Risk Premia in Gold Leasing Markets

Anh Le* Haoxiang Zhu †

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Abstract

Gold has been used as money throughout history and remains today as an important global reserve currency. Unlike interest rates of major fiat currencies, gold lease rates – interests paid in gold for borrowing gold – are rarely studied. Using one- to twelve-month gold lease rates, we find that the term structure of gold lease rates is upward sloping, implying positive risk premia on average. Moreover, risk premia on gold loans are highly time varying, leading to a strong rejection of the expectation hypothesis. By contrast, we cannot reject the expectation hypothesis using comparable U.S. dollar interest rate data. Our evidence reveals fundamental differences between gold lease rates and interest rates of fiat currencies.

*Kenan–Flagler Business School, University of North Carolina at Chapel Hill, anh_le@unc.edu
†Sloan School of Management, MIT, zhuh@mit.edu

1 Introduction

Gold has long served as a global reserve currency. According to World Gold Council (2013), gold accounts for about 13%, equivalent to $1.56 trillion, of the $12 trillion total reserves held by central banks, as of the second quarter of 2012. This amount makes gold the third largest reserve currency globally, only smaller than the US dollar and the Euro, but larger than the British Pound, Japanese Yen, and Swiss Franc. In three joint statements issued in 1999, 2004, and 2009, fifteen central banks led by the ECB affirmed that "gold will remain an important element of global monetary reserves."[1] Despite the important reserve-currency status of gold, and despite the vast volumes of research into the interest rate of fiat reserve currencies (e.g. the US dollar), surprisingly little is known about the "gold interest rates."

In this paper we investigate the dynamics of interest rates paid in gold for borrowing gold. In the gold market, these interest rates are called the gold lease rates. For example, an investor who lends 100 ounces of gold for 12 months at the lease rate of 2% will get back 102 ounces of gold at the maturity of the loan. The gold lease rates are, therefore, analogous to the interest rates of the US dollar. Just like the dollar interest rates capture investors' return for lending in dollar, the gold lease rates capture the returns—in term of gold—earned by central banks and investors for lending gold that is held as part of their reserves and asset allocation. Alternatively, the gold lease rates capture the opportunity costs for central banks and investors who hold gold passively without lending it. Embedded in the term structure of gold lease rates are premia required by investors for taking gold lending risks. As evident in Figure 1, gold lease rates display substantial volatility over time.[2] Characterizing the risk premia in gold lease rates, as well as comparing them to the risk premia in US dollars interest rates at comparable maturities, are the primary objective of our study.

Our theoretical benchmark for studying risk premia in gold lease rates is the expectation hypothesis (**EH**). The EH is one of the most established theories of interest rates. Applied to the gold lending market, the EH posits that excess returns on gold loans are not predictable, or equivalently, the risk premium in the gold leasing market is constant. Moreover, although EH has been rejected by many studies using long-term interest rates of major fiat currencies,[3] evidence regarding EH for short-term interest rates is far from conclusive. For example, Longstaff (2000) and Downing and Oliner (2007) find strong support for the EH in US repo market and US commercial paper market, whereas Buraschi and Menini (2002) show violations of EH for the repo "specialness" in Germany.[4] Given these conflicting findings on

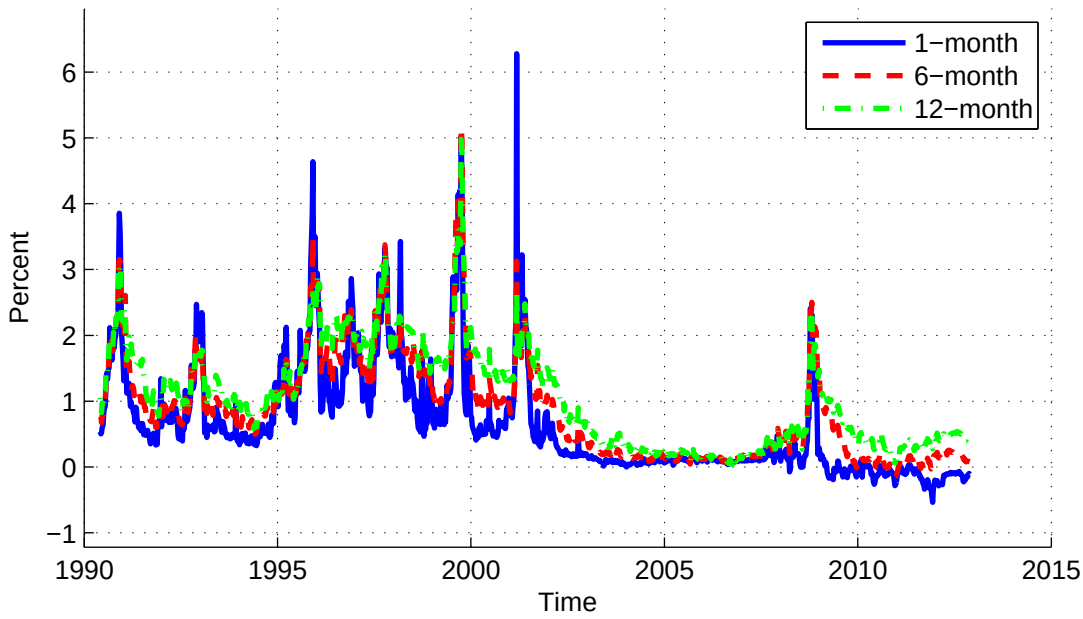
[1] See ECB press releases http://www.ecb.eu/press/pr/date/1999/html/pr990926.en.html, http://www.ecb.eu/press/pr/date/2004/html/pr040308.en.html, and http://www.ecb.eu/press/pr/date/2009/html/pr090807.en.html.

[2] During the recent financial crisis, short term gold lease rates occasionally become negative. In order to mitigate effects of possible abnormal market behaviors during this episodes, in our main analysis we start the data sample in June 1990 and end it in August 2007, right before the financial crisis.

[3] For U.S. evidences, see Dai and Singleton (2002) and references therein. For international evidences, see Jotikasthira, Le, and Lundblad (2013) and references therein.

[4] Using general-collateral US-dollar repo rates with maturities from one day to three months, Longstaff (2000) finds that "term [repo] rates are almost unbiased estimate of average overnight rates," therefore lending strong support for the EH. In markets for US commercial papers with maturities up to 270 days, Downing and

Figure 1: Time series of gold lease rates from June 1990 to November 2012.



short-term interest rates on the US dollar and the Euro, a natural question is whether the expectation hypothesis accurately describes the joint dynamics of gold lease rates at different maturities.

Using the available gold interest rates up to one year in maturity,[5] we find that the expectation hypothesis is decisively rejected in the gold leasing market. That is, risk premia in gold leasing market are strongly time-varying, even at short maturities. By sharp contrast, the expectation hypothesis cannot be rejected for US dollar LIBOR rates at the same maturities.[6]

We characterize the time-varying risk premia in gold leasing market through a series of tests. Our first step is the classical Campbell and Shiller (1991) regression. For an investor who lends gold for a longer term (e.g. 12 month) by borrowing gold for a shorter term (e.g. 1

Oliner (2007) also find supportive evidence for a "generalized expectation hypothesis"—which incorporates interest rate risk, default risk, and liquidity risk—after correcting for the year-end spike of the term premia. By contrast, in German government bond markets, Buraschi and Menini (2002) document violations of the EH in the term structure of the "specialness," defined as the difference between general-collateral repo rates and special-collateral repo rates.

[5]Since no central bank sets the target interest rate on gold loans, gold lease rates are determined in the marketplace and are published by the London Bullion Market Association for 1-, 2-, 3-, 6-, and 12-month maturities.

[6]LIBOR rates serve as a natural comparison with the gold lease rates because both rates are set in London with similar credit risk and overlapping maturities. Recently, regulators have uncovered attempts of LIBOR manipulation by a few LIBOR panel banks, starting in the third quarter of 2007. Because we stop the data sample in August 2007, LIBOR manipulation is unlikely to affect our results. We also verify that the survey-based LIBID (London Interbank Bid Rate) is very close to the eurodollar deposit rate.

month), the Campbell and Shiller regression tests whether the positive yield spread earned by the investor moves one-for-one with the mark-to-market loss on the long-term loan due to yield changes. If EH holds, regressing the mark-to-market loss on the yield spread should give a coefficient of one. In the gold leasing markets, we find that the regression coefficients are all significantly negative, ranging from about -0.4 for 8-week maturity to -1.7 for 52-week maturity, which is strong evidence for the rejection of the EH. The same regression applied to LIBOR data of the same maturities, however, gives coefficients not significantly different from 1. That is, the EH cannot be rejected in the LIBOR data.

Our second test is the Cochrane and Piazzesi (2005) regression, which uses all yields, not only the long-short spread, to predict excess returns in bond markets. We find that monthly excess returns on gold loans are strongly predictably by the gold lease rates, with adjusted R^2 between 17.5% and 24.8%. By contrast, the five LIBOR rates do not significantly predict excess returns on LIBOR loans, with adjusted R^2 between 2.6% and 5.1%.

Finally, we estimate a no-arbitrage dynamic term structure model of the gold lease rates. The zero-coupon gold lease rates are particularly convenient for the estimation of the model. We apply the framework of Joslin, Singleton, and Zhu (2011) (JSZ) and allow for filtering in estimation. Filtering provides a robust treatment of potential measurement errors in gold lease rates. The estimated market price of risk parameters are statistically and economically significant for the gold leasing markets, implying a time-varying risk premium and a rejection of the EH. The same term structure model applied to the LIBOR data yields insignificant market price of risk parameters and fails to reject the EH.

In addition to confirming our earlier regression-based analysis, the no-arbitrage models provide further insights regarding the risk-return trade-off in the gold lending market. We find that both level and slope risks are strongly priced. Nevertheless, the premia required by investors to take on these risks are only determined by the current steepness of the gold interest rate curves. As the curve steepens, risk premia increase. Over our sample period, the slope of the gold yield curve is highly time varying, mostly upward-sloping but occasionally inverted. The change of sign in the risk premia suggests that lending gold is usually viewed by the marginal investor as a risky investment, but under exceptional circumstances buying a gold bond can provide insurance-like benefits.

Neuberger (2001) suggests that 1999 Central Bank Agreement on Gold may have a large effect on risk premium in gold lease rates. As a robustness check, we separate our sample into two subsamples, June 1990–August 1999 and October 1999–August 2007, leaving out September 1999 when the first Central Bank Agreement on Gold was issued. In both subsamples, we reject the expectation hypothesis for gold lease rates but not for LIBOR.

The strong rejection of the expectation hypothesis in the gold leasing market, as well as the failure to reject it for the LIBOR data, reveals substantially different behaviors of risk premium in the two markets. We explore the mechanism behind this difference and find preliminary evidence that excess returns in the two markets load on distinct types of macro, liquidity, or volatility risks, among which the most notable are expected real GDP growth and expected inflation, as proxied by the Bluechip surveys. Controlling for gold lease rates and other market conditions, a 1% increase in the expected real GDP growth raises

the expected excess return on gold loans by 1.15% per year. By contrast, a 1% increase in expected inflation increases the LIBOR loan excess returns by 0.52% per year. However, the expected real GDP growth does not predict excess returns of LIBOR loans; nor does the expected inflation predict excess returns on gold loans.

The rest of the paper is organized as follows. We provide an overview of gold lease rates and the data we use in Section 2 and Section 3. In Section 4 we test a generalized version of the expectation hypothesis by the Campbell and Shiller (1991) and Cochrane and Piazzesi (2005) regressions. Section 5 estimates a term structure model of the gold lease rates. Section 6 explores the effects of macro, liquidity and volatility conditions for the excess returns. Section 7 concludes.

