

# Rare booms and disasters in a multi-sector endowment economy\*

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## **Abstract**

Why do value stocks have higher expected returns than growth stocks, in spite of having lower risk? Why do these stocks exhibit positive abnormal performance while growth stocks exhibit negative abnormal performance? This paper offers a rare-events based explanation, that can also account for facts about the aggregate market. Patterns in time-series predictability offer independent evidence for the model's conclusions.

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# 1 Introduction

This paper introduces a representative agent asset pricing model in which the endowment and the aggregate dividend are subject to large rare negative shocks (disasters) and large rare positive shocks (booms). We consider a two-sector model for the economy: the growth sector is the claim to the stream of dividends arising from the rare booms, while the value sector is the claim to the remaining dividend stream. The two sectors add up to the aggregate market. We show that this parsimonious model can explain important features of stock market data. As shown in earlier work, a time-varying probability of rare disasters can account for the high equity premium, high stock market volatility and return predictability exhibited by the aggregate market.<sup>1</sup> Beyond addressing these earlier points, our work also explains the cross-section of stock returns.

The possibility of rare booms has received little attention in comparison to rare disasters. This may be because the implications of rare booms for the equity premium, a focus of earlier work, are relatively minor. Because of decreasing marginal utility, the representative agent requires little compensation for bearing the risk of rare booms, even if they are large.<sup>2</sup> However, when assets have varying exposure to the booms, the impact on the cross-section can be substantial. The model implies that investors are willing to hold the growth portfolio despite its low return because of the small possibility of a high payout. The growth portfolio has a high covariance with the market because it is subject to a time-varying risk of booms as well as a time-varying risk of disaster; once a boom occurs the resulting dividend stream has the same disaster exposure as the rest of the economy. In fact, the model accurately predicts that the growth portfolio has a market beta greater

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<sup>1</sup>For the equity premium result, see Rietz (1988), Longstaff and Piazzesi (2004), and Barro (2006). For the volatility and predictability results, see Gabaix (2012), Gourio (2011) and Wachter (2012).

<sup>2</sup>In recent work, Bekaert and Engstrom (2010) propose a model in which the economy is also subject to shocks in which bad events predominate and shocks in which good events predominate. Their model differs from ours in that they focus on explaining aggregate market and consumption moments with an agent with habit-like preferences.

than one while the value portfolio has a market beta less than one. This combination of high betas with low expected returns allows the model to explain the striking failure of the Capital Asset Pricing Model (CAPM) observed in the data (Fama and French (1992)).

Our model introduces several innovations beyond those described above. First, we model disasters and booms as influencing the drift rate of fundamentals, rather than fundamentals directly. This allows our model to capture the fact that disasters and booms unfold slowly, as emphasized by Constantinides (2008). The assumption of recursive utility implies that there is still a substantial equity premium.<sup>3</sup> Second, we introduce a novel way to model value and growth assets that allows the dividends on value to grow more slowly than those of the aggregate market, but still implies value and growth add up to the market, and price ratios are stationary.

A number of other papers also offer risk-based explanations for the relatively high expected returns on value stocks (the value premium).<sup>4</sup> It is likely that the value premium has multiple causes, and it is not the purpose of this article to rule out other explanations. One difficulty with these risk-based explanations is that a value premium arises because returns on the value portfolio are more risky than the growth portfolio. This, however, is not the case in the data. In our model, growth is in fact more risky. We break the link between risk and return in two ways: first, while population returns on growth may be higher, in any given sample, it is not unlikely that a value premium will be observed in the data. Second, the risk in growth arises from rare booms, which occur in times of low

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<sup>3</sup>Bansal, Kiku, and Yaron (2010) also model large shocks to the growth rate in a setting with a constant probability of disaster. Nakamura, Steinsson, Barro, and Ursua (2011) also address the Constantinides (2008) critique; the focus of their empirical paper is to accurately capture the disaster distribution in complex setting where only numerical solutions are available. In contrast, the focus of this paper is to account for the aggregate market and cross-sectional moments using a relatively simple model with analytical solutions.

<sup>4</sup>For example, Ai and Kiku (2011), Berk, Green, and Naik (1999), Carlson, Fisher, and Giammarino (2004), Gârleanu, Kogan, and Panageas (2011), Gomes, Kogan, and Zhang (2003), Hansen, Heaton, and Li (2008), Novy-Marx (2010), Santos and Veronesi (2010) and Zhang (2005).

marginal utility. Hence investors do not require compensation for bearing this risk.<sup>5</sup>

Besides addressing the sign and magnitude of the value premium, our model can also account for the time-series behavior of the value premium and its relation to the equity premium. As is well-known, the price-dividend ratio can predict excess returns on the aggregate market, implying that the equity premium is varying over time (Campbell and Shiller (1988)). The value spread can predict the return on the value-minus-growth portfolio, implying that it, too, has a time-varying risk premium (Cohen, Polk, and Vuolteenaho (2003)). However, these risk premiums appear to have little to do with one-another; the price-dividend ratio has almost no predictive power for the value spread. In our model, a two-factor structure for risk premia arise naturally, and it is thus capable of explaining this result.

The remainder of the paper is organized as follows. Section 2 describes and solves the model. Section 3 discusses the quantitative fit of the model to the data. Section 4 concludes.

## 2 Model

### 2.1 Endowment and preferences

We assume an endowment economy with an infinitely-lived representative agent. Aggregate consumption (the endowment) follows a diffusion process with time-varying drift:

$$\frac{dC_t}{C_t} = \mu_{C_t} dt + \sigma dB_{C_t}, \quad (1)$$

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<sup>5</sup>Other studies succeed in breaking the link between risk and return using mechanisms other than what we consider here. These include Campbell and Vuolteenaho (2004) and Campbell, Polk, and Vuolteenaho (2010), who model growth and value in an ICAPM setting, and Lettau and Wachter (2007), who assume an exogenous stochastic discount factor. These studies, however, do not assume a representative agent pricing assets in equilibrium in which cash flows must add up to the market.

where  $B_{Ct}$  is a standard Brownian motion. The drift of the consumption process is given by

$$\mu_{Ct} = \bar{\mu}_C + \mu_{1t} + \mu_{2t}, \quad (2)$$

where

$$d\mu_{jt} = -\kappa_{\mu_j}\mu_{jt}dt + Z_{jt}dN_{jt}, \quad (3)$$

for  $j = 1, 2$ . This model allows expected consumption growth to be subject to two types of (large) shocks. The rare events  $N_{jt}$  each follow a Poisson process (that is, for a given  $t$ ,  $N_{jt}$  has a Poisson distribution). In what follows, we will consider the first type ( $j = 1$ ) to be disasters, so that  $Z_{1t} \leq 0$  and the second type ( $j = 2$ ) to be booms, so that  $Z_{2t} \geq 0$ . When a disaster occurs, the process  $\mu_{1t}$  jumps downward. It then mean-reverts back (absent any other bad shocks). Likewise, when a boom occurs, the process  $\mu_{2t}$  jumps upward. It too reverts back. This model allows for smooth consumption (as in the data), that nonetheless goes through periods of extreme growth rates in one direction or another. Writing down two separate processes influencing expected consumption growth (as opposed to one process with two types of shocks) simplifies pricing of different sectors and allows disasters to be shorter-lived than booms, as the data suggest.

In what follows, the magnitude of the jumps will be random with a time-invariant distribution. That is,  $Z_{jt}$  has distribution  $\nu_j$ . We will use the notation  $E_{\nu_j}$  to denote expectations taken over the distribution  $\nu_j$ . The intensity of the Poisson shock  $N_j$  is governed by  $\lambda_{jt}$ , which is stochastic, and follows the process

$$d\lambda_{jt} = \kappa_{\lambda_j}(\bar{\lambda}_j - \lambda_{jt}) dt + \sigma_{\lambda_j}\sqrt{\lambda_{jt}} dB_{\lambda_{jt}}. \quad (4)$$

where  $B_{\lambda_{jt}}, j = 1, 2$  are independent Brownian motions, that are each independent of  $B_{Ct}$ . Furthermore, we assume that the Poisson shocks  $N_{jt}$  are independent of each other, and of the Brownian motions. Define  $\lambda_t = [\lambda_{1t}, \lambda_{2t}]^\top$ ,  $\mu_t = [\mu_{1t}, \mu_{2t}]^\top$ ,  $B_{\lambda t} = [B_{\lambda_{1t}}, B_{\lambda_{2t}}]^\top$  and  $B_t = [B_{Ct}, B_{\lambda t}^\top]^\top$ .<sup>6</sup>

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<sup>6</sup>We assume throughout that  $\kappa_{\mu_j}, \kappa_{\lambda_j}, \bar{\lambda}_j$  and  $\sigma_{\lambda_j}$ , for  $j = 1, 2$ , are strictly positive.

We assume the continuous-time analogue of the utility function defined by Epstein and Zin (1989) and Weil (1990), that generalizes power utility to allow for preferences over the timing of the resolution of uncertainty. The continuous-time version is formulated by Duffie and Epstein (1992); we use the case that sets the parameter associated with the elasticity of intertemporal substitution (EIS) equal to one. Define the utility function  $V_t$  for the representative agent using the following recursion:

$$V_t = E_t \int_t^\infty f(C_s, V_s) ds, \quad (5)$$

where

$$f(C_t, V_t) = \beta(1 - \gamma)V_t \left( \log C_t - \frac{1}{1 - \gamma} \log((1 - \gamma)V_t) \right). \quad (6)$$

We follow common practice in interpreting  $\gamma$  as risk aversion and  $\beta$  as the rate of time preference. We assume throughout that  $\gamma > 0$  and  $\beta > 0$ .

## 2.2 The value function

Let  $W_t$  denote the wealth of the representative agent and  $J(W_t, \mu_t, \lambda_t)$  the value function. In equilibrium, it must be the case that  $J(W_t, \mu_t, \lambda_t) = V_t$ . The following describes the value function and its properties. The proof of Theorem 1 is in Appendix B.

