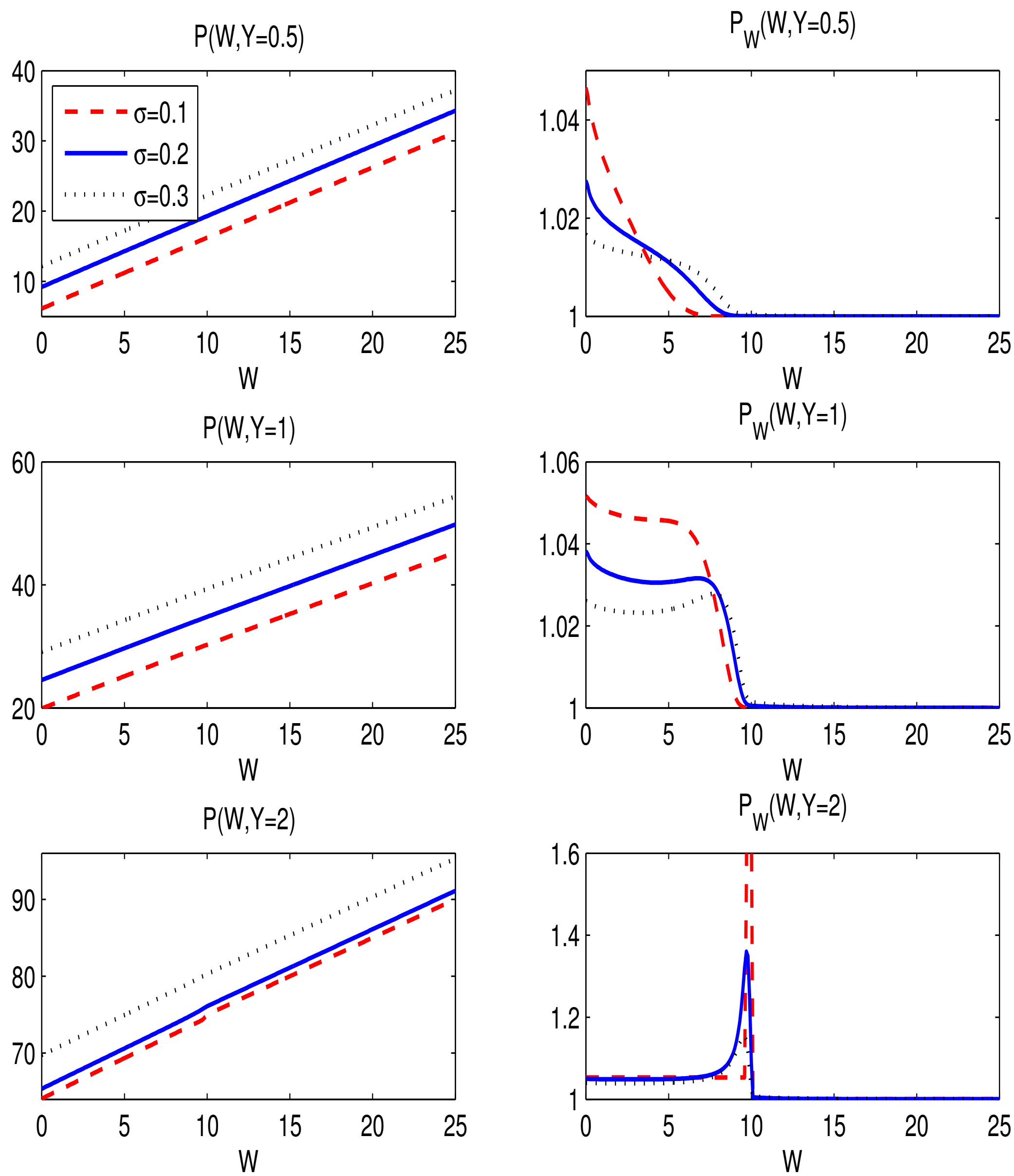

Figure 10: **Firm value before exercising the real option $P(W, Y)$ and the marginal value of liquid wealth for three different levels of volatility $\sigma = 0.1, 0.2, 0.3$.**