2 Gold Lease Rates

Gold lease rates are "gold interest rate" investors pay, in gold, for borrowing gold. The gold lease rates are defined as the difference between the US LIBOR and the "Gold Forward Offered Rate" (GOFO) at the same tenor. That is,

$$\text{Gold Lease Rate} = \text{LIBOR} - \text{GOFO}. \tag{1}$$

GOFOs are "rates at which dealers will lend gold on a swap basis against US dollars," according to the London Bullion Market Association (LBMA). In other words, GOFOs are the US dollar rates at which dealers are willing to pay in order to borrow US dollars, using physical gold as collateral.[7] At the maturity of the dollar loan collateralized by gold, the dollar borrower gets back his gold collateral and pays $X(1 + GOFO \times t)$, where X is the loan amount and t is the tenor of the loan. From the dollar borrower's perspective, the collateralized dollar loan can be decomposed into two trades: a spot sale of his gold at the price of X dollars and a forward repurchase of the same amount of gold at the price of $X(1 + GOFO \times t)$ dollars. Thus, GOFOs are also the gold forward rates measured in US dollars, just as the name implied.

GOFOs are determined daily by the LBMA for 1-, 2-, 3-, 6- and 12-month maturities. At 10.30 am London time, market makers in the London bullion markets enter their forward offered rates for all five maturities. A minimum of six contributors must enter rates in order for the means to be calculated. At 11.00 am, the mean is established for each maturity by discarding the highest and lowest quotations in each period and averaging the remaining rates. The current market making members of the LBMA include the Bank of Nova Scotia-ScotiaMocatta, Barclays Bank Plc, Deutsche Bank AG, HSBC Bank USA London Branch, Goldman Sachs, JP Morgan Chase Bank, Société Générale and UBS AG.[8]

[7] Although the LBMA definition may appear to imply that the gold borrower pays GOFO, it is actually the lender of gold, namely the borrower of US dollars, who pays the interest rate.

[8] See http://www.lbma.org.uk/pages/index.cfm?page_id=62.

Gold Lease Rates as Gold Interest Rates, Paid in Gold

We now show through a no-arbitrage argument that the gold lease rate, defined as the difference between the LIBOR and the GOFO, is indeed a "gold interest rate"—the interest rate that banks pay in physical gold for their unsecured gold borrowing. The gold lease rate is analogous to the US dollar LIBOR rate, which is the interest rate, in US dollar, that banks pay on their unsecured US dollar borrowing.

For simplicity, we work with a one-year horizon. Suppose that today's gold price is S dollars per ounce. By the definition of GOFO, the forward gold price in one year is $F = Se^{GOFO}$ dollars per ounce. Suppose that we own one ounce of gold today and implement the following trading strategy: We sell one ounce of gold and deposit the proceeds S dollars at the one-year dollar LIBOR rate; simultaneously, we enter a forward contract to buy back $e^{LIBOR}S/F = e^{LIBOR}S/(e^{GOFO}S) = e^{LIBOR-GOFO}$ ounces of gold in one year at the forward price of F dollars per ounce. Barring default risk and market frictions, this strategy is risk-free, self-financing and gives a one-year return of $LIBOR - GOFO$, in physical gold, on our initial investment on one ounce of gold. Therefore, the gold lease rate, $LIBOR - GOFO$, can be viewed as the "gold interest rate." A similar trading strategy can be implemented at other horizons and derive the term-structure of gold lease rate.

An alternative view of the gold lease rates is that they present the collateral value of gold. Because GOFO is the secured borrowing rate and LIBOR is the unsecured borrowing rate, the wedge should capture the value of gold as collateral.

3 Data

Our data sample starts in June 1990 and ends in August 2007. GOFO and LIBOR data are obtained from LBMA. We construct the gold lease rate from the GOFO and LIBOR data according to (1). All the spot rates are converted to a continuously-compounding basis. While LBMA's GOFO data start in July 1989, only since May 1990 do the GOFO data include all five maturities (i.e., 1-, 2-, 3-, 6- and 12-month). We also exclude the financial crisis in our analysis.

To mitigate the effect of potential outliers in the data, we winsorize the data as follows. We first construct the five demeaned principle components (PCs) of the gold lease rates at daily frequency. By construction, these five PCs are linear combinations of the five gold lease rates at different maturities. Next, we winsorize the PCs at the 1% level by their absolute values. Finally, we back out the implied gold lease rates from the winsorized PCs using the loading matrices in the original PC calculations. The same procedure is used on the LIBOR data. Our analysis is conducted using these winsorized data at either the weekly frequency (for our main analysis in Section 4 and Section 5) or the monthly frequency (in Section 6). Winsorizing the data weakens the return predictability.

4 Expectation Hypothesis

In this section, we ask whether the expectation hypothesis (EH) can provide an adequate characterization of the gold lease rates. We employ two most commonly used tests: (i) the Campbell and Shiller (1991) (CS) regression; and (ii) the Cochrane and Piazzesi (2005) (CP) regression.

4.1 The CS Regression

A generic property of yields data is that risk-premium adjusted expected changes in bond yields are proportional to the slope of the yield curve in the precise sense that (Dai and Singleton (2002)):

$$E\left[y_{t+1}^{n-1} - y_t^n - (c_{t+1}^{n-1} - c_t^{n-1}) + \frac{p_t^{n-1}}{n-1}|y_t\right] = \frac{y_t^n - r_t}{n-1}, \tag{2}$$

where y_t^n denotes the n-period continuously compounding yields at time t and r_t denotes the short rate (y_t^1) and

$$c_t^n \equiv y_t^n - \frac{1}{n}\sum_{i=0}^{n-1} E[r_{t+1}|y_t] \text{ and } p_t^n \equiv f_t^n - E[r_{t+n}|y_t] \tag{3}$$

are the yield and forward term premiums, respectively, and f_t^n denotes the forward rate for one-period loans commencing at date $t+n$. Under the EH, both the yield and forward premiums c_t^n and p_t^n are constant, for all n. Whence the coefficients $\phi_n = 1$ in the projections

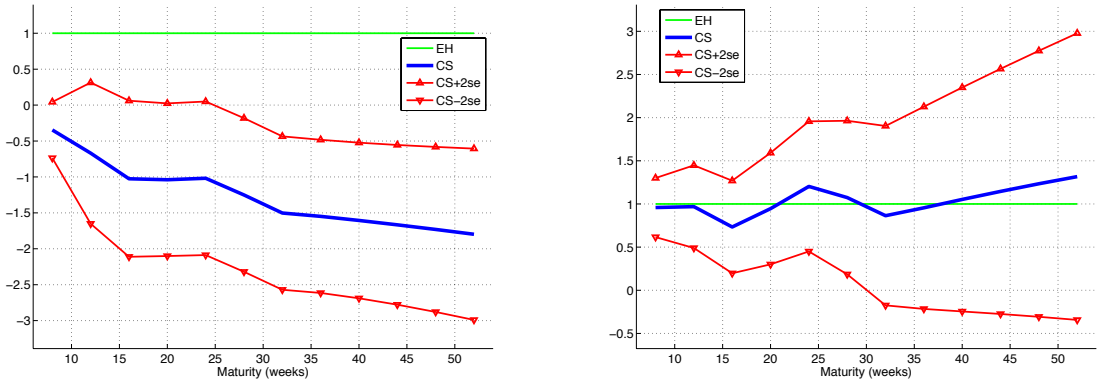
$$Proj\left[y_{t+1}^{n-1} - y_t^n | y_t^n - r_t\right] = \alpha_n + \phi_n\left(\frac{y_t^n - r_t}{n-1}\right), \text{ for all } n > 1, \tag{4}$$

where $Proj[\cdot|\cdot]$ denotes linear least-squares projection.

Using risk–free interest rate data, CS show that ϕ_n's are significantly different from ones and become increasingly negative for larger maturities n's. This pattern has been commonly considered among the most robust evidences against the EH. In applying this regression to the gold lease rates, we note that the yield and forward term premiums can contain compensations for default and/or liquidity risks. In this sense, the null hypothesis that we are testing is one in which premiums for all risks are constant. Following Downing and Oliner (2007), we refer to this as the generalized expectation hypothesis (GEH).

Figure 2 (a) shows the CS coefficients for the gold lease rates together with their two-standard-error bounds. The maturity n's are in increments of four weeks (so the short rate r_t corresponds to a maturity of four weeks) from 4- to 52-weeks, where the data are sampled at the weekly frequency. The standard errors are computed using the Newey and West (1987) covariance matrix with six lags. Interestingly, the CS regression loadings are uniformly below one even after taking into account statistical uncertainty. The downward pattern of the loadings as maturity increases curiously resembles what is commonly observed with risk free

Figure 2: CS Regression Coefficients.



(a) Gold Lease Rates

(b) LIBOR

interest rates of much longer maturities. More strikingly, this pattern is in stark contrast to Figure 2 (b), in which we plot the CS coefficients for the LIBOR. Along the maturity dimension, the CS coefficient estimates for the LIBOR very closely trace the unit line, which in turn is generously within the two-standard-error bounds.

In the context of the fed funds markets, results of Lange, Sack, and Whitesell (2003) suggest that patterns similar to Figure 2 (a) might be attributable to less transparent monetary policy in the early nineties during which the Federal Reserve bank did not publicly announce policy changes and the rationale for their actions following the conclusion of the FOMC meetings. In fact, Lange, Sack, and Whitesell (2003) find that after 1994, the fed funds rate became much more predictable and the EH performs much better. To check whether shifts in monetary policy over our sample, or more generally any structural break of similar nature, can explain the deviation from the EH observed for the gold lease rates, we split the our sample into sub-samples: before and after 1999.[9] The downward pattern of CS loadings found in Figure 2 (a) remains essentially intact for both sub-samples, suggesting that this pattern is unlikely sample specific.

In the context of the commercial paper (CP) markets, Downing and Oliner (2007) also found strong evidence rejecting the GEH. Interestingly, they found that the bulk of the evidence clusters around the month of December, a time frame during which, for window dressing reasons, CPs with relatively longer maturities are typically less preferred compared to cash and treasury bills. This gives rise to a surge in the premiums required to hold relatively long dated CPs around year-ends. Once this seasonal effect is controlled for, Downing and Oliner (2007) show that the GEH actually holds in the data. To map this into our setting, if for similar institutional reasons, banks may find it relatively more difficult to obtain gold

[9] We also shorten our sample by starting from 1994 and our results remain robust.

financing (gold becomes harder to borrow) over relatively longer terms around year-ends, the long term gold lease rates would similarly contain a high and predictable premium components around the same time. To check for this possibility, we drop all observations in the months of December and November from out sample, but Figure 2 (a) still remains essentially unchanged. This suggests that the deviations from the GEH observed with the gold lease rates cannot be explained by the year end effects.

Turning to the CS regression with LIBOR, note that the projection in (4) can be equivalently written as:

$$Proj[rx^n_{t+1}|y^n_t - r_t] = \text{constant} + (1 - \phi_n)(y^n_t - r_t) \tag{5}$$

where rx^n_{t+1} denotes the one-period excess return on an n-period bond. Viewed this way, the CS regression is equivalent to a test of whether the slope of the yield curve, $y^n_t - r_t$, possesses any predictive content for future excess returns. Accordingly, that ϕ_n's are indistinguishable from the unit line in Figure 2 (b) is not definite evidence that LIBOR risk premiums are constant. It is still possible that LIBOR risk premiums are time varying in directions that are essentially orthogonal to the slopes of the LIBOR yield curves. Using a richer set of conditioning information in addition to slopes may result in a more powerful test of the GEH. We consider one of these tests in the next subsection.

4.2 The CP Regression

Using risk-free interest rate data, CP run a predictive regression similar to (5) using a much richer, but still yield curve based, information set, being the five annual forwards rates with maturities ranging between one to five year. They indeed find that this richer information set leads to much stronger predictability for future bonds' excess returns. In this spirit, we implement the following regression:

$$E_t[rx^n_{t+4}(4)|\mathcal{L}_t] = \text{constant} + \beta\,\mathcal{L}_t \tag{6}$$

where $\mathcal{L}_t$ includes five zero yields, constructed from either the gold lease rates or LIBOR, with maturities: 4-, 8-, 12-, 26-, and 52-week; $rx^n_{t+4}(4)$ denotes the 4-week excess return on an n-period bond.