**Theorem 1.** *Assume parameter values satisfy Assumption 1. Then the value function  $J$  takes the following form:*

$$J(W_t, \mu_t, \lambda_t) = \frac{W_t^{1-\gamma}}{1-\gamma} I(\mu_t, \lambda_t), \quad (7)$$

where

$$I(\mu_t, \lambda_t) = \exp \{ a + b_\mu^\top \mu_t + b_\lambda^\top \lambda_t \}, \quad (8)$$

for vectors  $b_\mu = [b_{\mu_1}, b_{\mu_2}]^\top$  and  $b_\lambda = [b_{\lambda_1}, b_{\lambda_2}]^\top$ . The coefficients  $a$ ,  $b_{\mu_j}$  and  $b_{\lambda_j}$  for  $j = 1, 2$

take the following form:

$$a = \frac{1-\gamma}{\beta} \left( \bar{\mu}_C - \frac{1}{2}\gamma\sigma^2 \right) + (1-\gamma) \log \beta + \frac{1}{\beta} b_\lambda^\top (\kappa_\lambda * \bar{\lambda}) \quad (9)$$

$$b_{\mu_j} = \frac{1-\gamma}{\kappa_{\mu_j} + \beta}, \quad (10)$$

$$b_{\lambda_j} = \frac{1}{\sigma_{\lambda_j}^2} \left( \beta + \kappa_{\lambda_j} - \sqrt{(\beta + \kappa_{\lambda_j})^2 - 2E_{\nu_j} [e^{b_{\mu_j} Z_{jt}} - 1] \sigma_{\lambda_j}^2} \right). \quad (11)$$

Here and in what follows, we use the notation  $*$  to denote element-by-element multiplication of vectors of equal dimension.

As the next corollary shows, an investor is made better off (as measured by the value function), by an increase in the components of expected consumption growth or by an increase in the probability of a boom. The investor is made worse off by an increase in the probability of disaster.

**Corollary 2.** *The value function is increasing in  $\mu_{jt}$  for  $j = 1, 2$ , decreasing in  $\lambda_{1t}$ , and increasing in  $\lambda_{2t}$ .*

**Proof** To fix ideas, consider  $\gamma > 1$ . It suffices to show  $b_{\lambda_1} > 0$ ,  $b_{\lambda_2} < 0$ , and  $b_{\mu_j} < 0$  for  $j = 1, 2$ . It follows immediately from (10) that  $b_{\mu_j} < 0$ . Because  $Z_1 < 0$  and  $b_{\mu_1} < 0$ ,  $E_{\nu_1} [e^{b_{\mu_1} Z_{1t}} - 1] > 0$ . Therefore,

$$\sqrt{(\beta + \kappa_{\lambda_1})^2 - 2E_{\nu_1} [e^{b_{\mu_1} Z_{1t}} - 1] \sigma_{\lambda_1}^2} < \beta + \kappa_{\lambda_1}.$$

It follows that  $b_{\lambda_1} > 0$ . Because  $Z_2 > 0$  and  $b_{\mu_2} < 0$ ,  $E_{\nu_2} [e^{b_{\mu_2} Z_{2t}} - 1] < 0$ . Therefore,

$$\sqrt{(\beta + \kappa_{\lambda_2})^2 - 2E_{\nu_2} [e^{b_{\mu_2} Z_{2t}} - 1] \sigma_{\lambda_2}^2} > \beta + \kappa_{\lambda_2}$$

and  $b_{\lambda_2} < 0$ . □

The riskfree rate takes a particularly simple form:

**Corollary 3.** *Let  $r_t$  denote the instantaneous risk-free rate in this economy, then  $r_t$  is given by*

$$r_t = \beta + \mu_{Ct} - \gamma\sigma^2. \quad (12)$$

## 2.3 The aggregate market

Let  $D_t$  denote the dividend on the aggregate market. Assume that dividends follow the process

$$\frac{dD_t}{D_t} = \mu_{D_t} dt + \phi \sigma dB_{C_t}, \quad (13)$$

where

$$\mu_{D_t} = \bar{\mu}_D + \phi \mu_{1t} + \phi \mu_{2t}.$$

This structure allows dividends to respond by a greater amount than consumption to booms and disasters (this is consistent with the U.S. experience, as shown in Longstaff and Piazzesi (2004)). For parsimony, we assume that the parameter, namely,  $\phi$ , governs the dividend response to normal shocks, booms and disasters. This  $\phi$  is analogous to leverage in the model of Abel (1999), and we will refer to it as leverage in what follows.

### 2.3.1 Prices

We price equity claims using no-arbitrage and the state-price density. Duffie and Skiadas (1994) show that the state-price density  $\pi_t$  equals

$$\pi_t = \exp \left\{ \int_0^t \frac{\partial}{\partial V} f(C_s, V_s) ds \right\} \frac{\partial}{\partial C} f(C_t, V_t). \quad (14)$$

Let  $H(D_t, \mu_t, \lambda_t, \tau)$  denote the time  $t$  price of a single future dividend payment at time  $t + \tau$ . Then

$$H(D_t, \mu_t, \lambda_t, s - t) = E_t \left[ \frac{\pi_s}{\pi_t} D_s \right].$$

The following corollary gives the solution for  $H$  up to ordinary differential equations. This corollary is a special case of Theorem B.2, given in Appendix B.4.

**Corollary 4.** *The solution for the function  $H$  is as follows*

$$H(D_t, \mu_t, \lambda_t, \tau) = D_t \exp \{ a_\phi(\tau) + b_{\phi\mu}(\tau)^\top \mu_t + b_{\phi\lambda}(\tau)^\top \lambda_t \}, \quad (15)$$



where  $b_{\phi\mu}(\tau) = [b_{\phi\mu_1}(\tau), b_{\phi\mu_2}(\tau)]^\top$  and  $b_{\phi\lambda}(\tau) = [b_{\phi\lambda_1}(\tau), b_{\phi\lambda_2}(\tau)]^\top$ . Furthermore, for  $j = 1, 2$ ,

$$b_{\phi\mu_j}(\tau) = \frac{\phi - 1}{\kappa_{\mu_j}} (1 - e^{-\kappa_{\mu_j}\tau}), \quad (16)$$

while  $b_{\phi\lambda_j}(\tau)$  (for  $j = 1, 2$ ) and  $a_\phi(\tau)$  satisfy the following:

$$\frac{db_{\phi\lambda_j}}{d\tau} = \frac{1}{2}\sigma_{\lambda_j}^2 b_{\phi\lambda_j}(\tau)^2 + (b_{\lambda_j}\sigma_{\lambda_j}^2 - \kappa_{\lambda_j}) b_{\phi\lambda_j}(\tau) + E_{\nu_j} \left[ e^{b_{\mu_j}Z_{jt}} \left( e^{b_{\phi\mu_j}(\tau)Z_{jt}} - 1 \right) \right] \quad (17)$$

$$\frac{da_\phi}{d\tau} = \bar{\mu}_D - \bar{\mu}_C - \beta + \gamma\sigma^2(1 - \phi) + b_{\phi\lambda}(\tau)^\top (\kappa_\lambda * \bar{\lambda}) \quad (18)$$

with boundary conditions  $b_{\phi\lambda_j}(0) = a_\phi(0) = 0$ .

Let  $F(D_t, \mu_t, \lambda_t)$  denote the value of the market portfolio (namely, the price of the claim to the entire future dividend stream). Then

$$F(D_t, \mu_t, \lambda_t) = \int_0^\infty H(D_t, \mu_t, \lambda_t, \tau) d\tau.$$

Corollary 4 implies that the price-dividend ratio, which we will denote by a function  $G$ , can be written as

$$G(\mu_t, \lambda_t) = \int_0^\infty \exp(a_\phi(\tau) + b_{\phi\mu}(\tau)^\top \mu_t + b_{\phi\lambda}(\tau)^\top \lambda_t) d\tau. \quad (19)$$

The expressions in Corollary 4 show how prices respond to innovations in expected consumption growth and in changing disaster probabilities. Because  $\phi > 1$ , (16) shows that innovations to expected consumption growth increase the price-dividend ratio. The presence of the  $\phi - 1$  term shows that this is a trade-off between the effect of expected consumption growth on the riskfree rate and on dividend cash flows. In our recursive utility model, the cash flow effect dominates and asset prices fall during disasters and rise during booms. Moreover, the more persistent is the process for the mean (the lower is  $\kappa_{\mu_j}$ ), the greater is the effect of a change in  $\mu_{jt}$  on prices.<sup>7</sup> Finally, an increase in the probability of a disaster lowers the price-dividend ratio, while an increase in the probability of a boom raises it. These effects are summarized in the following corollary.

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<sup>7</sup>The derivative of (16) with respect to  $\kappa_{\mu_j}$  equals  $(\kappa_{\mu_j}\tau + 1)e^{-\kappa_{\mu_j}\tau} - 1$  which is negative, because  $e^{\kappa_{\mu_j}\tau} > \kappa_{\mu_j}\tau + 1$ .

**Corollary 5.** *The price-dividend ratio  $G(\mu_t, \lambda_t)$  is increasing in the components of expected consumption growth  $\mu_{jt}$  (for  $j = 1, 2$ ), decreasing in the probability of a disaster  $\lambda_{1t}$  and increasing in the probability of a boom  $\lambda_{2t}$ .*

The fact that  $G(\mu_t, \lambda_t)$  is increasing in  $\mu_{jt}$  follows immediately from the form of (16). The results for  $\lambda_{1t}$  and  $\lambda_{2t}$  are less obvious. We give a full proof in Appendix B and discuss the intuition here. Consider the ODE (17). The functions  $b_{\phi\lambda_j}(\tau)$  would be identically zero without the last term  $E_{\nu_j} \left[ e^{b_{\mu_j} Z_{jt}} \left( e^{b_{\phi\mu_j}(\tau) Z_{jt}} - 1 \right) \right]$ . It is this term that determines the sign of  $b_{\phi\lambda_j}(\tau)$ , and thus how prices respond to changes in probabilities.