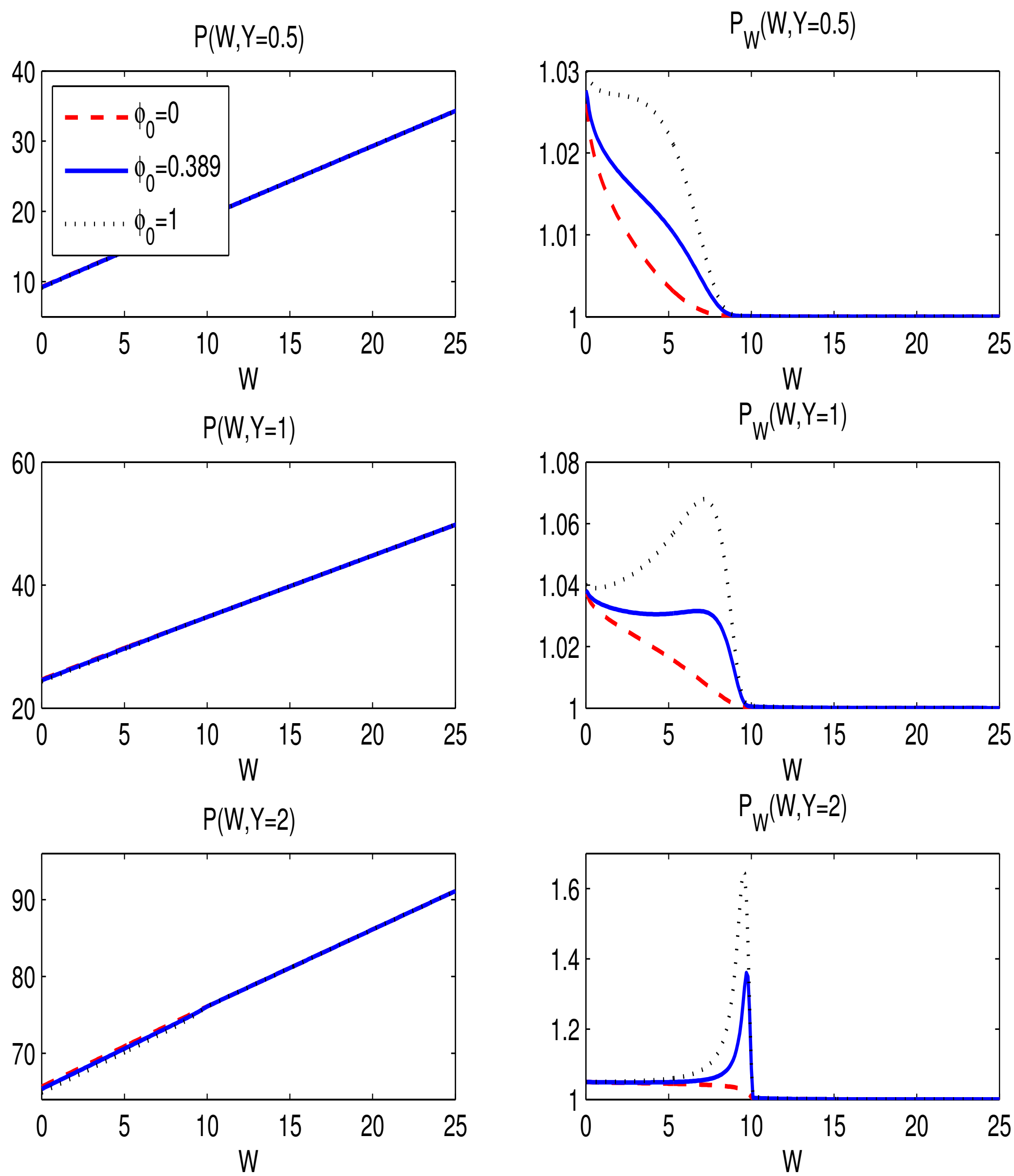

Figure 11: **Firm value before the growth option exercise, $P(W, Y)$, and the marginal value of liquidity P_W for three levels of fixed cost $\phi_0 = 0, 0.389, 1$.**