Panels A and B of Table 1 report the results of this regression for the gold lease rates and LIBOR, respectively. Even with a stronger conditioning set, the contrast between the predictability of gold bonds and LIBOR excess returns remains similarly striking. The adjusted R^2's across different maturities in Panel A fluctuate around 20%, compared to just more than 3% for Panel B. Additionally, the χ^2 tests of joint significance of the β coefficients are very strongly rejected for all regressions in Panel A. In contrast, the probability values (pval) for this test with the LIBOR are either very high or only borderline significant. As an example, taking the first row of each panel where we use the cross-maturity average excess returns as the dependent variable. Four out of five loadings show up significant with the χ^2 pval of essentially zero in Panel A. On the other hand, with a χ^2 test pval of 0.127, we cannot reject the hypothesis of no predictability for the corresponding regression in Panel B.

Table 1: Regression of 4-week excess returns onto yields with 4-, 8-, 12-, 26- and 52-week to maturity. $\overline{rx}$ denotes average excess returns across all maturities. Standard errors (bracketed) are computed using Newey West matrices with six lags. "χ^2 pval" refers to the probability value of the test of the joint significance of all the coefficient estimates. *, **, *** refer to the significance levels of 10%, 5%, and 1% respectively.

	4-week	8-week	12-week	26-week	52-week	χ^2 pval	Adj. R^2
Panel A: Gold Lease Rates							
$\overline{rx}$	-0.173^{***}	-0.486^{***}	0.795^{***}	-0.237^{*}	0.124	0.000	0.220
	(0.059)	(0.176)	(0.293)	(0.136)	(0.084)		
8-week	-0.091^{***}	0.038	0.098	-0.047	0.012	0.000	0.197
	(0.015)	(0.043)	(0.070)	(0.041)	(0.024)		
24-week	-0.142^{***}	-0.389^{***}	0.585^{**}	-0.041	0.010	0.001	0.175
	(0.049)	(0.142)	(0.235)	(0.120)	(0.072)		
40-week	-0.220^{***}	-0.686^{***}	1.096^{***}	-0.363^{**}	0.202^{*}	0.000	0.236
	(0.079)	(0.238)	(0.396)	(0.180)	(0.112)		
52-week	-0.273^{***}	-0.888^{***}	1.430^{***}	-0.526^{**}	0.286^{**}	0.000	0.248
	(0.102)	(0.316)	(0.527)	(0.219)	(0.136)		
Panel B: LIBOR							
$\overline{rx}$	0.084	0.128	-0.156	-0.171	0.124^{**}	0.127	0.036
	(0.066)	(0.191)	(0.220)	(0.117)	(0.059)		
8-week	-0.019	0.034	0.011	-0.045^{***}	0.020^{***}	0.045	0.051
	(0.013)	(0.021)	(0.020)	(0.014)	(0.007)		
24-week	0.047	0.107	-0.155	-0.025	0.032	0.597	0.026
	(0.048)	(0.139)	(0.171)	(0.103)	(0.047)		
40-week	0.132	0.188	-0.242	-0.256	0.191^{**}	0.084	0.039
	(0.095)	(0.279)	(0.316)	(0.166)	(0.085)		
52-week	0.204	0.219	-0.305	-0.336	0.238^{**}	0.122	0.039
	(0.138)	(0.363)	(0.418)	(0.230)	(0.119)		

5 No-arbitrage Term Structure Model of Gold Lease Rates

In this section, we lay down a standard discrete-time no-arbitrage term structure model of the gold lease rates. Our construction closely follows the canonical setup of Joslin, Singleton, and Zhu (2011) (JSZ), in which all normalizations for obtaining econometric identifications are implemented under the risk neutral measures. We then implement the full-information maximum-likelihood estimation and explore various implications of the model.

5.1 Modeling Setup

We let X_t be an N-element vector of state variables. The core of the risk-neutral setup consists of the risk neutral dynamics of states:

$$X_{t+1} = \lambda^{\mathbb{Q}} X_t + \sqrt{\Sigma_X}\epsilon^{\mathbb{Q}}_{t+1}, \tag{7}$$

with standard gaussian noise $\epsilon^{\mathbb{Q}}_{t+1}$ and the short rate specification:

$$r_t = r^{\mathbb{Q}}_{\infty} + \iota' X_t, \tag{8}$$

where r_t is the amount of gold (expressed as a percentage) earned on a gold loan over one unit of time interval. $\lambda^{\mathbb{Q}}$ is a diagonal matrix and, for simplicity, real-valued, and ι denotes a column vector of ones. It is important to stress that these normalizations, obtained through standard rotations of the state variables, are only for identification purposes.[10] Under the risk neutral measures, investors are indifferent between making a long term loan and rolling short term loans successively. This indifference allows us to derive the n-period gold lease rates $y_{n,t}$ for all n's as affine functions of the state variables:[11]

$$y_{n,t} = A_{n,X}(\lambda^{\mathbb{Q}}, r^{\mathbb{Q}}_{\infty}, \Sigma_X) + B_{n,X}(\lambda^{\mathbb{Q}})X_t, \tag{9}$$

where the loadings A_n, B_n can be computed as:

$$B_{n,X} = \frac{1}{n} diag((I - \lambda^{\mathbb{Q}})^{-1}(I - (\lambda^{\mathbb{Q}})^n))', \tag{10}$$

$$A_{n,X} = r^{\mathbb{Q}}_{\infty} - \frac{1}{2n} tr\left(\Sigma_X \sum_{i=0}^{n-1} i^2 B_{i,X} B'_{i,X}\right). \tag{11}$$

We let y_t be the $J \times 1$ vector of yields used in estimation. For each fixed $N \times J$ matrix W, we can write:

$$\mathcal{P}_t = W\, y_t = W\, A_X + W\, B_X X_t, \tag{12}$$

[10] The assumption that $\lambda^{\mathbb{Q}}$ is real-valued is slightly over-identifying since it rules out the possibility that $\lambda^{\mathbb{Q}}$ may be complex. Nevertheless, as is shown in JSZ, the real-valued case is often empirically adequate.

[11] To be precise, the n-period gold lease rate $y_{n,t}$ is per unit of time interval. That is, the gold price of a gold bond with n periods until maturity is given by $e^{-ny_{n,t}}$.

where A_X is obtained by stacking $A_{n,X}$ on top of each other for all n's used in the estimation. B_X is similarly obtained by stacking column vectors $B_{n,X}$ together. Equation (12) allows us to re-write the term structure model using $\mathcal{P}_t$ as the pricing factor:

$$y_t = A_{\mathcal{P}} + B_{\mathcal{P}}\mathcal{P}_t, \tag{13}$$

where

$$A_{\mathcal{P}} = A_X - B_{\mathcal{P}}(WA_X) \text{ and } B_{\mathcal{P}} = B_X(WB_X)^{-1}.$$

The risk-neutral dynamics of $\mathcal{P}_t$ is given by:

$$\mathcal{P}_{t+1} = \kappa_0^{\mathbb{Q}} + \kappa_1^{\mathbb{Q}}\mathcal{P}_t + \sqrt{\Sigma_{\mathcal{P}}}\epsilon_{t+1}^{\mathbb{Q}}, \tag{14}$$

where

$$\Sigma_P = (WB_X)\Sigma_X(WB_X)', \quad \kappa_1^{\mathbb{Q}} = (WB_X)\lambda^{\mathbb{Q}}(WB_X)^{-1} \text{ and } \kappa_0^{\mathbb{Q}} = (I - \kappa_1^{\mathbb{Q}})WA_X.$$

To complete the model, we specify the physical dynamics of $\mathcal{P}_t$ as:

$$\mathcal{P}_{t+1} = \kappa_0^{\mathbb{P}} + \kappa_1^{\mathbb{P}}\mathcal{P}_t + \sqrt{\Sigma_{\mathcal{P}}}\epsilon_{t+1}^{\mathbb{P}}, \tag{15}$$

where $\epsilon_{t+1}^{\mathbb{P}}$ follows a standard normal distribution, and $\kappa_0^{\mathbb{P}}$ and $\kappa_1^{\mathbb{P}}$ are both free parameters. As in JSZ, for each loading matrix W, the model is fully characterized by the parameter set $\Theta = \{\kappa_0^{\mathbb{P}}, \kappa_1^{\mathbb{P}}, r_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \Sigma_{\mathcal{P}}\}$.

5.2 Estimation Strategies

In fitting our model to the data, we employ two estimation strategies. The first approach is in the spirit of Chen and Scott (1993) in that exactly N combinations of yields are assumed to be observed perfectly. This allows for the state factors to be conveniently inverted from the observed yields portfolios. In the second approach, we relax this assumption and allow all yields to be observed with errors and estimate the model through Kalman filtering.

The Chen and Scott (1993) approach

For the first strategy, we assume that the pricing portfolios, $\mathcal{P}_t$, are observed perfectly, whereas the remaining $J - N$ yields portfolios, $\mathcal{P}_t^e = W^e y_t$, are observed with i.i.d. errors. As is standard, W^e is rotated so that each of its rows, scaled to have unit L^2 norm, is orthogonal to each other and to each row of W. Following practices in the literature (see, for example, Joslin, Le, and Singleton 2012), we assume that the observation errors are uncorrelated and share one common variance. Specifically,

$$\mathcal{P}_t^{e,o} = \mathcal{P}_t^e + e_t \text{ with } e_t \sim N(0, \sigma_e^2 I_{J-N}), \tag{16}$$

where the superscript ^{o} differentiates the observed series from its theoretical counterpart.

The log likelihood of observing the entire time series of $(\mathcal{P}_t', \mathcal{P}_t^{e,o\prime})'$ is:

$$\mathcal{L} = \sum_t f(\mathcal{P}_{t+1}, \mathcal{P}_{t+1}^{e,o} | \mathcal{I}_t), \tag{17}$$

where f denotes the log conditional density and $\mathcal{I}_t$ denotes all information up to time t. Due to the Markovian property of the states, $\mathcal{I}_t$ is equivalently informative to $\mathcal{P}_t$ (which is perfectly observed). Applying the product rule to (17) and using the i.i.d. property of the error e_t, we can decompose the right hand side of (17) into two components:

$$\sum_t f(\mathcal{P}_{t+1}, \mathcal{P}_{t+1}^{e,o} | \mathcal{I}_t) = \sum_t f(\mathcal{P}_{t+1} | \mathcal{P}_t) + \sum_t f(\mathcal{P}_{t+1}^{e,o} | \mathcal{P}_{t+1}). \tag{18}$$

The first component captures the time series dynamics of pricing portfolios, $\mathcal{P}_t$, while the second component captures the cross-sectional fit. Due to the Gaussian assumption, both components can be written down analytically:

$$\sum_t f(\mathcal{P}_{t+1} | \mathcal{P}_t) = -\frac{T}{2} log((2\pi)^N |\Sigma_{\mathcal{P}}|) - \frac{1}{2} \sum_t ||\Sigma_P^{-1/2} (\mathcal{P}_{t+1} - (\kappa_0^{\mathbb{P}} + \kappa_1^{\mathbb{P}} \mathcal{P}_t)||_2, \tag{19}$$

$$\sum_t f(\mathcal{P}_{t+1}^{e,o} | \mathcal{P}_{t+1}) = -\frac{T}{2} log((2\pi)^{J-N} \sigma_e^{2(J-N)}) - \frac{1}{2\sigma_e^2} \sum_t ||\mathcal{P}_{t+1}^{e,o} - W^e (A_{\mathcal{P}} + B_{\mathcal{P}} \mathcal{P}_{t+1})||_2, \tag{20}$$

where T refers to the sample length and $||.||_2$ denotes the L^2 norm.

Maximum likelihood estimation requires the optimization of $\mathcal{L}$ over the parameter set Θ and the error variance parameter σ_e. As shown in JSZ, the global estimates of $\kappa_0^{\mathbb{P}}$ and $\kappa_1^{\mathbb{P}}$ are exactly the same as those obtained by fitting a VAR(1) process to $\mathcal{P}_t$. Additionally, in the Appendix, we show how we could concentrate out $r_\infty^{\mathbb{Q}}$, σ_e analytically and $\Sigma_{\mathcal{P}}$ quasi-analytically. As such, the optimization is effectively only over the N diagonal elements of $\lambda^{\mathbb{Q}}$. Moreover, using the regression approach of Joslin and Le (2011), we obtain excellent starting guesses for $\lambda^{\mathbb{Q}}$. The end result is an instantaneous and very robust convergence for our estimation.