To fix ideas, consider disasters ( $j = 1$ ). The last term in (17) can itself be written as a sum of two terms:

$$E_{\nu_1} \left[ e^{b_{\mu_1} Z_{1t}} \left( e^{b_{\phi\mu_1}(\tau) Z_{1t}} - 1 \right) \right] =$$

$$- \underbrace{E_{\nu_1} \left[ \left( e^{b_{\mu_1} Z_{1t}} - 1 \right) \left( 1 - e^{b_{\phi\mu_1}(\tau) Z_{1t}} \right) \right]}_{\text{Risk premium effect}} + \underbrace{E_{\nu_1} \left[ e^{b_{\phi\mu_1}(\tau) Z_{1t}} - 1 \right]}_{\text{Cash flow and riskfree rate effect}} \quad (20)$$

The first of the terms in (20) is one component of the equity premium, indeed it is what we will refer to as the static disaster premium, terminology that we discuss in more detail in the next section.<sup>8</sup> When the risk of a disaster increases, the static equity premium increases. Because an increase in the discount rate lowers the price-dividend ratio, this term appears in (20) with a negative sign. The second term in (20) is the expected price response in the event of a disaster.<sup>9</sup> It represents the combined effect of the disaster on cash flows and on the riskfree rate. The net effect is negative, as described above. Thus the response of equity values to changes in the probability of a disaster is determined by a risk premium effect, and a (joint) cash flow and riskfree rate effect. Both effects turn out to be negative; our calibration implies that they are roughly of equal magnitude (the full risk premium however is of much greater magnitude since it also includes compensation for time-varying  $\lambda_{1t}$ ). A similar structure holds for booms. However, in the case of booms,

<sup>8</sup>More precisely, this is the static disaster premium for zero-coupon equity with maturity  $\tau$ .

<sup>9</sup>Again, more precisely, it is the price response of zero-coupon equity with maturity  $\tau$ .

the joint riskfree-rate and cash flow effect is positive, and it dominates the risk premium effect.<sup>10</sup>

### 2.3.2 The equity premium

Here, we give an expression for the instantaneous equity premium and discuss its properties. This will be useful in understanding the quantitative results in Section 3.

First, we define the *jump operator*, which denotes how a process responds to an occurrence of a rare event. Namely, let  $X_t$  be any pure diffusion process ( $X_t$  can be a vector), and let  $\mu_{jt}$ ,  $j = 1, 2$  be defined as above. Consider a scalar, real-valued function  $h(\mu_{1t}, \mu_{2t}, X_t)$ . Define the jump operator  $\mathcal{J}$  as follows:

$$\begin{aligned}\mathcal{J}_1(h(\mu_{1t}, \mu_{2t}, X_t)) &= h(\mu_1 + Z_1, \mu_2, X_t) \\ \mathcal{J}_2(h(\mu_{1t}, \mu_{2t}, X_t)) &= h(\mu_1, \mu_2 + Z_2, X_t).\end{aligned}$$

Further, define

$$\bar{\mathcal{J}}_j(h(\mu_{1t}, \mu_{2t}, X_t)) = E_{\nu_j} \mathcal{J}_j(h(\mu_{1t}, \mu_{2t}, X))$$

for  $j = 1, 2$ , and

$$\bar{\mathcal{J}}(h(\mu_{1t}, \mu_{2t}, X_t)) = [\bar{\mathcal{J}}_1(h(\mu_{1t}, \mu_{2t}, X_t)), \bar{\mathcal{J}}_2(h(\mu_{1t}, \mu_{2t}, X_t))]^\top.$$

Using Ito's Lemma and the definition above, we can write the process for the aggregate stock price  $F_t = F(D_t, \mu_t, \lambda_t)$  as follows:

$$\frac{dF_t}{F_{t-}} = \mu_{F,t} dt + \sigma_{F,t} dB_t + \sum_j \frac{\mathcal{J}_j(F_t)}{F_{t-}} dN_{jt}.$$

The instantaneous expected return is the expected change in price, plus the dividend yield:

$$r_t^m = \mu_{F,t} + \frac{D_t}{F_t} + \frac{1}{F_t} \lambda_t^\top \bar{\mathcal{J}}(F_t). \quad (21)$$

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<sup>10</sup>The relative magnitude of these terms can be seen by comparing the risk premiums with the observed expected returns in samples when no jumps occur (namely Figures 5 and 7 with Figures 9 and 10). The term on the left hand side of (20) corresponds to the observed static premium in no-jump samples while the first term on the right hand side corresponds to the static premium in population.

**Corollary 6.** *The equity premium relative to the risk-free rate  $r$  is*

$$r_t^m - r_t = \phi\gamma\sigma^2 - \underbrace{\sum_j \lambda_{jt} E_{\nu_j} \left[ (e^{b_{\mu_j} Z_{jt}} - 1) \frac{\mathcal{J}_j(G_t)}{G_t} \right]}_{\text{static rare event premium}} - \underbrace{\sum_j \lambda_{jt} \frac{1}{G_t} \frac{\partial G}{\partial \lambda_j} b_{\lambda_j} \sigma_{\lambda_j}^2}_{\lambda\text{-premium}}. \quad (22)$$

As Corollary 6 shows, the equity premium is the sum of three terms. The first is the standard term arising from the consumption Capital Asset Pricing Model (CCAPM) of Breenden (1979). The second term is the premium directly attributable to rare events. It arises from the co-movement in prices and in marginal utility when one of these events occurs. We will call this term the *static rare event premium* (we include the negative sign in the definition of the premium). This term can itself be divided into the *static disaster premium* and the *static boom premium*:

$$\begin{aligned} \text{static disaster premium:} & \quad -\lambda_{1t} E_{\nu_1} \left[ (e^{b_{\mu_1} Z_{1t}} - 1) \frac{\mathcal{J}_1(G_t)}{G_t} \right] \\ \text{static boom premium:} & \quad -\lambda_{2t} E_{\nu_2} \left[ (e^{b_{\mu_2} Z_{2t}} - 1) \frac{\mathcal{J}_2(G_t)}{G_t} \right] \end{aligned}$$

If a rare event occurs, instantaneous current dividends do not change, but future dividends do. This is why the formulas above contain the price dividend ratio  $G_t$  (it would also be correct to substitute  $G_t$  with  $F_t$ ). Note that this is the premium that would obtain if the probability of the rare event  $\lambda_{jt}$  were constant. It is for this reason that we refer to these terms as the static rare event premium.<sup>11</sup>

Finally, the third term in (22) represents the compensation the investor requires for bearing the risk of changes in the rare event probabilities (again, the definition should be viewed as including the negative sign). Accordingly, we call this the  $\lambda$ -premium. This term can also be divided into the compensation for time-varying disaster probability (the  $\lambda_1$ -premium) and compensation for time-varying boom probability (the  $\lambda_2$ -premium). Note that under power utility, only the CCAPM term would appear in the risk premium. This is because, in the power utility model, only the instantaneous co-movement with consumption matters for risk premia, not changes to the consumption distribution.

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<sup>11</sup>However, the term “static premium” is somewhat of a misnomer here, since even the direct effect of rare events on the price-dividend ratio is a dynamic one.

We next address the question of how these various terms contribute to the equity premium. The following corollary describes the signs of these terms:

**Corollary 7.** 1. *The static disaster and boom premiums are positive.*

2. *The  $\lambda_1$ -premium (the premium for time-varying disaster probability) is positive. The  $\lambda_2$ -premium (the premium for time-varying boom probability) is also positive.*

**Proof** To show the first statement, recall that  $b_{\mu_j} < 0$  for  $j = 1, 2$  (Corollary 2). First consider disasters ( $j = 1$ ). Note  $Z_1 < 0$ , so  $e^{b_{\mu_1} Z_{1t}} - 1 > 0$ . Furthermore, because  $G$  is increasing in  $\mu_1$  (Corollary 5),  $\mathcal{J}_1(G_t) < 0$ . It follows that the static disaster premium is positive. Now consider booms ( $j = 2$ ). Because  $Z_2 > 0$ ,  $e^{b_{\mu_2} Z_{2t}} - 1 < 0$ . Because  $G$  is increasing in  $\mu_2$ ,  $\mathcal{J}_2(G_t) > 0$ . Therefore the static boom premium is also positive.

To show the second statement, first consider disasters ( $j = 1$ ). Recall that  $b_{\lambda_1} > 0$  (Corollary 2). Further,  $\partial G / \partial \lambda_1 < 0$  (Corollary 5). For booms ( $j = 2$ ), each of these quantities takes the opposite sign. The result follows.  $\square$

The intuitive content of Corollary 7 is that both booms and disasters increase the risk of equities for the representative agent. They do so both because of the direct (static) effect stemming from happens to equities in these events, and because of an indirect (dynamic) effect, due to what happens to equities (as a result of rational forecasts of what *would* happen in these events) during normal times.

It is also useful to consider the return the econometrician would observe in a sample without rare events. We will distinguish these expected returns using the subscript nj (“no jump”). This expected return is simply given by the drift rate in the price, plus the dividend yield

$$r_{\text{nj},t}^m = \mu_{F,t} + \frac{D_t}{F_t}.$$

Based on this definition, the fact that  $\frac{\bar{\mathcal{J}}(F_t)}{F_t} = \frac{\bar{\mathcal{J}}(G_t)}{G_t}$  and on Corollary 6, these expected returns can be calculated as follows:

**Corollary 8.** *The observed expected excess return in a sample without jumps is*

$$r_{nj,t}^m - r_t = \phi\gamma\sigma^2 - \sum_j \lambda_{jt} E_{\nu_j} \left[ e^{b_{\mu_j} Z_{jt}} \frac{\mathcal{J}_j(G_t)}{G_t} \right] - \sum_j \lambda_{jt} \frac{1}{G_t} \frac{\partial G}{\partial \lambda_j} b_{\lambda_j} \sigma_{\lambda_j}^2 \quad (23)$$

This expression differs from (22) in that the contribution directly due to rare events is equal to  $-\sum_j \lambda_{jt} E_{\nu_j} \left[ e^{b_{\mu_j} Z_{jt}} \frac{\mathcal{J}_j(G_t)}{G_t} \right]$  as opposed to  $-\sum_j \lambda_{jt} E_{\nu_j} \left[ (e^{b_{\mu_j} Z_{jt}} - 1) \frac{\mathcal{J}_j(G_t)}{G_t} \right]$ . We will refer to the  $j = 1$  term as the observed static disaster premium in a sample without jumps and the  $j = 2$  term as the observed static boom premium in a sample without jumps.