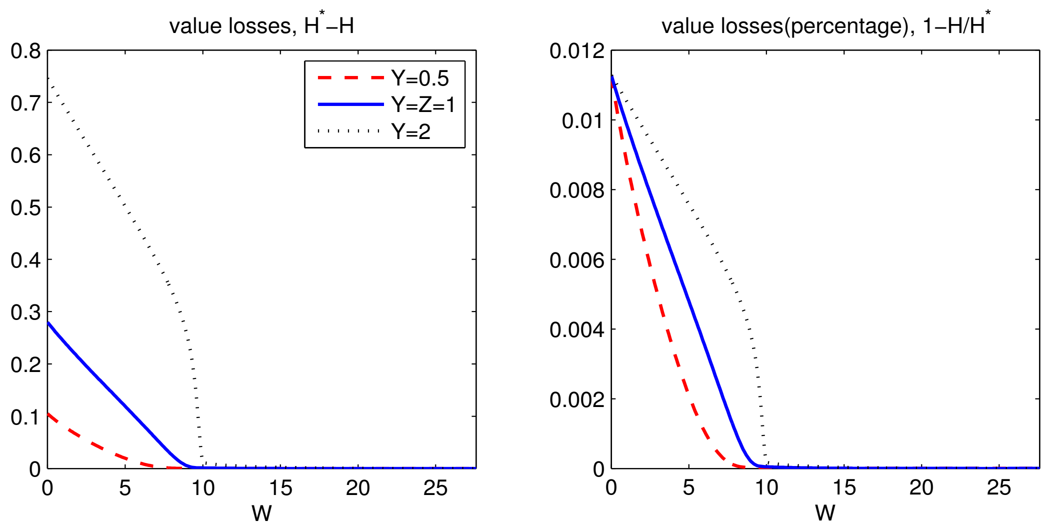

Figure 12: **The value losses (in level and percentage terms) due to financial constraints before the growth option exercise,** $H^*(Y) - H(W,Y)$ **and** $1 - H(W,Y)/H^*(Y)$ **for three levels of revenue** $Y = 0.5, 1, 2$**. The financing cost parameters are** $\phi_0 = 0.389$ **and** $\phi_1 = 0.053$.

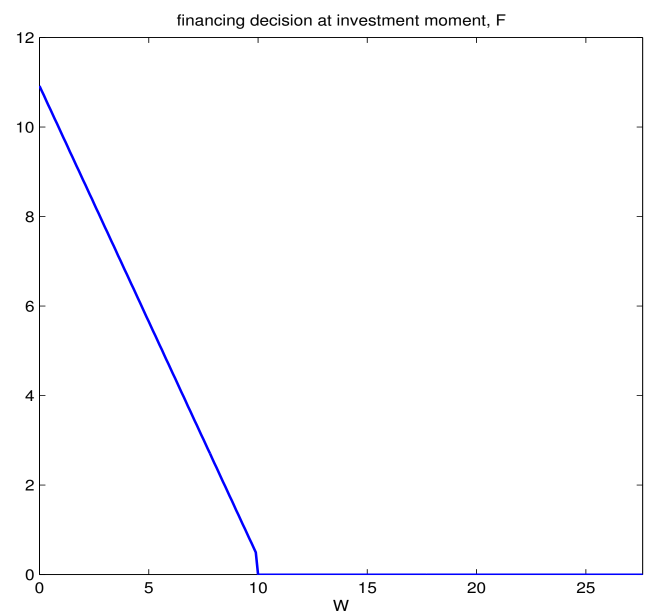

Figure 13: **The financing amount at the moment of investment. The parameter values are $\phi_0 = 0.389$ and $\phi_1 = 0.053$.**