Estimation with filtering

For the second strategy, we relax the assumption that $\mathcal{P}_t$ is perfectly observed and assume that all yields are observed with i.i.d. errors whose joint distribution is non-degenerate. To set up the filtering problem, we use (15) as the state equation. For the observation equations, we again assume that the observational errors across maturities are uncorrelated and share one common variance:

$$y_t^o = A_{\mathcal{P}} + B_{\mathcal{P}} \mathcal{P}_t + e_t \text{ and } e_t \sim N(0, \sigma_e^2 I_J), \tag{21}$$

with $A_{\mathcal{P}}$ and $B_{\mathcal{P}}$ given in equation (13). Equipped with the state and observation equations, the likelihood of observing the entire panel of yields can be analytically computed through

standard recursive procedures. Maximum likelihood estimation similarly requires the optimization of this likelihood over the same parameter set Θ and σ_e. Although the numerical optimization will be over the full parameter vector (since analytical concentration is no longer feasible), estimates from the first estimation approach can be used as the starting guesses. For our choices of the loading matrix W, these guesses are again excellent and deliver efficient and robust estimation.

5.3 Implementation

In implementing the estimation, we use the gold lease rates, for all available maturities (1-, 2-, 3-, 6-, and 12-month) from June 1990 to August 2007, constructed as described in Section 3. Notably, these rates correspond to straight gold loans without coupons; hence no bootstrapping is required. We sample our data at the weekly frequency – each data point is closest to the end of week – so the time interval for our model is $\Delta t = 1/52$. The weekly sampling of our data results in a time series with 900 data points.

Applying the principal component (PC) analysis to our gold lease rates data reveals that the first two PCs explain 99.7% of all variations, with the first PC explaining 96.7%. Given this evidence, we set the number of pricing factors to $N = 2$. Additionally, we choose the loading matrix W of the pricing portfolios to correspond to the those of the first two PCs.[12]

For a comparable benchmark, we also fit our model to the LIBOR dataset over the same sample, at the weekly frequency, for an identical maturities set, and with the same number of pricing factors ($N = 2$). We also choose the loading matrix W to correspond to the first two PCs of LIBORs, although the actual numerical values of W will be slightly different between LIBORs and the gold lease rates.

5.4 Parameter Estimates

Table 2 reports the estimates for $\kappa_1^{\mathbb{P}}$, $\kappa_1^{\mathbb{Q}}$, and the Cholesky decomposition of the covariance matrix $\Sigma_{\mathcal{P}}$ for the gold lease rates (Panel A) and the LIBOR (Panel B). For each of the datasets, we report the estimates (Ests) together with corresponding robust standard errors (s.e.) using both the Chen and Scott (1993) approach (WOE) and the filtering approach (FY).[13]

[12] As far as the first estimation approach is concerned, Joslin, Le, and Singleton (2012) show that this choice of the loading matrix is the most innocuous for it is likely to deliver estimates that are closest to those obtained by Kalman filtering. An important caveat here is Joslin, Le, and Singleton (2012)'s results are based on the interest rates data in the U.S., sampled at the monthly frequency. They also discuss conditions under which the filtering and the Chen and Scott (1993) approaches may be nearly equivalent. At the heart of these conditions is the relative difference between the magnitudes of time-series and cross-sectional errors, which can be quite different for distinct datasets and/or data frequencies.

[13] To compute these robust standard errors, we first express the first order conditions with respect to the parameter vector as relevant moment conditions in the Hansen (1982)'s GMM framework. We then use six lags in computing the Newey and West (1987) covariance matrix, which is prewhitened and recolored in accordance with Andrews and Monahan (1992) to treat possible autocorrelation in the residual errors.

Table 2: Parameter Estimates

		$\kappa_1^{\mathbb{P}}$		$\kappa_1^{\mathbb{Q}}$		$\Sigma_{\mathcal{P}}$	
Panel A: Gold Lease Rate							
WOE	Ests	0.978***	0.019	0.995***	0.060***	0.160***	
		0.027**	0.903***	-0.001	0.955***	−0.114***	0.137***
	s.e.	0.011	0.017	0.002	0.003	0.018	
		0.011	0.018	0.004	0.008	0.019	0.011
FY	Ests	0.979***	0.015	0.995***	0.058***	0.154***	
		0.026***	0.934***	-0.001	0.959***	−0.108***	0.097***
	s.e.	0.011	0.018	0.002	0.003	0.017	
		0.010	0.015	0.004	0.008	0.020	0.011
Panel B: LIBOR							
WOE	Ests	0.997***	0.059***	0.999***	0.051***	0.082***	
		-0.003	0.964***	−0.003***	0.970***	0.007	0.106***
	s.e.	0.002	0.011	0.000	0.002	0.006	
		0.002	0.011	0.001	0.006	0.011	0.008
FY	Ests	0.997***	0.059***	0.999***	0.051***	0.077***	
		-0.003	0.979***	−0.003***	0.972***	0.008	0.082***
	s.e.	0.002	0.011	0.000	0.002	0.006	
		0.002	0.009	0.001	0.006	0.009	0.006

For brevity, we have omitted the estimates of the intercept parameters ($\kappa_0^{\mathbb{P}}$, $r_\infty^{\mathbb{Q}}$) – these are available upon request. The pricing errors implied by the models are economically small with σ_e estimated at 7 basis points (5 basis points) for the gold lease rates data (the LIBOR data). These estimates are smaller than or comparable to estimates obtained in prior studies of the term structure of interest rates in the U.S. and other countries.

Two interesting observations emerge from the reported estimates. First, a quick comparison between the diagonal values of $\kappa_1^{\mathbb{Q}}$ and of $\kappa_1^{\mathbb{P}}$ suggests that the degree of persistence under $\mathbb{Q}$ is relatively higher than that under $\mathbb{P}$ for the gold interest rates, whereas the persistence in the LIBOR data are similar under the $\mathbb{Q}$ and $\mathbb{P}$ measures. Explicit eigenvalue calculations of these matrices confirm this is indeed the case, for both the WOE and FY estimations. Since risk premia in no-arbitrage models are intimately connected to the difference between the physical and risk-neutral drifts, the difference in persistences under the two measures is informative of time-varying risk premia embedded in the gold lease rates – an issue we take on in depth in the next subsection. Second, while the estimates under WOE and FY are similar, they are not identical, perhaps with the exception of $\kappa_1^{\mathbb{Q}}$. The differences in estimates capture the differences between the observed yields portfolios $\mathcal{P}_t^o$ and their filtered counterpart. As we discuss shortly, the differences across WOE and FY estimates become

more pronounced when these parameter estimates are transformed into those that govern the model's market prices of risks (reported in Table 3).

Figure 3 compares the observed series and its filtered counterpart of PC1 (subfigure (a)) and PC2 (subfigure (b)) of the gold lease rates. Similar to Joslin, Le, and Singleton (2012), we find that filtering is inconsequential for the level factor PC1 – the two graphs of subfigure (a) are on top of one another. Turning to the slope factor PC2, although the observed and filtered series track each other relatively well, filtering does have some effect in smoothing out some extreme movements of the slope. This suggests that observed PC2 may have nontrivial, albeit small, measurement errors.

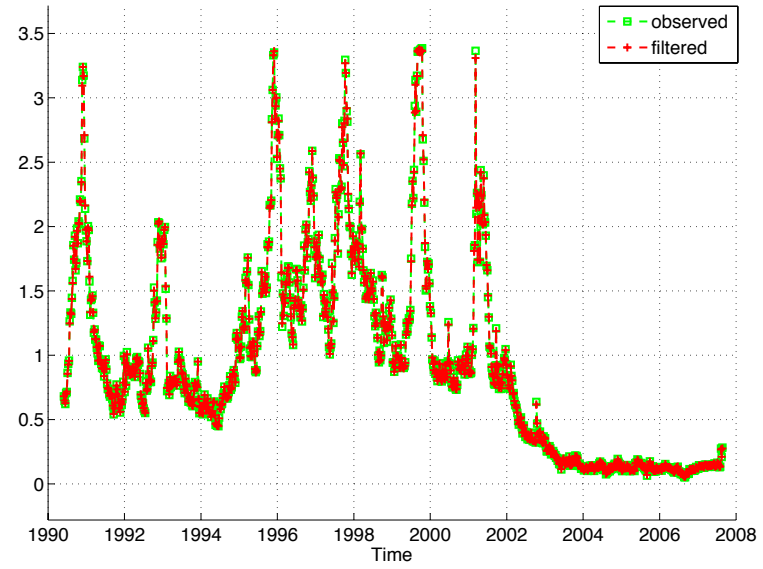
Using the FY estimates, we plot in Figure 4 the expected yield curve (subfigure (a)) and the term structure of rates volatility (subfigure (b)) implied by the no-arbitrage model against their sample counterparts for all maturities in one-week increment from one week to one year. Maturities not in our data (which contain only one-, two-, three-, six- and twelve-month maturities) are filled by a standard bootstrapping technique that assumes constant forward rates between available maturities (see, for example, Fama and Bliss (1987) for details).

The no-arbitrage model does a very good job in replicating both the upward pattern of the (zero-coupon) yield curve as well as the downward slope in rates volatility. Expected yield implied by the no-arbitrage model deviates from the corresponding sample averages by no more than six basis points across all maturities. The upward-sloping pattern of the yield curve depends on the sign of the average risk premia implied by the model. We examine the model-implied dynamics of risk premia in detail in the next subsection.

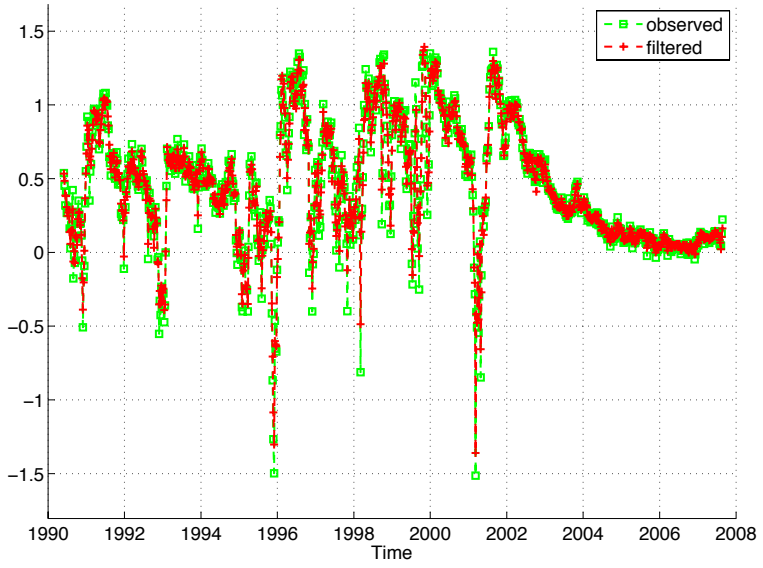
The model-implied volatilities curve misses their empirical counterparts by no more than two basis points. The downward-sloping pattern of volatility is partly due to the stationarity of the states under the pricing ($\mathbb{Q}$) measures. If the eigenvalues of the risk-neutral feedback matrix $\kappa_1^{\mathbb{Q}}$ were outside of the unit circle, the yield loadings would be exploding as maturities increase, giving rise to an upward term structure of volatility. More subtly, the flexibility of the risk neutral setup, as opposed to the physical dynamics, allows our model to match the speed at which long-maturity yields converge to their limits (as maturities goes infinity), which in turn determines how fast or slowly volatility decreases in maturity.[14]

[14] Backus and Zin (1993) provides an example in which a no-arbitrage model, with constant market prices of risks, can grossly miss the volatility curve implied by the data. In their example, due to the constancy of the market prices of risks that shackles the physical and risk-neutral persistences together, their no-arbitrage model lacks the flexibility afforded by our setup.

Figure 3: Observed and filtered time series of PC1 and PC2 of the gold lease rates.