**Corollary 9.** *The observed static disaster premium in a sample without jumps is positive. The observed static boom premium in a sample without jumps is negative.*

**Proof** The result follows from the fact that  $G$  is increasing in  $\mu_1$  and  $\mu_2$ , and hence  $\mathcal{J}_1(G) < 0$  and  $\mathcal{J}_2(G) > 0$ .  $\square$

Note that the observed disaster premium is positive, just like the true disaster premium. However, the observed boom premium is negative, the opposite sign to the true boom premium.<sup>12</sup>

## 2.4 Growth and value sectors

The value sector is defined as the claim to cash flows that are not subject to the positive jumps, but are otherwise identical to those of the market. We will use the superscript  $v$  to denote processes related to the value sector and the subscript  $g$  to denote processes related to the growth sector. The dividend process for the value sector is as follows:

$$\frac{dD_{t,s}^v}{D_{t,s}^v} = \mu_{D_s}^v ds + \phi\sigma dB_{C_s}, \quad (24)$$

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<sup>12</sup>We refer to these as the observed premiums to distinguish them from the true risk premiums (note that, unlike true risk premiums, they do not in fact represent a return for risk). In practice, it will be nearly impossible to distinguish the separate terms in (23). The terminology “observed static disaster premium” and “observed static boom premium” is used for convenience, not to suggest that these terms can in fact be observed separately from other parts of the expected excess return.

where  $\mu_{Dt}^v = \bar{\mu}_D + \phi\mu_{1t}$ , and with the boundary condition  $D_{t,t}^v = D_t$ . The price of the value sector claim can be determined in the same way as the price of the claim to the aggregate market (see Corollary 10 below).

The growth sector is defined as the residual. Let  $D_{t,s}^g = D_s - D_{t,s}^v$ . Define  $F_{t,s}^g$  to be the price of the growth claim. Then, by the absence of arbitrage,

$$F_{t,s}^g = F_s - F_{t,s}^v.$$

As long as there are no positive jumps, the dividend on the value claim and the aggregate market are identical. However, when a positive jump takes place, the market dividend begins to diverge permanently from the value dividend. The dividend on the value sector will henceforth grow at a lower rate than the aggregate dividend, with the dividend on the growth claim comprising the difference.

In this setting, thinking of the value and the growth claim as long-lived assets would imply a value claim that makes up a vanishingly small portion of the aggregate market as time passes. The asset pricing implications of defining the value claim in this way would not be very interesting. Therefore, we do not think of the value claim as being a long-lived asset (indeed, because markets are complete, the actual assets that are specified do not affect the equilibrium). If one wishes to think of long-lived assets, the following interpretation may be helpful (though note that given that the value and growth claim are priced by no-arbitrage, this interpretation is not necessary): Every time there is a positive jump, the growth sector is disbanded. Some of the capital is used to start a new growth sector, and some goes into the rest of the economy. The value of the claims to the new growth and value sectors are adjusted so that the owners of the previous growth sector still receive the value of the claim to the (previous) growth dividends. In effect, the owners of the growth sector are diluting the owners of the value sector in the event of a positive jump.

### 2.4.1 Prices

Let  $H^v(D_{t,s}^v, \mu_s, \lambda_s, \tau)$  denote the time  $t$  price of a single future value sector dividend payment at time  $s + \tau$ . Recall that  $\pi_t$  is the state-price density, defined in (14). As in the case of the aggregate market,

$$H^v(D_{t,s}^v, \mu_s, \lambda_s, u - s) = E_s \left[ \frac{\pi_u}{\pi_s} D_{t,u}^v \right].$$

Furthermore,

$$F^v(D_{t,s}^v, \mu_s, \lambda_s) = \int_0^\infty H^v(D_{t,s}^v, \mu_s, \lambda_s, \tau) d\tau. \quad (25)$$

The following corollary is a special case of Theorem B.2, given in Appendix B.4.

**Corollary 10.** *The solution for the function  $H^v$  is as follows:*

$$H^v(D_{t,s}^v, \mu_s, \lambda_s, \tau) = D_{t,s}^v \exp \{ a_\phi^v(\tau) + b_{\phi\mu}^v(\tau)^\top \mu_s + b_{\phi\lambda}^v(\tau)^\top \lambda_s \},$$

where  $b_{\phi\mu}^v(\tau) = [b_{\phi\mu_1}^v(\tau), b_{\phi\mu_2}^v(\tau)]^\top$  and  $b_{\phi\lambda}^v(\tau) = [b_{\phi\lambda_1}^v(\tau), b_{\phi\lambda_2}^v(\tau)]^\top$ . Furthermore,

$$b_{\phi\mu_1}^v(\tau) = \frac{\phi - 1}{\kappa_{\mu_1}} (1 - e^{-\kappa_{\mu_1}\tau}) \quad (26)$$

$$b_{\phi\mu_2}^v(\tau) = -\frac{1}{\kappa_{\mu_2}} (1 - e^{-\kappa_{\mu_2}\tau}), \quad (27)$$

while  $b_{\phi\lambda_j}^v(\tau)$  (for  $j = 1, 2$ ) and  $a_\phi(\tau)$  satisfy

$$\frac{db_{\phi\lambda_j}^v}{d\tau} = \frac{1}{2} \sigma_{\lambda_j}^2 b_{\phi\lambda_j}^v(\tau)^2 + (b_{\lambda_j} \sigma_{\lambda_j}^2 - \kappa_{\lambda_j}) b_{\phi\lambda_j}^v(\tau) + E_{\nu_j} \left[ e^{b_{\mu_j} Z_{jt}} \left( e^{b_{\phi\mu_j}^v(\tau) Z_{jt}} - 1 \right) \right], \quad (28)$$

$$\frac{da_\phi^v}{d\tau} = \bar{\mu}_D - \bar{\mu}_C - \beta + \gamma \sigma^2 (1 - \phi) + b_{\phi\lambda_j}^v(\tau)^\top (\kappa_\lambda * \bar{\lambda}) \quad (29)$$

with boundary conditions  $b_{\phi\lambda_j}^v(0) = a_\phi(0) = 0$ .

It follows from (25) and Corollary 10 that the price-dividend ratio on the value sector is

$$G^v(\mu_t, \lambda_t) = \int_0^\infty \exp(a_\phi^v(\tau) + b_{\phi\mu}^v(\tau)^\top \mu_t + b_{\phi\lambda}^v(\tau)^\top \lambda_t) d\tau. \quad (30)$$

The dynamics of this price-dividend ratio are given by the following:



**Corollary 11.** *The price-dividend ratio for the value claim  $G^v(\mu_t, \lambda_t)$  is increasing in  $\mu_{1t}$ , decreasing in  $\mu_{2t}$ , and decreasing in the probability of a rare event  $\lambda_{jt}$ , for  $j = 1, 2$ .*

Though the dividends on the value sector are not exposed to positive jumps, the value sector still depends on  $\mu_{2t}$  and therefore on  $\lambda_{2t}$  because of the effect of  $\mu_{2t}$  on the riskfree rate.

## 2.4.2 Risk premia

Risk premia on the value claim can be derived similarly to those on the aggregate market. As we will see, however, they behave quite differently.<sup>13</sup>

**Corollary 12.** *The value sector premium relative to the risk-free rate  $r$  is*

$$r_t^v - r_t = \phi\gamma\sigma^2 - \sum_j \lambda_{jt} E_{\nu_j} \left[ (e^{b_{\mu_j} Z_{jt}} - 1) \frac{\mathcal{J}_j(G_t^v)}{G_t^v} \right] - \sum_j \lambda_{jt} \frac{1}{G_t^v} \frac{\partial G^v}{\partial \lambda_j} b_{\lambda_j} \sigma_{\lambda_j}^2 \quad (31)$$

The three terms in (31) have an analogous interpretation to those for the market premium, and can also be signed.

**Corollary 13.** *1. The static disaster premium for the value sector is positive.*

*2. The static boom premium for the value sector is negative.*

*3. The  $\lambda_1$ -premium on the value sector is positive.*

*4. The  $\lambda_2$ -premium on the value sector is negative.*

Finally, the following corollary characterizes the observed expected return in a sample without jumps

**Corollary 14.** *The observed expected excess return on the value sector in a sample without jumps is*

$$r_{nj,t}^v - r_t = \phi\gamma\sigma^2 - \sum_j \lambda_{jt} E_{\nu_1} \left[ e^{b_{\mu_j} Z_{jt}} \frac{\mathcal{J}(G_t^v)}{G_t^v} \right] - \sum_j \lambda_{jt} \frac{1}{G_t^v} \frac{\partial G^v}{\partial \lambda_j} b_{\lambda_j} \sigma_{\lambda_j}^2 \quad (32)$$

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<sup>13</sup>The proofs of these results are directly analogous to those for the market, and therefore we do not repeat them.

Both the terms corresponding to disaster and boom risk in this expression are positive. As in the case of the aggregate market, the sign of the disaster component is the same as in the risk premium, while the sign of the boom component is reversed.