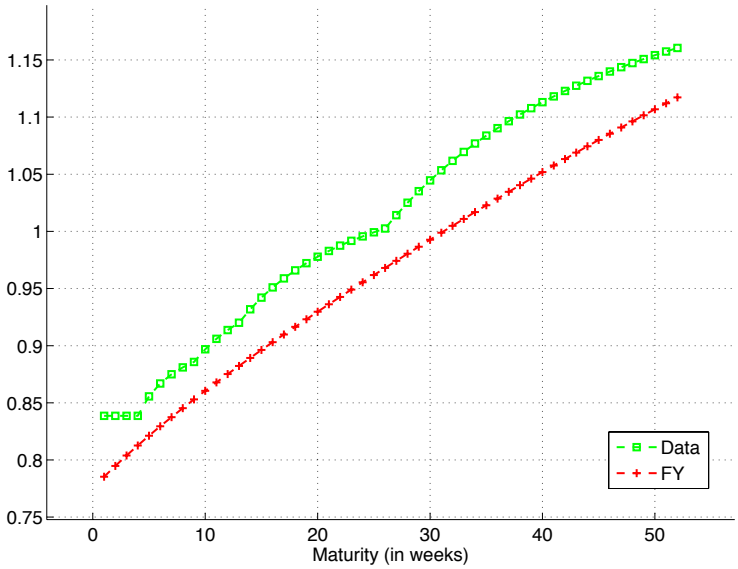


(a) PC1

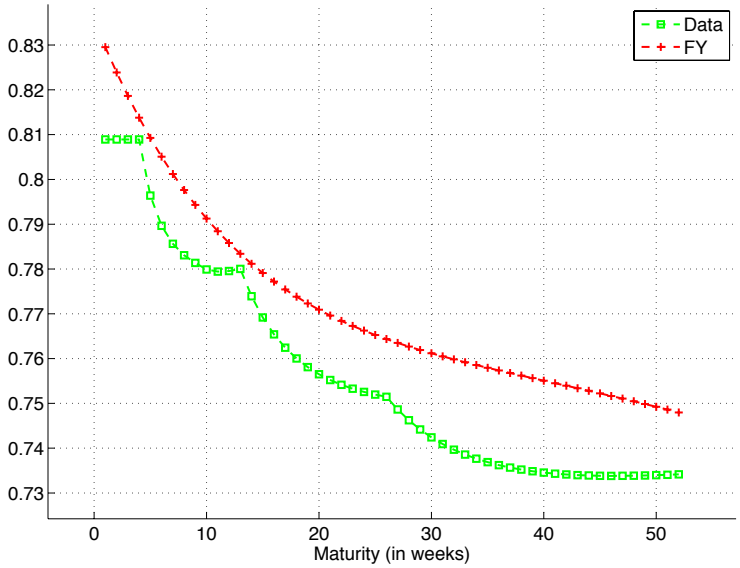


(b) PC2

Figure 4: Expected yields and term structure of yields volatility implied by the no-arbitrage model of gold lease rates. Estimation is done by Kalman filtering.



(a) Mean



(b) Volatility

5.5 Expectation Hypothesis and Time Varying Risk Premia

In this section, we consider the models' implication for the dynamics of risk premia and its consequence for the expectation hypothesis. At the heart of this analysis is the time-varying property of risk premia and their determinants.

We consider both variance risks (VAR) and standard deviation risks (STD). We let $\Sigma_{\mathcal{P}}^{1/2}$ be the Cholesky decomposition of the covariance matrix $\Sigma_{\mathcal{P}}$, and let $\mu_t^{\mathbb{P}}$ and $\mu_t^{\mathbb{Q}}$ be the physical and risk-neutral conditional means of the states $\mathcal{P}_t$, respectively:

$$\mu_t^{\mathbb{P}} = E_t^{\mathbb{P}}[\mathcal{P}_{t+1}] = \kappa_0^{\mathbb{P}} + \kappa_1^{\mathbb{P}}\mathcal{P}_t, \tag{22}$$

$$\mu_t^{\mathbb{Q}} = E_t^{\mathbb{Q}}[\mathcal{P}_{t+1}] = \kappa_0^{\mathbb{Q}} + \kappa_1^{\mathbb{Q}}\mathcal{P}_t. \tag{23}$$

The market prices of variance risks and standard-deviation risks can be computed, respectively, as:

$$\Lambda_t^{VAR} = \Sigma_{\mathcal{P}}^{-1}(\mu_t^{\mathbb{P}} - \mu_t^{\mathbb{Q}}) = \Sigma_{\mathcal{P}}^{-1}\left(\kappa_0^{\mathbb{P}} - \kappa_0^{\mathbb{Q}}\right) + \underbrace{\Sigma_{\mathcal{P}}^{-1}\left(\kappa_1^{\mathbb{P}} - \kappa_1^{\mathbb{Q}}\right)}_{\Lambda^{VAR}}\mathcal{P}_t, \tag{24}$$

$$\Lambda_t^{STD} = \Sigma_{\mathcal{P}}^{-1/2}(\mu_t^{\mathbb{P}} - \mu_t^{\mathbb{Q}}) = \Sigma_{\mathcal{P}}^{-1/2}\left(\kappa_0^{\mathbb{P}} - \kappa_0^{\mathbb{Q}}\right) + \underbrace{\Sigma_{\mathcal{P}}^{-1/2}\left(\kappa_1^{\mathbb{P}} - \kappa_1^{\mathbb{Q}}\right)}_{\Lambda^{STD}}\mathcal{P}_t. \tag{25}$$

Intuitively, (24) and (25) capture the extra expected returns required by investors for each extra "unit" in variance and standard deviation, respectively. In our setup, both (24) and (25) are affine in the state vector $\mathcal{P}_t$. The standard deviation risk formulation is more prevalent in existing term structure models, although the variance risk is easier to generalize (see, for example, Le, Singleton, and Dai 2010). The sensitivities of the two market prices of risk with respect to the state $\mathcal{P}_t$ are:

$$\Lambda^{VAR} \equiv \Sigma_{\mathcal{P}}^{-1}\left(\kappa_1^{\mathbb{P}} - \kappa_1^{\mathbb{Q}}\right), \tag{26}$$

$$\Lambda^{STD} \equiv \Sigma_{\mathcal{P}}^{-1/2}\left(\kappa_1^{\mathbb{P}} - \kappa_1^{\mathbb{Q}}\right). \tag{27}$$

Table 3 reports the market prices of risk parameters, Λ^{VAR} and Λ^{STD}, estimated from the no-arbitrage models for both the gold lease rates (Panel A) and the LIBOR data (Panel B). Since we are primarily interested in the time-varying nature of the market prices of risks, we omit the intercept parameters in (24) and (25) in Table 3.

As we see from Panel A of Table 3, both entries in the second column of the market prices of risk matrix are significantly negative, whereas the (2,1) entry is borderline and the (1,1) entry is not significant. This is the case regardless of the estimation method, WOE or FY. In stark contrast, all of the entries estimated using the LIBOR data, reported in Panel B, are individually insignificant, and they sometimes change signs across the two estimation methods. Moreover, when put together, estimates from LIBOR data do not pass the χ^2 test of joint significance. As reported in the columns labeled χ^2, the probability values (p-values) of the χ^2 test for LIBOR data are always larger than 50%, irrespective of the estimation method (WOE or FY) or the type of risk (VAR or STD). By contrast, the p-values of the same tests for the gold interest rates, reported in Panel A, are essentially zeros.

Table 3: Market prices of risks parameters

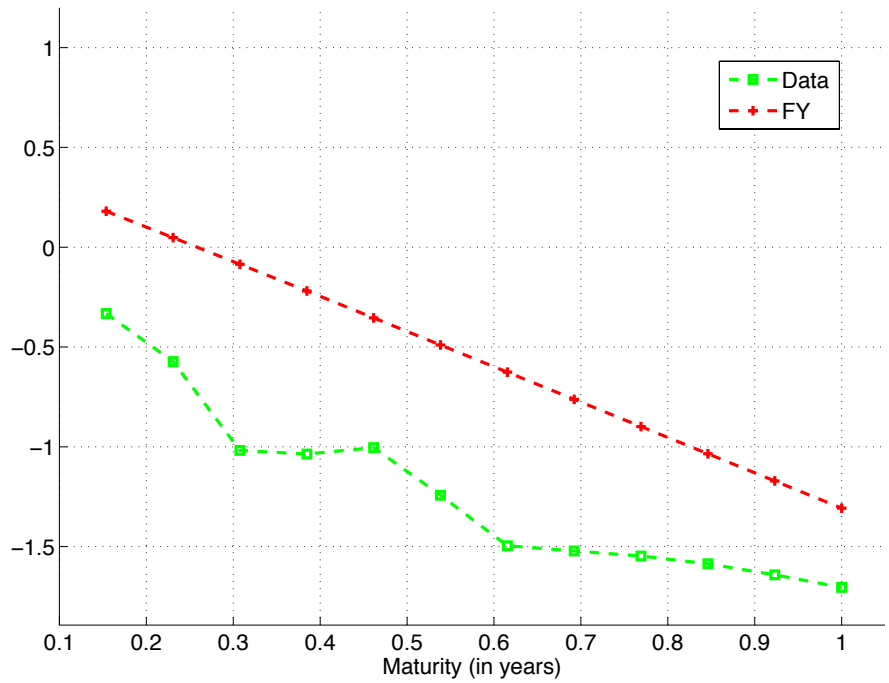
		VAR risk		χ^2	STD risk		χ^2
Panel A: Gold Lease Rate							
WOE	Ests	-0.073	-4.595^{***}	0.000	-0.106	-0.251^{**}	0.000
		0.831^{*}	-4.276^{***}		0.114^{*}	-0.585^{***}	
	s.e.	0.536	0.994		0.067	0.119	
		0.486	0.719		0.066	0.081	
FY	Ests	0.527	-5.936^{***}	0.000	-0.100	-0.281^{**}	0.000
		1.680^{*}	-5.889^{***}		0.163^{**}	-0.571^{***}	
	s.e.	0.829	1.452		0.072	0.127	
		0.947	1.440		0.082	0.100	
Panel B: LIBOR							
WOE	Ests	-0.303	1.144	0.546	-0.025	0.090	0.599
		-0.003	-0.579		-0.000	-0.062	
	s.e.	0.220	1.340		0.018	0.115	
		0.185	0.913		0.020	0.098	
FY	Ests	-0.322	1.313	0.579	-0.025	0.109	0.596
		0.008	1.000		0.001	0.082	
	s.e.	0.255	1.596		0.020	0.124	
		0.300	1.458		0.025	0.123	

The failure to reject the null hypothesis that the market prices of LIBOR risks are constant is consistent with our earlier finding, reported in Figure 2 (b), that the Campbell and Shiller (1991) (CS) regression coefficients are indistinguishable from ones. On the flipped side, the strongly time varying nature of the market prices of gold lending risks suggests that the corresponding CS regression coefficients will deviate from the unit line. Not immediately clear from our estimates, however, is the extent to which the model-implied CS regression coefficients match their sample counterparts.

Figure 5 plots the CS coefficients implied by the no-arbitrage model of gold lease rates, together with those implied by the data (which are also reported in Figure 2 (a)).[15] The model-implied regression coefficients (red crosses) clearly stay closer to the empirical counterparts (green diamonds) than to the unit line implied by the expectation hypothesis. Although the red line lies strictly above the green line for all maturities, based on evidences from existing studies using U.S. interest rates data (for example Dai and Singleton 2002), our model does a relatively good job in capturing the predictability implicit in the CS regressions. The goodness-of-fit of our model is comparable to previously estimated Gaussian term structure

[15] Dai and Singleton (2002) provide the analytical expressions to compute the CS coefficients from model parameters.

Figure 5: CS regression implied by the no-arbitrage model of gold lease rates. Estimation is done by Kalman filtering.



models of interest rates.

Given the significant market prices of risk parameters that are estimated from the gold lease rates data, a natural question entails: what do those estimates capture economically? To answer this question, it is instructive to examine the exact connection between the market prices of risk matrix and the model-implied risk premia (or, equivalently, the model-implied expected excess returns). For example, consider a generic one-period security whose payoff next period is $e^{-c'\mathcal{P}_{t+1}}$, where c is a constant vector that determines the exposure of this security to the factor risk in next period's state variable $\mathcal{P}_{t+1}$. The continuously compounded expected excess return on the security is given by:

$$\begin{aligned} E_t^{\mathbb{P}}\left[\log\left(\frac{e^{-c'\mathcal{P}_{t+1}}}{E_t^{\mathbb{Q}}[e^{-r_t}e^{-c'\mathcal{P}_{t+1}}]}\right)\right] - r_t &= -\,c'(\mu_t^{\mathbb{P}} - \mu_t^{\mathbb{Q}}) + \text{constant}, \\ &= -\,c'\Sigma_{\mathcal{P}}\Lambda^{VAR}\mathcal{P}_t + \text{constant}. \end{aligned} \tag{28}$$

That is, the time-varying part of the expected excess return for holding this security is equal to the quantity of variance risks, $-c'\Sigma_{\mathcal{P}}$, multiplied by the market prices of variance risks, $\Lambda^{VAR}\mathcal{P}_t$.