**Corollary 15.** *In a sample without jumps, the observed disaster and boom premiums for the value sector are positive.*

The corollaries in this section state that the premiums related to disaster risk (the static disaster premium and the  $\lambda_1$ -premium) are positive for the value sector, just as they are for the aggregate market. The premiums related to boom risk (the static boom premium and the  $\lambda_2$ -premium) are negative for the value sector, though they are positive for the aggregate market. In population, the expected returns on the value sector will therefore be lower than those on the aggregate market. In a sample without jumps, however, this effect may be (and, for reasonable parameter values, will be) reversed. The reason is that the static boom premium switches signs: in a sample without booms, it is negative for the aggregate market, but positive for the value sector. This will produce an observed value premium.

## 3 Quantitative results

### 3.1 Calibration

#### 3.1.1 Data

To calibrate the rare events, we use international consumption data described in detail in Barro and Ursua (2008), and updated by Barro and Ursua to include data on 43 countries. These data contain annual observations on real, per capita consumption; start dates vary from early in the 19th century to the middle of the 20th century.

Our aggregate market data come from CRSP. We define the market return to be the gross return on the value-weighted CRSP index. Dividend growth is computed from the

dividends on this index. The price-dividend ratio is price divided by the previous 12 months of dividends to remove the effect of seasonality in dividend payments (in computing this dividend stream, we assume that dividends on the market are not reinvested). We compute market returns and dividend growth in real terms by adjusting for inflation using changes in the consumer price index (also available from CRSP). For the government bill rate, we use real returns on the 3-month Treasury Bill. We also use real, per capital expenditures on non-durables and services for the U.S., available from the Bureau of Economic Analysis. These data are annual, begin in 1947, and end in 2010. Focusing on post-war data allows for a clean comparison between U.S. data and hypothetical samples in which no rare events take place.

Data on value and growth portfolio are from Ken French's website. CRSP stocks are sorted annually into deciles based on their book-to-market ratios. Our growth claim is an extreme example of a growth stock; it is purely a claim to positive extreme events and nothing else. In the data, it is more likely that growth stocks are a combination of this claim and the value claim. To avoid modeling complicated share dynamics, we identify the growth claim with the decile that has the lowest book-to-market ratio, while the value claim consists of a portfolio (with weights defined by market equity) of the remaining nine deciles. A standard definition of the value spread is the log book-to-market ratio of the value portfolio minus the log book-to-market ratio of the growth portfolio (Cohen, Polk, and Vuolteenaho (2003)). In our endowment economy, book value can be thought of as the dividend. However, the dividend on the growth claim is identically equal to zero (though of course this claim has future non-zero dividends), and for this reason, there is no direct analogue of the value spread. We therefore compute the value spread in the model as the log dividend-price ratio on the value portfolio minus the log dividend-price ratio on the aggregate market. For comparability, we compute the same quantity in the data. Where our non-standard definition might be an issue is our predictability results; we have checked that these results are robust to the more standard data definition.

### 3.1.2 Parameter values

We report parameter values in Table 1. Average consumption growth and the volatility of consumption growth equal their post-war averages over a set of developed countries as in Barro (2006). These are both about 2%. We calibrate dividend growth to be slightly higher: 3.55%. Given the construction of CRSP dividends, there no reason to assume that dividends and consumption should grow at the same rate. Indeed, CRSP dividends do not include repurchases; presumably these imply that dividends are likely to be higher some time in the future, and that the sample mean is not a good indicator of the true mean. For this reason, we choose the mean of the dividend growth distribution that is implied by the level of the price-dividend ratio in the data.

Leverage,  $\phi$ , is chosen to be 3.5. This implies that the volatility of log dividends is 3.5 times that of log consumption. In our data, the ratio is 4.66. However, this value would most likely imply too great a response of dividends to consumption disasters; we therefore choose a smaller and more conservative value. We choose a low rate of time preference to obtain a realistic government bill rate.<sup>14</sup> Relative risk aversion is equal to 3.

The average probability of a disaster is chosen to be 2.86%, which is the value calibrated by Barro and Ursua (2008) for OECD countries.<sup>15</sup> The persistence in the price-dividend ratio is nearly entirely determined by the persistence in the disaster probability. We therefore choose a low rate of mean reversion:  $\kappa_{\lambda_1} = 0.11$ . With this choice, the median small-sample value of the persistence of the price-dividend ratio is 0.78; the value in the data is 0.92. This suggests the possibility of lowering  $\kappa_{\lambda_1}$  still further (which would increase the effect of disaster risk on the equity premium and volatility); however, insisting that the model fit the very large degree of persistence in the data greatly widens the parameter range at which the value function fails to exist. The volatility  $\sigma_{\lambda_1}$  is chosen to be 9.4%, which leads

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<sup>14</sup>Further lowering this value leads there to be no solution to the investor's optimization problem.

<sup>15</sup>We calibrate the size of the disasters to the full set of samples and the average probability to the OECD subsample. In both cases, we are choosing the more conservative measure, because the OECD sub-sample has rarer, but more severe disasters.

to a realistic volatility for the aggregate market.

The disaster distribution, and the mean reversion in the disaster component of the expected consumption growth ( $\kappa_{\mu_1}$ ) are chosen to fit the distribution of consumption declines, reported in Table 2 and the left panels Figures 1 and 2. These results suggest that the consumption growth reverts to its normal level relatively quickly, suggesting a high value for  $\kappa_{\mu_1}$  (we choose 1.0). To calculate the size of the jumps, we assume a power law distribution (see Gabaix (2009) for a discussion of the properties of power law distributions). Following Barro and Ursua (2008), we consider 10% as the smallest magnitude of the disaster. Our calibration procedure suggests a power law parameter of 7 (the lower this parameter, the heavier the tail of the power law). Barro and Jin (2011) find similar results using maximum likelihood.<sup>16</sup> Table 2 also reports the distribution of declines in a model in which all the decline takes place immediately. This model fits the data less well, substantially over-predicting the number of large declines at the one-year horizon.

We follow a similar strategy for booms (data for large positive consumption events are reported in Table 3 and the right panels of Figures 1 and 2). Parameter values for the  $\lambda_{2t}$  process are chosen to fit the mean, volatility, and mean reversion in the value spread as reported in Table 7. Booms in the data do not seem to be as heavy-tailed as disasters, but they die out somewhat more slowly. We choose a minimum value of 5%, a mean reversion coefficient of 0.60, and a power law parameter of 20. Our results are not sensitive to the precise choices of these parameter values.

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<sup>16</sup>Barro and Jin (2011) estimate that the power law parameter is 6.86%. They also argue that the distribution is better characterized by a double power law, with a lower exponent for larger disasters. In this sense our choice of a single power with a coefficient of 7 is conservative.

## 3.2 Prices and expected returns as functions of the state variables

### 3.2.1 Prices

Figures 3 and 4 show terms in the expressions for the price-dividend ratio on the market (19) and the corresponding quantity for the value claim (30). These expressions are an integral of exponential-linear terms. Each of these terms can be interpreted as the ratio of the price of a zero-coupon equity claim to the current dividend. The integral is over  $\tau$ , which can be interpreted as the maturity of these claims. Figure 3 shows the functions  $b_{\phi\mu_j}(\tau)$  and  $b_{\phi\mu_j}^v(\tau)$  as a function of  $\tau$ , and Figure 4 does the same for the functions  $b_{\phi\lambda_j}(\tau)$  and  $b_{\phi\lambda_j}^v(\tau)$ . The persistence of the state variables, combined with the effect of the duration of the claims implies that the magnitude of these functions is increasing in  $\tau$ , as the figures show.

We first discuss the effect of variation in the mean of consumption on the price-dividend ratios. It is useful to discuss this first, because the effect of  $\mu$  on the price is ultimately what determines the effect of  $\lambda$ . Note that both  $b_{\phi\mu_1}(\tau)$  and  $b_{\phi\mu_2}(\tau)$  are positive, reflecting the fact that the market is exposed to both positive and negative jumps in dividend growth. Greater average dividend growth, whether it arises from the absence of a disaster or the presence of a boom, increases the price-dividend ratio. Both terms converge to their limits in a relatively short time, reflecting the fact that neither booms nor disasters are highly persistent in the model. The fact that  $b_{\phi\mu_2}(\tau)$  takes longer to converge reflects the greater persistence of booms than disasters, as does the fact that  $b_{\phi\mu_2}(\tau)$  is larger in magnitude than  $b_{\phi\mu_1}(\tau)$  (because in fact the distribution of immediate responses is larger for disasters than for booms).

The response of the value claim to disasters, reflected in  $b_{\phi\mu_1}^v(\tau)$  is nearly the same as that of the market as a whole. However, the response to booms is quite different. The reason, of course, is that the cash flows on the value claim are not exposed to booms. Indeed, the price of the value claim is decreasing in  $\mu_{2t}$  because of the effect of  $\mu_{2t}$  in

interest rates. As explained above, the price response to  $\mu_{jt}$  is determined by the tradeoff between the cash flow effect and the interest rate effect. Because  $\phi > 1$ , the cash-flow effect dominates for both types of shocks for the aggregate market. For the value claim, the cash-flow effect dominates for  $\mu_{1t}$ . However, there is no cash flow effect for  $\mu_{2t}$  on the value claim (this can be seen by comparing Equation 26 with 27; the first of these terms has a  $\phi$  while the second does not). Thus the riskfree rate effect implies that an increase in expected consumption growth arising from booms decreases the price of the value claim.

Figure 4 shows the functions  $b_{\phi\lambda_j}(\tau)$  (which multiply  $\lambda_{jt}$  in the expression for the market price-dividend ratio) and  $b_{\phi\lambda_j}^v(\tau)$  (which multiply  $\lambda_{jt}$  in the expression for the price-dividend ratio on the value claim).  $b_{\phi\lambda_1}(\tau)$  and  $b_{\phi\lambda_1}^v(\tau)$  are negative, implying that an increase in the probability of a disaster lowers prices. These coefficients are similar, though slightly greater in magnitude for the market portfolio because of the greater duration of this claim.

Note first that  $b_{\phi\lambda_2}(\tau)$  is positive, implying that an increase in the probability of a boom increases the value of the market. The magnitude of this effect is about half the size of that of disasters. The reason is that there is asymmetry in the value function regarding booms and disasters. Consider the last term in (17):  $E_{\nu_j} \left[ e^{b_{\mu_j} Z_{jt}} \left( e^{b_{\phi\mu_j}(\tau) Z_{jt}} - 1 \right) \right]$ . The magnitude of this function is determined in large part by this relatively simple expression. For booms, the term  $b_{\mu_j} Z_{jt}$  is negative, implying that the immediate effect of a positive jump on prices, given by  $e^{b_{\phi\mu_j}(\tau) Z_{jt}} - 1$ , is scaled down.<sup>17</sup> For disasters, however,  $b_{\mu_j} Z_{jt}$  is positive, implying that the effect of a negative jump is scaled up. Finally, an increase in the probability of a boom decreases the price of the value claim, because of the riskfree rate effect described above.

### 3.2.2 Risk premia

Figures 5–10 decompose risk premia into the compensation for the various sources of risk in the economy. These results are useful for understanding the simulation results that follow.

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<sup>17</sup>The expression  $e^{b_{\phi\mu_j}(\tau) Z_{jt}} - 1$  gives the percent change in the price of zero-coupon equity with maturity  $\tau$ .

Figure 5 shows the equity premium (left panel) and the risk premium on the value claim (right panel) as a function of the probability of a disaster. The risk premium (defined in Section 2.3.2) represents the expected instantaneous return on the asset less the riskfree rate.

In both panels, the solid line shows the full risk premium. This can be decomposed into the static rare event premium and the  $\lambda$ -premium (the compensation for time-varying risk of rare events). The static rare event premium is shown by the dashed line. The static rare event premium can itself be decomposed into the static boom premium and the static disaster premium. The static disaster premium is shown by the dashed-dotted line. Finally, there is the premium for risk in consumption in normal times, as would obtain in the CCAPM (dashed line).

As Figure 5 shows, the CCAPM premium is negligible, not surprisingly, given the low value of risk aversion. Both the static rare event premium and the full premium are increasing in the probability of a disaster. While the static rare event premium is substantial, the full premium is more than twice as large, indicating that the risk of time-varying rare event risk is important. For the market portfolio, the static disaster premium lies below the rare event premium, indicating that the static boom premium is positive; however for the value claim it is negative. In both cases, it is small in comparison with the other components (at least when the probability of a boom is fixed at its mean). The static boom premium arises from the co-movement of marginal utility and prices during rare events. Holding all else equal, marginal utility changes less in response to a boom than to a disaster.

Figure 6 shows the total  $\lambda$ -premium and the disaster component, the  $\lambda_1$ -premium. As this figure shows, nearly the entire  $\lambda$ -premium is accounted for by disaster risk. The  $\lambda_2$ -premium is negligible. Why this difference? Recall that the  $\lambda_1$ -premium is given by

$$\underbrace{-b_{\lambda_1}}_{\text{price of risk}} \times \underbrace{\frac{1}{G} \frac{\partial G}{\partial \lambda_1} \sigma_{\lambda_1}^2 \lambda_{1t}}_{\text{risk loading}}.$$

An analogous expression holds for the  $\lambda_2$ -premium. From evaluating the terms in this



expression, we see that two forces contributing to make the compensation for time-varying disasters much greater than for booms. First, the price of risk for time-varying disasters is much larger in magnitude;  $b_{\lambda_1}$  is 11.7, while  $b_{\lambda_2}$  is -3.9. Second, changes in the probability of disaster have a much greater effect on the price-dividend ratio than do changes in the probability of a boom (that is,  $\partial G/\partial\lambda_1$  is about twice the magnitude of  $\partial G/\partial\lambda_2$ ).

Figures 7 and 8 repeat Figures 5 and 6, except that risk premia are shown as functions of the probability of a boom. The main conclusion from these figures is the same; except when the probability of a boom is very high, the booms have little contribution to risk premiums.

It is tempting to conclude from this analysis that the presence of booms will have little impact on the cross-section of asset returns. However, while booms have a relatively small impact on true risk premia, their impact on observed risk premia can be large. Whether the sample contains jumps or not makes little difference for disasters, as comparing Figure 5 with Figure 9 shows. However, for booms, the difference is substantial. Because the  $\lambda_2$ -premium is the same (for a given value of the state variables) regardless of whether booms take place or not, the entire difference must arise from the static boom premium. As shown in Sections 2.3.2 and 2.4.2, the static boom premium switches sign, depending on whether booms are observed or not: In population, the boom premium is positive. However, this value is more than entirely due to the realized return should a boom take place. In normal times, the investors receive a lower-than-average return. Figure 10 shows that, for the market, the premium for booms lowers the equity premium by 1% (per annum) when the probability is at its average value, and possibly much more as the probability of a boom increases. Because the value claim is only exposed to boom risk through the effect on discount rates, the effect is much smaller and in the opposite direction.

### 3.3 Simulation results

In what follows, we consider the population and small-sample properties of the model. Both require a stationary distribution for the rare event probabilities. We show this stationary distribution in Figure 11. The solid line shows the probability density function for the disaster probability  $\lambda_1$ , while the dashed line shows the probability density function for the boom probability  $\lambda_2$ . The mean of the disaster probability is greater, as can be seen from the fact that the solid line lies above the dotted line for most of the relevant range. However, the boom probability is more skewed; the chance of unusually high values of the probability is greater for booms than for disasters. This can be seen from the fact that, for the tail of the distribution, the dashed line lies above the solid line.<sup>18</sup>

To evaluate the quantitative success of the model, we simulate monthly data for 600,000 years, and also simulate 10,000 60-year samples. For each sample, we initialize the  $\lambda_{jt}$  processes using a draw from the stationary distribution. In the tables, we report population values for each statistic, percentile values from the small-sample simulations, and percentile value for the subset of small-sample simulations that do not contain jumps. It is this subset of simulations that is the most interesting comparison for postwar data.

#### 3.3.1 The aggregate market

Table 4 reports moments of log growth rates of consumption and dividends. There is little skewness or kurtosis in postwar annual consumption data.<sup>19</sup> Postwar dividend growth exhibits somewhat more skewness and kurtosis. The simulated paths of consumption and dividends for the no-jump samples are, by definition, normal, and the results reflect this. However, the full set of simulations does show significant non-normality; the median kurtosis is seven for consumption and dividend growth. Kurtosis exhibits a substantial small-

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<sup>18</sup>As Cox, Ingersoll, and Ross (1985) discuss, the stationary distribution for  $\lambda_{jt}$  is Gamma with shape parameter  $2\kappa_j\bar{\lambda}_j/\sigma_{\lambda_j}^2$  and scale parameter  $\sigma_{\lambda_j}^2/(2\kappa_j)$ . This characterization simplifies drawing from the stationary distribution.

<sup>19</sup>In the definition of kurtosis that we use, three is the value for the normal distribution.

sample bias. The last column of the table reports the population value of this measure, which is 37.

Table 5 reports simulation results for the aggregate market. The model is capable of explaining most of the equity premium: the median value among the simulations with no disaster risk is 4.8%; in the data it is 7.2%. Moreover, the data value is below the 95th percentile of the values drawn from the model indicating the data value is not high enough to reject the model at the 10% level.

Several other recent papers note that the equity premium can be explained by allowing for consumption disasters. However, this paper departs from most of the literature in that the disasters are to expected rather than realized consumption growth. Our results thus speak to a debate concerning whether properly accounting for the smoothness of consumption growth, and the multiperiod nature of disasters, greatly reduces their effect. Barro (2006) calibrates the disaster sizes using a peak-to-trough measure of disasters. In the data, these disasters typically unfold over several years. Barro's model, and that used by a number of subsequent papers treats the disasters as occurring instantaneously. Constantinides (2008) and Julliard and Ghosh (2011) show that if instead the annual declines in consumption are used, the disasters explain only a small portion of the equity premium. In effect, converting the disasters to annual from multiperiod increases their frequency, but greatly reduces their size. Further increasing the frequency to monthly and beyond further reduces the effect. This debate recalls earlier concerns raised in response to the rare disaster model of Rietz (1988) (see Mehra and Prescott (1988)).

To understand this debate, it is necessary to distinguish between two different ways of confronting the problem of the different frequency of consumption and returns. One response is to model both the consumption data and the returns as occurring at the same frequency. Indeed, Barro (2006) notes that changing the frequency at which returns are measured has very little effect on the model's ability to explain the equity premium. That is, if one's goal is to explain long-horizon returns using long-horizon consumption growth,

the disaster risk model is successful.<sup>20</sup> There are some drawbacks, however. Most of the literature focuses on the equity premium that is observable at short horizons. More importantly, explaining long-horizon returns in this way implicitly assumes a decision interval for agents that spans several years. This is not realistic.

A second response is to explicitly model the consumption declines as taking place over several periods, while allowing a realistically short decision interval. If one assumes that consumption growth is iid, but that there are more, smaller, disasters, then certainly it is difficult to explain the equity premium as noted above. If one considers these consumption declines as happening together, a power utility model with leverage below risk aversion would actually have greater difficulty in explaining the equity premium than in the iid case, because prices rise when further consumption declines become more likely. Equity thereby becomes a disaster hedge.<sup>21</sup>

How can one reconcile the fact that the model can explain multi-year returns (assuming a buy-and-hold investor) but not single-year returns (assuming an investor who can trade at realistic intervals)? Moreover, it seems odd, intuitively, that agents would not somehow take into account that disaster-years occur together. In fact, this result is a knife-edge property of power utility. Moving beyond power utility, even slightly (as in this paper; risk aversion and the EIS are not very different) implies that the agent takes more than just the instantaneous innovation to consumption growth into account when pricing assets. Indeed as Hansen (2012) notes, the recursive utility investor takes the long run into account when pricing assets, similarly to the power utility investor with a long decision interval. Thus by making consumption smooth and allowing disasters to unfold slowly, we offer a plausible description of consumption dynamics that confronts the problem raised by Constantinides

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<sup>20</sup>Constantinides (2008) discusses this precise issue. However, in his equation that addresses the long-horizon return and consumption growth problem, he does not take into account the fact that reducing the frequency raises the probability of disasters; for example, going from one to three years increases the probability of a disaster by a factor of three.