To further illustrate the economic intuition, consider now a security whose risk exposure is such that $c'\Sigma_{\mathcal{P}} = (1, 0)$, i.e., the only relevant source of risk is the variance risk of the

level factor. For such a security, the time variation in risk premia is driven by the product of the first row of Λ^{VAR} and the state vector $\mathcal{P}_t$. Combined with the fact that only the second column of Λ^{VAR} is significantly estimated (see Table 3), it implies that the slope factor, or PC2, is the only determinant of premia that compensates the level risk. Likewise, for a slope-risk-only security (i.e. $c'\Sigma_{\mathcal{P}} = (0, 1)$), slope is the only factor relevant for risk premia because the (2,1) entry of Λ^{VAR}, which picks up the level factor's effect on risk premia, is insignificant.

The negative signs for the (1,2) and (2,2) entries of Λ^{VAR}, after interacting with the negative sign accompanying yields when converted to bond prices, translate into a positive relationship between slope and expected excess returns. Holding all else constant, an increase in the steepness of the gold interest rates curve leads to an increase in the required compensation for bearing gold lending (variance) risks.

Our line of reasoning also applies to standard deviation risks. In short, the negative entries observed for the second column of Λ^{STD} imply that both level and slope factors' standard deviation risks are priced. The only factor that determines the compensation for these risks is the steepness of the gold interest rates curve. The steeper the curve, the higher is the risk premium required for taking on gold lending (standard deviation) risks.

To visually confirm the linkage between slope and expected excess returns, we perform the following exercise. We take each maturity used in the estimation (one-, two-, three-, six-, and twelve-month), compute the annualized one-week expected excess returns implied by the FY model,[16] and plot the cross-maturity averages in Figure 6. Next, we plot on the same graph the filtered time series of the slope (PC2), which is normalized to have the same mean and standard deviation as the first series. Evidently, the two series bear remarkable resemblances with a correlation of 0.99. On the other hand, the correlation between level and the model-implied expected excess returns is essentially zero.

It is evident from Figure 6 that risk premia in gold leasing markets are mostly positive, with the exception of the last three years of the sample. This implies that the average term structure of gold lease rates must be upward sloping, as is seen in Figure 4 (a).[17]

Equally notable from Figure 6 are the occasional short episodes during which risk premia become negative but quickly turn positive again. This suggests that buying a gold bond (or lending gold) is often viewed by the marginal investor as a risky investment (due to both slope

[16]For a gold loan with n weeks to maturity, the one-week expected excess return can be computed as

$$nA_{\mathcal{P},n} - (n-1)A_{\mathcal{P},n-1} - A_{\mathcal{P},1} - (n-1)B_{\mathcal{P},n-1}\kappa_0^{\mathbb{P}} + \left(nB_{\mathcal{P},n} - (n-1)B_{\mathcal{P},n-1}\kappa_1^{\mathbb{P}} - B_{cP,1}\right)\mathcal{P}_t,$$

where $A_{\mathcal{P},n}$ and $B_{\mathcal{P},n}$ are the intercept and loadings for the n-week maturity.

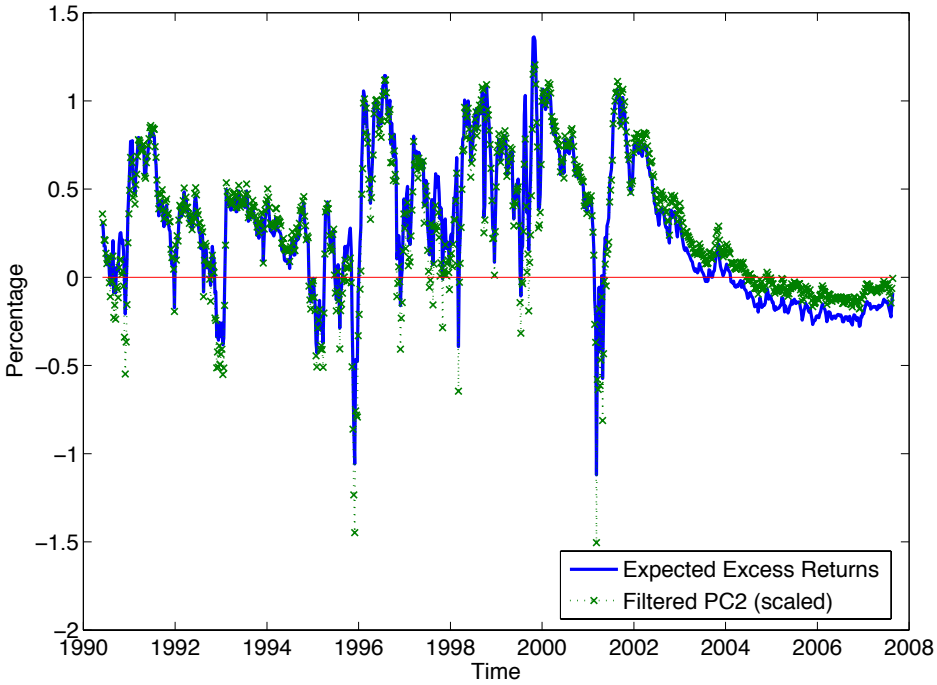
[17]For a formal derivation, let us denote the one-week excess return on the n-period gold loan by

$$xr_{n,t} = ny_{n,t} - (n-1)y_{n-1,t+1} - r_t,$$

and note that

$$\frac{1}{n}\sum_{i=0}^{n-1} xr_{n-i,t+i} = y_{n,t} - \frac{1}{n}\sum_{i=0}^{n-1} r_{t+i},$$

Figure 6: Expected weekly-excess returns (annualized) and filtered slope.



and level risk), but once in a while buying a gold bond can provide some insurance benefits (thus the negative risk premium). For example, the cluster of negative risk premia observed during December 1994 and for much of 1995 coincides with Orange County's bankruptcy, which, according to the New York Times on December the 8th 1994, "shook the market for public borrowing across the country." It is perhaps possible that under such extraordinary circumstances, investment in gold bonds provide insurance-like benefits for holders of gold, just like a US treasury bond is a safe haven for keeping US dollars in times of crisis. It is important to note, however, that unlike treasury bonds, gold bonds are typically not free of default risks. As a result, Figure 6 suggests that the crisis-insurance benefits of gold bonds can more than offset default risk, even during market stress.

which, by applying the expectation operator to both sides, yields:

$$\frac{1}{n}\sum_{i=0}^{n-1} E[xr_{n-i,t+i}] = E[y_{n,t} - r_t].$$

Clearly, the average slope, as measured by $E[y_{n,t} - r_t]$, must be positive, implying an upward-sloping average yield curve, so long as the risk premia are also positive on average.

5.6 Sub-sample Analysis

In this subsection we conduct a robustness check of our main result. Neuberger (2001) suggests that the 1999 Central Bank Agreement on gold may be pivotal for the predictability embedded in gold interest rates. As a robustness check, we divide our sample into two sub-samples, June 1990–August 1999 and October 1999–August 2007. We leave out September 1999, the monthly in which the first Central Bank Agreement on gold was issued.

Table 4: Sub-sample Analysis. Market prices of risks parameters implied by the no-arbitrage model of gold lease rates.

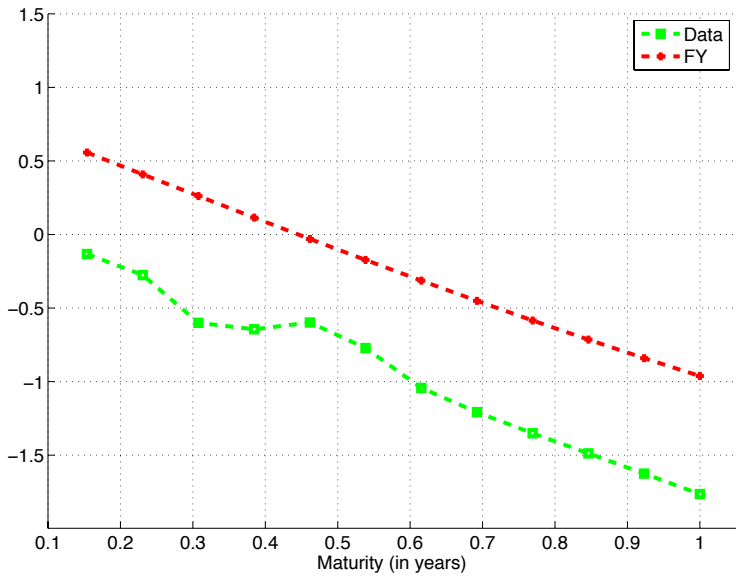
		Jun 1990 – Aug 1999			Oct 1999 – Aug 2007		
		STD risk		χ^2	STD risk		χ^2
WOE	Ests	-0.074	-0.160	0.000	-0.423^{**}	-0.605^{***}	0.002
		0.153	-0.529^{***}		0.245	-0.551^{**}	
	s.e.	0.094	0.117		0.181	0.165	
		0.104	0.102		0.288	0.259	
FY	Ests	-0.065	-0.200	0.001	-0.444^{*}	-0.651^{***}	0.000
		0.231^{*}	-0.457^{***}		0.229	-0.513^{**}	
	s.e.	0.100	0.124		0.261	0.227	
		0.131	0.119		0.345	0.247	

For both of the sub-samples, the market price of standard deviation risks parameters continue to be jointly significant for the gold lease rates, as we see from the joint χ^2 statistics of Table 4. The slope is again the dominant determinant of risk premia. Similar to the full-sample results, the slope loadings (the second column of Λ^{STD}) are, except for one entry, significantly negative with broadly comparable magnitudes. The (borderline) non-significance of the (1,2) entry of Λ^{STD} in the first sub-sample is at least partly due to the weaker statistical power given the shorter sample. Notably, in the second sub-sample, there is some evidence that the level factor, in addition to slope, may play a role in determining premia for compensating level risks.

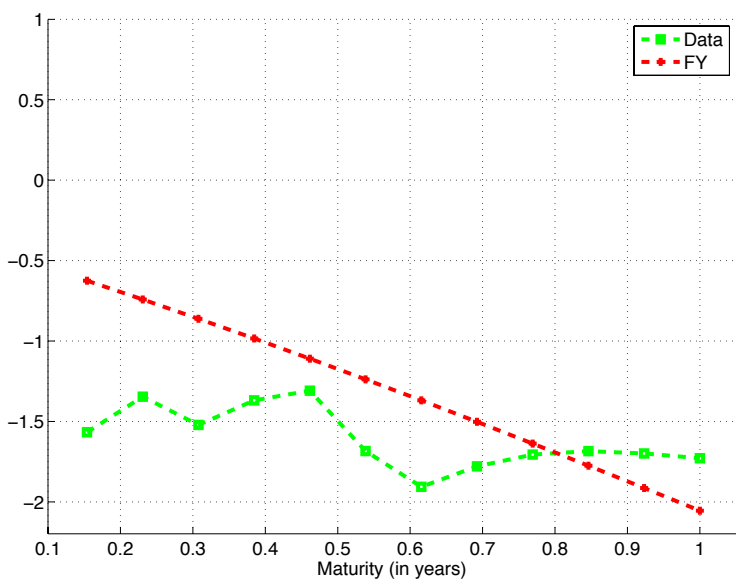
Figure 7 reports the CS regression coefficients which, for both sub-samples, again lie significantly below the unit line. Importantly, the no-arbitrage models estimated for each sub-sample are again able to replicate to a reasonable extent these deviations from the expectation hypothesis.