<sup>21</sup>This point is made in various contexts by Gourio (2008), Nakamura, Steinsson, Barro, and Ursua (2011) and Wachter (2012).

(2008) and others, but we can still explain a substantial fraction of the equity premium.

Before moving on to the cross-section, we note two limitations to the model's fit to the data. First, the government bond yield in the model is higher than in the data (2.9% vs. 1.25%). This fit could be improved by allowing a fraction of the disaster to hit consumption immediately (or a larger fraction than in the present calibration to hit within the first three months). In fact, results reported in Table 2 suggest that this might better fit the behavior of disasters in the data, and, provided that the fraction of the disaster that hits instantaneously would be relatively small, would not raise concerns regarding the discussion of consumption smoothness above. This effect would be straightforward to implement in the model, but would substantially complicate the notation and exposition without changing any of the underlying economics. We should also note that Treasury bill returns may in part reflect liquidity at the very short end of the yield curve (Longstaff (2000)); the model does a better job of explaining the return on the one-year bond.<sup>22</sup> Second, while the model can account for a substantial fraction of the volatility of the price-dividend ratio (the volatility puzzle, reviewed in Campbell (2003)), it cannot explain all of it, at least if we take the view that the postwar series in a sample without rare events. This is a drawback that the model shares with other models attempting to explain aggregate prices using time-varying moments (see the discussion in Bansal, Kiku, and Yaron (2012) and Beeler and Campbell (2012)) but parsimoniously-modeled preferences. It arises from strong general equilibrium effects: time-varying moments imply cash flow, riskfree rate, and risk premium effects, and one of these generally acts as an offset to the other two, limiting the effect time-varying moments have on prices. One possible response is that some behavior of the prices (i.e. the "bubble" in the late 1990s) may be beyond the reach of this type of model. Certainly this is a fruitful area for further research.

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<sup>22</sup>The model predicts a near-zero volatility for returns on this bill in samples without disasters. This is not a limitation, since the volatility in returns in the data is due to inflation, which is not captured in the model.

### 3.3.2 Unconditional moments of value and growth portfolios

Table 6 reports cross-sectional moments. Recall that the data moments are constructed using the growth portfolio as the top decile formed by sorting on book-to-market and the value portfolio as the remaining nine deciles. The resulting difference between the value and the growth portfolio is 1.34%. In samples without jumps, the model easily accounts for this difference; the median value is in fact 2.16%. The higher expected return does not come about because of an increase in volatility: the standard deviation of returns on the value portfolio in the model is in fact far lower than the standard deviation of growth returns. Moreover, the model correctly captures the relative Sharpe ratios of value and growth, as well as the Sharpe ratio on the value-minus-growth strategy. In population, the value premium is negative because growth stocks are in fact more risky than value in the model. However, this population number is not necessarily relevant for calibration in a rare events model; among the full set of simulated paths, the 95 percent critical value of the value premium is 3.35%, far above what is measured in the data. If the value premium does not represent a return for risk, what in the model makes it arise? As explained in Sections 2.4.2 and 3.2.2, it is because investors are willing to accept a lower return on growth in most periods, in return for an occasional very high payout.<sup>23</sup>

The model also exhibits negative alphas for the growth portfolio and positive alphas for the value portfolio. The betas for the growth portfolio are above one, while the betas for the value portfolio are below one. Both of these patterns are what is found in the data, and both represent a point of difficulty for many general equilibrium models of the cross-section. Indeed, the alphas and betas for the growth portfolio are more extreme than in the data.<sup>24</sup>

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<sup>23</sup>A value premium can also be observed in many, but not all developed economies, as reported by Fama and French (1992). Over their relatively short sample period, like the U.S., these countries do not appear to have experienced large booms.

<sup>24</sup>In this sense, the model is in fact too successful in explaining the relatively betas and volatilities of value and growth stocks. A realistic extension of the model might involve a channel by which value stocks would become more volatile and growth less volatile. One such channel could be that value stocks have greater declines in disasters. This would of course increase the value premium. We have chosen not to

This reflects the extreme nature of the growth portfolio in the model. Interestingly, the pattern for alphas and betas does not just characterize the median sample in the no-jump simulations, it also characterizes the median sample in the full set of simulations, as well as in population. Thus, unlike the value premium, the results do not arise from in-sample biases.

The discussion of prices and risk premia in Sections 3.2.1 and 3.2.2 is useful in understanding why the betas on growth stocks are above one, and why the alphas are negative. First note that growth stocks are quite volatile because they account for the entire market's loading on the risk of booms. In the model, growth represents a highly levered claim on the innovations of the economy. Risk premia, on the other hand, arise almost entirely from disasters. They arise both from the co-movement of marginal utility and asset prices during disasters themselves, and from the covariance of asset prices with the risk of disasters during normal times. There is a large endogenous asymmetry between the effects of disasters and booms, stemming from the fact that the investor's marginal utility is relatively insensitive to positive events. Thus growth stocks have high volatility, but not the kind of volatility that leads to risk premia.

### 3.3.3 Return predictability

In a recent survey, Cochrane (2011) notes that time-varying risk premia are a common feature across asset classes. However, variables that predict excess returns in one asset class often fail in another, suggesting that more than one economic mechanism lies behind this common predictability.<sup>25</sup> For example, as the tables below show, the price-dividend ratio is a significant predictor of aggregate market returns, but fails to predict the value-minus-growth return. On the other hand, the value spread predicts the value-minus-growth

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model this mechanism here to focus attention on the effect of rare booms on the cross-section. However, combining the mechanism that we introduce here, with other mechanisms that increase the value premium by increasing risk would be of interest for future work.

<sup>25</sup>Lettau and Wachter (2011) show that if a single factor drives risk premia, then population values of predictive coefficients should be proportional across asset classes.

return, but it is less successful than the price-dividend ratio at predicting the aggregate market return.

Table 8 shows the results of regressing the aggregate market portfolio return on the price-dividend ratio in the actual and simulated data. Not surprisingly given earlier work (Wachter (2012)), the model can reproduce the data finding that the price-dividend ratio predicts excess returns. This result arises from the fact that a high value of the disaster probability is followed, on average, high returns, because a higher than average premium compensates investors for taking on greater risk. As described above, a high disaster probability also pushes down the price-dividend ratio. A time-varying boom probability lowers the effect of predictability, since in a sample without jumps, times of higher-than-average boom probabilities signify lower-than-average returns. However, this effect is not large enough to overturn the effect of disasters. Note that in the full set of simulations, predictability is still present, but it is smaller. This is because more of the variance of stock returns arises from the (more volatile) realized dividends during these periods. In population, the magnitude of predictability is smaller, reflecting the well-known small-sample bias in predictive regressions.

In the data, the market return can also be predicted by the value spread, though with substantially smaller  $t$ -statistics and  $R^2$  values (Table 9). The model also captures the sign and the relative magnitude of this predictability; in a sample without jumps, the median  $R^2$  is 3% at the 1-year horizon, compared with a data value of 5%. The coefficient implies that high realizations of the value spread are associated with low future market returns. Like the price-dividend ratio, the value spread is a function of the probability of disaster, so the intuition above goes through in this case. The reason is that the market is somewhat more sensitive to changes in disaster risk than the value spread (though the cash flow effects are similar) because of its greater duration. Thus the price of the value claim declines by less than the price of the market when the risk of a disaster rises. Of course, the value spread is also determined by the boom probability, which has minimal effects on the market expected return. This is why the  $R^2$  values are much lower in this case.



Table 10 shows that, in contrast to the market portfolio, the value-minus-growth return cannot be predicted by the price-dividend ratio. The data coefficient is positive and insignificant. This fact represents a challenge for models that seek to simultaneously explain market returns and returns in the cross-section since the forces that explain time-variation in the equity premium also lead to time-variation in the value premium (e.g. Lettau and Wachter (2011), Santos and Veronesi (2010)); this reasoning would lead the coefficient to be negative. The present model does, however, predict a positive coefficient. A high value of the price-dividend ratio on the market indicates a relatively high probability of a boom. In samples without rare events, the return on growth will be lower than the return on value when the boom probability is high. In the population, the coefficient is negative (and quite small); times of high  $\lambda_2$  precede periods of high returns on growth when jumps occur with their proper frequency.<sup>26</sup>

One might think that the reason that the value-minus-growth return cannot be predicted by the price-dividend ratio is that it is not very predictable. This is, however, not the case. Table 11 shows that, as in the data, the value spread predicts the value-minus-growth return with a positive sign in samples without jumps. The median  $R^2$  value at a 1-year horizon is 9%, compared with a data value of 10%. At a 5-year horizon, the value in the model is 34%, it is 21% in the data. The intuition is the same as for the regressions on the price-dividend ratio. When the probability of a boom is high (but the boom does not occur), the realized return on value is high relative to growth. The  $R^2$  values are much higher than for the price-dividend ratio because the value spread is primarily driven by the probability of a boom, while the price-dividend ratio is only driven by this probability to

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<sup>26</sup> The median coefficient across all simulations is also positive, on account of small-sample bias. This bias arises from the negative correlation between shocks to the price-dividend ratio and shocks to the value-minus-growth return. Shocks to the disaster probability decrease the price-dividend ratio; both value and growth returns fall, but growth falls by more because of its higher duration. Shocks to the boom probability increase the price-dividend ratio; value returns fall but growth returns rise. This bias is conceptually the same as for regressions of the market portfolio on the price-dividend ratio (see Stambaugh (1999)), but, because the correlation is negative rather than positive, it is in the opposite direction.

a small extent.<sup>27</sup>

To summarize, the joint predictive properties of the price-dividend ratio and the value spread would be quite difficult to explain with a model in which single factor drives risk premia; they therefore constitute independent evidence of a multiple-factor structure of the kind presented here.