We find that the different behavior of risk premia in gold lease rates and LIBOR persist in each sub-sample. For LIBOR, the estimates of Λ^{STD} continue to be jointly insignificant in each sub-sample; so are results using Λ^{VAR}. The CS coefficients using the LIBOR data are again indistinguishable from the unit line for each sub-sample. These results using LIBOR data are omitted for brevity and are available upon request.

Figure 7: Sub-sample analysis. CS regression implied by the no-arbitrage model of gold lease rates. Estimation is done by Kalman filtering.



(a) Jun 1990 – Aug 1999



(b) Oct 1999 – Aug 2007

6 Macro, Liquidity and Volatility Conditions

Evidence from Section 4 and Section 5 demonstrates sharply different behaviors of risk premia in gold lease rates versus LIBOR at the same maturities. A natural question is why. In this section we provide a preliminary analysis of this question by exploring the effects of macroeconomic, liquidity, and volatility conditions for risk premia in the two markets.

We complement the gold lease rate data with data on macroeconomic, liquidity and volatility conditions. Because realized macroeconomic activities are reported with a lag, we use forecasts of inflation and real GDP growth from Bluechip. Bluechip forecasts provide, at the monthly frequency, survey-based annualized inflation estimates and real GDP estimate for the next 1, 2, 3, 4, 5 and 6 quarters.[18] Given the noise in short-term inflation and GDP forecasts, we calculate the expected inflation ($Einf$) and expected real GDP growth ($Ergdp$) as the average of the corresponding 3-, 4-, 5- and 6-quarter forecasts. Our liquidity variables include the bond liquidity measures by Fontaine and Garcia (2012), Wright and Zhou (2009), and Hu, Pan, and Wang (2012). Finally, we include the VIX index from CBOE/CME to capture equity volatility and the Merrill Option Volatility Estimate (MOVE) index to capture treasury bond yield volatility.[19] Both the VIX and the MOVE indices measure implied volatilities with a horizon of about one month.

Table 5 tabulates the contemporaneous correlation matrix of the variables, their means, and their standard deviations. We use the monthly frequency because of the macroeconomic variables. Overall, we have 207 months from June 1990 to August 2007.

Table 5: Summary statistics of monthly data. Row 1 to row 11 report the correlation matrix of the variables. The last two rows report their means and standard deviations. The PCs are demeaned.

	Gold PC1	Gold PC2	LIBOR PC1	LIBOR PC2	Einf	Ergdp	GF	HPW	WZ	VIX	MOVE
Gold PC1	1.00	0.00	0.54	−0.13	0.28	−0.63	0.19	0.39	0.09	0.26	0.17
Gold PC2		1.00	0.06	0.17	−0.04	−0.13	−0.04	0.35	−0.24	0.31	0.20
LIBOR PC1			1.00	0.00	0.46	−0.77	0.57	0.11	0.36	0.02	−0.26
LIBOR PC2				1.00	0.25	−0.02	0.21	0.02	0.21	−0.37	0.07
Einf					1.00	−0.45	−0.01	0.24	0.38	−0.28	0.09
Ergdp						1.00	−0.47	−0.04	−0.51	−0.02	0.05
GF							1.00	−0.11	0.48	−0.05	−0.30
HPW								1.00	−0.09	0.43	0.40
WZ									1.00	−0.41	−0.17
VIX										1.00	0.36
MOVE											1.00
Mean	0	0	0	0	0.0278	0.0289	0.194	3.01	0.0159	18.9	98.8
St.Dev.	0.0168	0.0032	0.0392	0.003	0.0056	0.0051	0.1405	1.041	0.0834	6.43	19.4

As in Section 4 we denote by y_t^n the period-t zero rate of a n-period gold loan, but in this section one period is one month. The excess return of holding an n-period gold loan for one

[18] Ang, Bekaert, and Wei (2007) find that survey data provide better inflation forecasts than macro variables and asset markets do.

[19] The MOVE index is a yield curve weighted index of the normalized implied volatility on 1-month Treasury options. For more details, see http://www.rimes.com/merrill-option-volatility-estimate-indices.

month is

$$rx_{t+1}^n \equiv y_t^n \frac{n}{12} - y_{t+1}^{n-1} \frac{n-1}{12} - y_t^1 \frac{1}{12} = \frac{y_t^n - y_t^1}{12} + (y_t^n - y_{t+1}^{n-1}) \frac{n-1}{12}. \tag{29}$$

This excess return is measured in gold. We run the predictive regression for the average excess return across maturities $n \in \{2, 3, 6, 12\}$:

$$\bar{rx}_{t+1} \equiv \frac{1}{4} \sum_n rx_{t+1}^n = \bar{\alpha} + \bar{\beta} \cdot [PC, Einf, Ergdp, GF, HPW, WZ, VIX, MOVE]_t + \bar{\epsilon}_{t+1}. \tag{30}$$

We also run the same regression for excess returns for individual maturities:

$$rx_{t+1}^n = \alpha^n + \beta^n \cdot [PC, Einf, Ergdp, GF, HPW, WZ, VIX, MOVE]_t + \epsilon_{t+1}^n. \tag{31}$$

In these regressions, the PCs are normalized to have mean zero, and the variables GF, HPW, WZ, VIX, and $MOVE$ are demeaned and then scaled by their standard deviation.

Table 6 reports the results of regression (30) applied to average monthly excess returns for gold lease rates and LIBOR. For the ease of exposition, the reported coefficients in the tables are multiplied by 100. These regressions reveal two salient features of the excess returns.

First, regressions with only PCs as predictors (i.e. the first two columns of the two regressions) confirm our earlier evidence of the expectation hypothesis. The PCs of gold lease rate significantly predict excess returns of gold loans. PC1 and PC2 explain almost 10% of the average excess returns, and all five PCs explain 13%. Notably, PC2 is the only significant predictor, which is consistent with our estimation results in the term structure model of Section 5. By contrast, LIBOR excess returns cannot be explained by PC1 and PC2 of LIBOR at all. Once the other three PCs are included, PC3 becomes significant, and adjusted R^2 increases from zero to 5.2%.

Second, distinct macro, liquidity, and volatility factors explain the excess returns of gold loans and LIBOR loans, as we see in the last column of the two regressions in Table 6. For gold lease rates, only the real GDP growth forecast and bond volatility are significant predictors. A 1% increase in the real GDP growth forecast increases the average monthly excess return of gold loans by 9.6 bps per month, or 1.15% per year. That is, excess returns on gold loans move almost one-for-one with expected real GDP growth. A one standard-deviation increase in the treasury bond volatility, measure by $MOVE$, lowers the gold loan excess returns by 1.7 bps per month (or 20 bps per year), which is suggestive of investors' precautionary saving. Moreover, after controlling for those additional variables, PC1 becomes significant, and PC2 remains significant and even becomes larger economically. Overall, the additional variables increase the adjusted R^2 from 13.1% to 19.4%.

For LIBOR excess returns, adding the macro, liquidity, and volatility factors shows a different picture. On the one hand, once these additional factors are included, LIBOR PC3 loses its predictive power, whereas PC4 and PC5 become significant. On the other hand, the two additional significant predictors are expected inflation $Einf$ and the WZ jump measure. A 1% increase in the inflation forecast raises the average excess return on the LIBOR loan by 4.36 bps per month (or 52.3 bps per year), which is intuitive as investors demand higher

nominal returns to compensate a higher expected inflation. A one-standard deviation increase in the WZ liquidity measure reduces the excess return by 3.3 bps per month (or 39.6 bps per year), which is consistent with the notation that LIBOR returns contain a liquidity premium. Overall, the additional variables increase the adjusted R^2 from 5.2% to 20.8%, which shows that most of information regarding excess returns for LIBOR loans are not spanned by the LIBOR PCs themselves.

Table 7 and Table 8 report results from excess return regressions for individual maturities (31). The results are qualitatively similar to those in average-return regressions. The loading on significant predictors generally increase with maturity, as we have expected.

Overall, our analysis in this section reveals that information regarding the risk premia in gold leasing markets comes mostly from the gold lease rates themselves, especially PC2, although the expected real GDP growth also seems important. By contrast, most information regarding the risk premia in LIBOR comes from expected inflation and liquidity.

Table 6: Average excess return regression (30) applied to gold lease rates and LIBOR. The left-hand side is the average monthly excess return of holding a gold loan or a LIBOR loan with 2-, 3-, 6-, and 12-month maturities. The PCs are normalized to have mean zero. Variables GF, HPW, WZ, VIX, and $MOVE$ are normalized to have mean zero and standard deviation one. All the regression coefficients are multiplied by 100. The t-statistics are reported in parentheses, and statistical significance at the 10%/5%/1% levels are denoted by $^{*}/^{**}/^{***}$, respectively.

	Gold lease rates			LIBOR		
Const	0.024***	0.024***	−0.241*	0.024***	0.024***	−0.096
	(3.423)	(3.496)	(−1.864)	(4.255)	(4.409)	(−1.470)
PC1	0.551	0.555	3.025***	0.183	0.182	0.267
	(0.876)	(0.916)	(2.618)	(1.461)	(1.498)	(1.068)
PC2	10.252***	10.265***	16.714***	−0.721	−0.696	0.372
	(3.730)	(4.049)	(4.924)	(−0.310)	(−0.314)	(0.160)
PC3		11.355	6.499		25.913***	4.550
		(0.917)	(0.574)		(3.016)	(0.569)
PC4		−21.266	−21.271		1.430	29.827**
		(−1.438)	(−1.454)		(0.103)	(2.216)
PC5		20.384	25.078*		−24.326	−45.483**
		(1.527)	(1.923)		(−1.039)	(−2.410)
Einf			−0.516			4.357***
			(−0.253)			(2.888)
Ergdp			9.631***			−0.071
			(2.851)			(−0.044)
GF			−0.005			−0.010
			(−0.563)			(−1.184)
HPW			−0.016			0.010
			(−1.142)			(1.440)
WZ			0.018			−0.033***
			(1.147)			(−4.328)
VIX			−0.005			−0.000
			(−0.402)			(−0.003)
MOVE			−0.017**			−0.008
			(−2.585)			(−1.357)
R2 adj	0.098	0.131	0.194	−0.001	0.052	0.208

Table 7: Excess return regression (31) applied to gold lease rates. The left-hand side is the monthly excess return of holding a 2-, 3-, 6-, or 12-month gold loan for 1 month. The PCs are normalized to have mean zero, and the variables GF, HPW, WZ, VIX, and $MOVE$ are demeaned and then scaled by their respective standard deviation. All the coefficients are multiplied by 100. The t-statistics are reported in parentheses, and statistical significance at the 10%/5%/1% levels are denoted by $^{*}/^{**}/^{***}$, respectively.

	2-month gold loan			3-month gold loan			6-month gold loan			12-month gold loan		
Const	0.007***	0.007***	−0.032	0.013***	0.013***	−0.044	0.024***	0.024***	−0.247	0.051***	0.050***	−0.641**
	(2.750)	(2.945)	(−0.764)	(2.997)	(3.103)	(−0.574)	(2.937)	(2.966)	(−1.522)	(3.608)	(3.697)	(−2.522)
PC1	0.454*	0.457**	1.114***	0.688*	0.692*	1.633**	0.778	0.778	3.375**	0.285	0.296	5.977***
	(1.669)	(2.037)	(2.896)	(1.676)	(1.855)	(2.106)	(1.069)	(1.080)	(2.368)	(0.220)	(0.230)	(2.604)
PC2	0.155	0.163	1.816	2.386	2.396	4.684**	11.073***	11.075***	17.346***	27.393***	27.425***	43.010***
	(0.120)	(0.168)	(1.463)	(1.465)	(1.543)	(2.184)	(3.216)	(3.440)	(4.184)	(5.064)	(5.351)	(6.266)
PC3		−6.911*	−5.130		−5.568	−3.436		−16.069	−11.767		−16.872	−5.664
		(−1.787)	(−1.436)		(−0.727)	(−0.499)		(−1.094)	(−0.877)		(−0.635)	(−0.233)
PC4		−15.924***	−15.864***		−11.931	−11.591		−3.778	−2.194		−53.432	−55.433*
		(−3.822)	(−3.685)		(−1.627)	(−1.549)		(−0.228)	(−0.133)		(−1.596)	(−1.681)
PC5		2.235	4.416		25.623***	28.247***		14.249	18.582		39.428	49.065*
		(0.304)	(0.664)		(2.710)	(3.211)		(0.822)	(1.115)		(1.379)	(1.677)
Einf			−0.946			−1.230			−0.974			1.085
			(−1.299)			(−1.118)			(−0.389)			(0.260)
Ergdp			2.263**			3.151			10.308**			22.801***
			(2.112)			(1.466)			(2.451)			(3.394)
GF			−0.003			−0.003			−0.005			−0.006
			(−1.048)			(−0.616)			(−0.546)			(−0.397)
HPW			−0.002			−0.003			−0.012			−0.046*
			(−0.406)			(−0.288)			(−0.726)			(−1.680)
WZ			0.007			0.007			0.017			0.041
			(1.197)			(0.626)			(0.888)			(1.404)
VIX			−0.002			−0.004			−0.004			−0.007
			(−0.518)			(−0.675)			(−0.317)			(−0.332)
MOVE			−0.005**			−0.009**			−0.020**			−0.034**
			(−2.384)			(−2.280)			(−2.420)			(−2.577)
R2 adj	0.031	0.156	0.198	0.038	0.101	0.126	0.084	0.092	0.143	0.154	0.192	0.264

Table 8: Excess return regression (31) applied to LIBOR. The left-hand side is the monthly excess return of holding a 2-, 3-, 6-, or 12-month LIBOR loan for 1 month. The PCs are normalized to have mean zero, and the variables GF, HPW, WZ, VIX, and $MOVE$ are demeaned and then scaled by their respective standard deviation. All the coefficients are multiplied by 100. The t-statistics are reported in parentheses, and statistical significance at the 10%/5%/1% levels are denoted by $^{*}/^{**}/^{***}$, respectively.

	2-month LIBOR loan			3-month LIBOR loan			6-month LIBOR loan			12-month LIBOR loan		
Const	0.005***	0.005***	−0.011	0.011***	0.011***	−0.030	0.023***	0.023***	−0.099	0.055***	0.055***	−0.242
	(5.456)	(5.544)	(−0.714)	(6.062)	(6.235)	(−1.320)	(4.260)	(4.416)	(−1.524)	(3.718)	(3.853)	(−1.440)
PC1	0.017	0.017	−0.022	0.090**	0.089**	0.022	0.204	0.204*	0.153	0.422	0.420	0.914
	(0.665)	(0.665)	(−0.464)	(2.174)	(2.216)	(0.244)	(1.647)	(1.715)	(0.613)	(1.241)	(1.267)	(1.421)
PC2	−0.095	−0.089	−0.458	−0.686	−0.671	−1.207	−1.412	−1.415	−1.448	−0.690	−0.612	4.602
	(−0.228)	(−0.214)	(−0.982)	(−0.840)	(−0.848)	(−1.460)	(−0.599)	(−0.628)	(−0.632)	(−0.115)	(−0.108)	(0.757)
PC3		2.600	−0.202		8.066**	1.463		23.065**	0.262		69.921***	16.676
		(1.567)	(−0.131)		(2.386)	(0.476)		(2.405)	(0.029)		(3.308)	(0.848)
PC4		2.965	−0.813		0.208	−8.675*		−11.737	−42.493***		2.844	−67.325**
		(1.027)	(−0.307)		(0.040)	(−1.801)		(−0.754)	(−2.823)		(0.084)	(−2.015)
PC5		−3.127	−6.624		0.155	−7.787		−31.121	−55.824***		−63.210	−111.698**
		(−0.666)	(−1.573)		(0.020)	(−1.154)		(−1.234)	(−2.612)		(−1.075)	(−2.395)
Einf			0.660**			1.594***			4.901***			10.271***
			(1.975)			(3.060)			(3.213)			(2.617)
Ergdp			−0.083			−0.095			−0.471			0.364
			(−0.255)			(−0.161)			(−0.296)			(0.089)
GF			0.001			0.002			−0.005			−0.040*
			(0.925)			(0.582)			(−0.552)			(−1.672)
HPW			0.004**			0.007***			0.015**			0.013
			(2.456)			(2.682)			(2.026)			(0.784)
WZ			−0.005***			−0.011***			−0.036***			−0.079***
			(−3.528)			(−3.984)			(−4.769)			(−4.043)
VIX			−0.003			−0.004			−0.005			0.011
			(−1.370)			(−1.298)			(−0.596)			(0.530)
MOVE			−0.001			−0.003			−0.007			−0.022
			(−1.007)			(−1.207)			(−1.158)			(−1.418)
R2 adj	−0.007	0.010	0.103	0.013	0.052	0.187	0.003	0.053	0.224	−0.004	0.052	0.201

7 Conclusion

In this paper we investigate the dynamics of gold interest rates – interests paid in gold for borrowing gold. We find that risks in both the level and the slope of gold interest rates are strongly priced. Moreover, the premia required by investors to take on these risks only depend on the current steepness of the gold interest rate curves. Over our sample period, slope is highly time varying, mostly upward-sloping but occasionally inverted. The time-varying nature of slope and its close connection to risk premia in turn lead to a strong rejection of the expectation hypothesis. Interestingly, the expectation hypothesis cannot be rejected for comparable US interest rate data over the same sample period. This stark contrast reveals fundamental differences between gold interest rates and interest rates of major fiat currencies.

As central banks around the world continue to hold gold as part of their currency reserves, it seems all the more important to better understand the risk-return dynamics in the gold lending market. While our analysis provides initial evidence of risk premium behavior in this market, many questions remain open. For example, what are the economic contents of the slope factor that make it so important in determining the risk premia required by investors for taking on gold lending risks? Additionally, since gold is widely perceived as a storage of value and not tied to any specific currency or sovereign country, to what extent do gold interest rates reflect the (shadow) term structure of real interest rates for the global economy? We leave these questions for future research.

A Estimation of No-Arbitrage Term Structure Models

In this section, we show how to analytically concentrate out $\kappa_0^{\mathbb{P}}$, $\kappa_1^{\mathbb{P}}$, σ_e, and $r_\infty^{\mathbb{Q}}$ for the WOE estimation.

A.1 Notations

Continuing from Section 5, we write:

$$A_{n,X} = r_\infty^{\mathbb{Q}} + tr(\Sigma_X C_{n,X}) \text{ where } C_{n,X} = -\frac{1}{2n}\sum_{i=0}^{n-1} i^2 B_{i,X} B_{i,X}'. \tag{32}$$

It is important to note that both $B_{n,X}$ and $C_{n,X}$ are only functions of $\lambda^{\mathbb{Q}}$.

Obviously, $B_{n,\mathcal{P}}$ is only dependent on $\lambda^{\mathbb{Q}}$. To see clearly how $r_\infty^{\mathbb{Q}}$, $\Sigma_{\mathcal{P}}$ and λ^Q influence $A_{n,\mathcal{P}}$, we first write:

$$\begin{aligned}
A_{n,X} =& r_\infty^{\mathbb{Q}} + tr(\Sigma_X C_{n,X}) && (33)\\
=& r_\infty^{\mathbb{Q}} + tr(V^{-1}\Sigma_{\mathcal{P}}(V')^{-1}C_{n,X}) && (34)\\
=& r_\infty^{\mathbb{Q}} + tr(\Sigma_{\mathcal{P}}(V')^{-1}C_{n,X}V^{-1}) && (35)\\
=& r_\infty^{\mathbb{Q}} + vech((V')^{-1}C_{n,X}V^{-1})vec(\Sigma_{\mathcal{P}}) && (36)\\
=& r_\infty^{\mathbb{Q}} + D_{n,X}vec(\Sigma_{\mathcal{P}}) && (37)
\end{aligned}$$

where $D_{n,X} = vech((V')^{-1}C_{n,X}V^{-1})$ and $V = WB_X$.

Stacking $A_{n,X}$ for all maturities used in the estimation, we obtain:

$$A_X = \iota_J r_\infty^{\mathbb{Q}} + D_X vec(\Sigma_{\mathcal{P}}). \tag{38}$$

where D_X is obtained by staking the relevant row vectors $D_{n,X}$ on top of one another. ι_J denotes a J-size vector of ones.

Now plugging $A_{n,X}$ and A_X into $A_{n,\mathcal{P}}$, we have:

$$\begin{aligned}
A_{n,\mathcal{P}} =& r_\infty^{\mathbb{Q}} + D_{n,X}vec(\Sigma_{\mathcal{P}}) - B_{n,X}'V^{-1}W(\iota_J r_\infty^{\mathbb{Q}} + D_X vec(\Sigma_{\mathcal{P}})) && (39)\\
=& (1 - B_X'V^{-1}W\iota_J)r_\infty^{\mathbb{Q}} + (D_X - B_X'V^{-1}WD_X)vec(\Sigma_{\mathcal{P}}) && (40)\\
=& E_n r_\infty^{\mathbb{Q}} + tr(\Sigma_{\mathcal{P}}F_n) && (41)
\end{aligned}$$

where

$$E_n = 1 - B_{n,X}'V^{-1}W\iota_J. \tag{42}$$

and F_n is obtained by appropriately collapsing the row vector $D_{n,X} - B_{n,X}'V^{-1}WD_X$ into a square matrix. It is important to note that both E_n and F_n are only dependent on $\lambda^{\mathbb{Q}}$.

Stacking $A_{n\mathcal{P}}$ for all maturities n used in estimation, we obtain:

$$A_{\mathcal{P}} = Er_\infty^{\mathbb{Q}} + F, \tag{43}$$

where E is obtained by stacking up E_n's; and F is obtained by stacking up $tr(\Sigma_{\mathcal{P}}F_n)$, for relevant n's.

A.2 $\kappa_0^{\mathbb{P}}$ and $\kappa_1^{\mathbb{P}}$

We know from JSZ that the global estimates of $\kappa_0^{\mathcal{P}}$ and $\kappa_1^{\mathcal{P}}$ can be obtained from a VAR of $\mathcal{P}_{t+1}$ on $\mathcal{P}_t$:

$$\begin{aligned} \kappa_1^{\mathbb{P}} &= E_T[\mathcal{P}_{t+1}^e(\mathcal{P}_t^e)'](E_T[\mathcal{P}_t^e(\mathcal{P}_t^e)'])^{-1}, & (44) \\ \kappa_0^{\mathbb{P}} &= E_T[\mathcal{P}_t] - \kappa_1 E_T[\mathcal{P}_t]. & (45) \end{aligned}$$

where $E_T[.]$ denotes sample average and $\mathcal{P}_t^e = \mathcal{P}_t - E_T[\mathcal{P}_t]$.

A.3 σ_e

Likewise, the global estimate of σ_e can be obtained from the first order condition of the likelihood function:

$$\sigma_e^2 = \frac{1}{T(J-N)} \sum_t ||\mathcal{P}_{t+1}^e - W^e(A_{\mathcal{P}} + B_{\mathcal{P}}\mathcal{P}_{t+1})||_2 \tag{46}$$

A.4 $r_\infty^{\mathbb{Q}}$

The first order condition with respect to $r_\infty^{\mathbb{Q}}$ is:

$$\left(E_T[\mathcal{P}_{t+1}^e] - W^e(A_{\mathcal{P}} + B_{\mathcal{P}}E_T[\mathcal{P}_{t+1}])\right)' W^e \frac{\partial A_{\mathcal{P}}}{\partial r_\infty^{\mathbb{Q}}} = 0 \tag{47}$$

Utilizing the earlier representation: $A_{\mathcal{P}} = Er_\infty^{\mathbb{Q}} + F$, we have

$$\left(E_T[\mathcal{P}_{t+1}^e] - W^e(F + B_{\mathcal{P}}E_T[\mathcal{P}_{t+1}]) - W^e E r_\infty^{\mathbb{Q}}\right)' W^e E = 0 \tag{48}$$

which is linear in $r_\infty^{\mathbb{Q}}$. Collecting terms, we have:

$$r_\infty^{\mathbb{Q}} = \frac{\left(E_T[\mathcal{P}_{t+1}^e] - W^e(F + B_{\mathcal{P}}E_T[\mathcal{P}_{t+1}])\right)' W^e E}{||W^e E||_2}. \tag{49}$$

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