## 4 Conclusion

This paper has addressed the question of how growth stocks can have both low returns and high risk, as measured by variance and covariance with the market portfolio. It does so within a framework that is also consistent with what we know about the aggregate market portfolio; namely the high equity premium, high stock market volatility, and time-variation in the equity premium. The problem can be broken into two parts: why is the expected return on growth lower, and why is the abnormal return relative to the CAPM negative? This latter question is important, because one does not want to increase expected return through a counterfactual mechanism.

This paper answers the first of these questions as follows: Growth stocks have, in population, a slightly higher expected return. In finite samples, however, this return may be measured as lower. The answer to the second question is different, because the abnormal return relative to the CAPM appears both in population and samples characterized by a value premium. The abnormal return result arises because risk premia are determined by two sources of risk, each of which is priced very differently by the representative agent. Covariance during disasters, and covariance with the changing disaster probability is assigned a high price by the representative agent because marginal utility is low in these states.

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<sup>27</sup>In population, the effect works in the opposite direction because high values of the boom probability predict low returns on value relative to growth. The resulting  $R^2$  coefficients are very small. For the set of all simulations, the median coefficient is again positive because of small-sample bias, as explained in footnote 26.

However, growth stock returns are highly influenced by booms, and by the time-varying probability of booms. Because marginal utility is low in boom states, the representative agent does not require compensation for holding this risk. This two-factor structure is also successful in accounting for the joint predictive properties of the market portfolio and of the value-minus-growth return.

A number of extensions of the present framework are possible. In this paper, we have specified the growth and the value claim in a stark manner. Extending our results to a setting with richer firm dynamics would allow one to answer a broader set of questions. Further, we have chosen a relatively simple specification for the latent variables driving the economy. An open question is how the specification of these variables affects the observable quantities. We leave these interesting topics to future research.

# Appendix

## A Required conditions on the parameters

**Assumption 1.**

$$(\kappa_{\lambda_j} + \beta)^2 \geq 2\sigma_{\lambda_j}^2 E_{\nu_1} [e^{b_{\mu_j} Z_j} - 1] \quad j = 1, 2.$$

**Assumption 2.**

$$(b_{\lambda_2} \sigma_{\lambda_2}^2 - \kappa_{\lambda_2})^2 \geq 2\sigma_{\lambda_2}^2 E_{\nu_2} \left[ e^{b_{\mu_2} Z_2} \left( e^{\frac{\phi-1}{\kappa_{\mu_2}} Z_2} - 1 \right) \right].$$

**Assumption 3.**

$$\bar{\mu}_D - \bar{\mu}_C - \beta + \gamma \sigma^2 (1 - \phi) - \sum_j \frac{\kappa_{\lambda_j} \bar{\lambda}_j}{\sigma_{\lambda_j}^2} \left( \zeta_{\phi_j} - \kappa_{\lambda_j} + b_{\lambda_j} \sigma_{\lambda_j}^2 \right) < 0,$$

where

$$\zeta_{\phi_j} = \sqrt{(b_{\lambda_j} \sigma_{\lambda_j}^2 - \kappa_{\lambda_j})^2 - 2E_{\nu_j} \left[ e^{b_{\mu_j} Z_j} \left( e^{\frac{\phi-1}{\kappa_{\mu_j}} Z_j} - 1 \right) \right]} \sigma_{\lambda_j}^2.$$

Assumption 1 is required for the solution for  $J(W_t, \mu_t, \lambda_t)$  to be real-valued. Assumption 2 is required for  $b_{\phi \lambda_2}(\tau)$  to converge as  $\tau$  approaches infinity. Without this assumption, the price-dividend ratio market does not have a finite solution. Note that the analogous condition for  $j = 1$  is satisfied automatically because  $Z_1 < 0$  and hence  $e^{\frac{\phi-1}{\kappa_{\mu_1}} Z_1} < 1$ . Furthermore, the analogous condition for the value claim is satisfied automatically; this condition replaces  $e^{\frac{\phi-1}{\kappa_{\mu_2}} Z_2}$  with  $e^{-\frac{1}{\kappa_{\mu_2}} Z_2}$  which is less than one. Assumption 3 states that the asymptotic slope of  $a_\phi(\tau)$  is negative. This is required for convergence of the price-dividend ratio on the market. If this condition is satisfied, the analogous condition for the value function is satisfied automatically.<sup>28</sup>

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<sup>28</sup>Specifically, define

$$\zeta_{\phi_2}^v = \sqrt{(b_{\lambda_2} \sigma_{\lambda_2}^2 - \kappa_{\lambda_2})^2 - 2E_{\nu_2} \left[ e^{b_{\mu_2} Z_2} \left( e^{-\frac{1}{\kappa_{\mu_2}} Z_2} - 1 \right) \right]} \sigma_{\lambda_2}^2$$

Then  $\zeta_{\phi_2}^v > \zeta_{\phi_2}$ .

## B Detailed derivation of the model

This Appendix derives the results given in the main text. The derivations generalize those in Wachter (2012), where there is a single disaster probability, and the shocks are to realized consumption growth. In what follows, there are two time-varying jump probabilities, and, more importantly, the jumps are in expected consumption growth. Like the results in the earlier paper, the derivations here assume that the EIS parameter is equal to one, and, based on this assumption, lead to solutions that are in closed-form up to a system of ordinary differential equations.<sup>29</sup>

### B.1 Notation

Let  $X_t$  be a pure diffusion process, and let  $\mu_{jt}$ ,  $j = 1, 2$  be defined as above. Consider a scalar, real-valued function  $h(\mu_{1t}, \mu_{2t}, X_t)$ . Define

$$\begin{aligned}\mathcal{J}_1(h(\mu_{1t}, \mu_{2t}, X_t)) &= h(\mu_1 + Z_1, \mu_2, X_t) \\ \mathcal{J}_2(h(\mu_{1t}, \mu_{2t}, X_t)) &= h(\mu_1, \mu_2 + Z_2, X_t)\end{aligned}$$

Further, define

$$\bar{\mathcal{J}}_j(h(\mu_{1t}, \mu_{2t}, X_t)) = E_{\nu_j} \mathcal{J}_j(h(\mu_{1t}, \mu_{2t}, X_t))$$

for  $j = 1, 2$ , and

$$\bar{\mathcal{J}}(h(\mu_{1t}, \mu_{2t}, X_t)) = [\bar{\mathcal{J}}_1(h(\mu_{1t}, \mu_{2t}, X_t)), \bar{\mathcal{J}}_2(h(\mu_{1t}, \mu_{2t}, X_t))]^\top.$$

In what follows, we will use the notation  $*$  to denote element-by-element multiplication for two vectors of equal length. We will use  $x^2$  notation for a vector  $x$  to denote the square of each element in  $x$ . For example,  $\sigma_\lambda^2$  will denote the vector  $[\sigma_{\lambda_1}^2, \sigma_{\lambda_2}^2]^\top$ .

Finally, because the process  $\lambda$  are independent, the second cross-partial derivatives do not enter into equations that determine the price. Given a function  $h(\lambda, X)$ , we will use the notation  $\partial h / \partial \lambda$  to denote the  $1 \times 2$  vector  $[\partial^2 h / \partial \lambda_1^2, \partial^2 h / \partial \lambda_2^2]$ .

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<sup>29</sup>Using log-linearization, Eraker and Shaliastovich (2008) and Benzoni, Collin-Dufresne, and Goldstein (2011) find approximate solutions to related continuous-time jump-diffusion models when the EIS is not equal to one.

## B.2 The value function

**Proof of Theorem 1** Let  $S$  denote the value of a claim to aggregate consumption, and conjecture that the price-dividend ratio for the consumption claim is constant:

$$\frac{S_t}{C_t} = l,$$

for some constant  $l$ . This relation implies that  $S_t$  satisfies

$$\frac{dS_t}{S_t} = \frac{dC_t}{C_t} = \mu_{C_t} dt + \sigma dB_{C_t}. \quad (\text{B.1})$$

Consider an agent who allocates wealth between  $S$  and the risk-free asset. Let  $\alpha_t$  be the fraction of wealth in the risky asset  $S_t$ , and let  $c_t$  be the agent's consumption. The wealth process is then given by

$$dW_t = (W_t \alpha_t (\mu_{C_t} - r_t + l^{-1}) + W_t r_t - c_t) dt + W_t \alpha_t \sigma dB_{C_t},$$

where  $r_t$  denote the instantaneous risk-free rate. Optimal consumption and portfolio choice must satisfy the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \sup_{\alpha_t, c_t} \left\{ \frac{\partial J}{\partial W} (W_t \alpha_t (\mu_{C_t} - r_t + l^{-1}) + W_t r_t - c_t) + \frac{\partial J}{\partial \lambda} (\kappa_\lambda * (\bar{\lambda} - \lambda_t)) - \frac{\partial J}{\partial \mu} (\kappa_\mu * \mu_t) \right. \\ \left. + \frac{1}{2} \frac{\partial^2 J}{\partial W^2} W_t^2 \alpha_t^2 \sigma^2 + \frac{1}{2} \left( \frac{\partial^2 J}{\partial \lambda^2} \right)^\top (\sigma_\lambda^2 * \lambda_t) + \lambda_t^\top \bar{\mathcal{J}}(J(W_t, \mu_t, \lambda_t)) + f(c_t, V) \right\} = 0, \quad (\text{B.2}) \end{aligned}$$

where, as defined in Appendix B.1,

$$\begin{aligned} \frac{\partial^2 J}{\partial \lambda^2} &= \left[ \frac{\partial^2 J}{\partial \lambda_1^2}, \frac{\partial^2 J}{\partial \lambda_2^2} \right]^\top \\ \sigma_\lambda^2 &= [\sigma_{\lambda_1}^2, \sigma_{\lambda_2}^2]^\top. \end{aligned}$$

In equilibrium,  $\alpha_t = 1$  and  $c_t = C_t = W_t l^{-1}$ . Substituting these policy functions into (B.2) implies

$$\begin{aligned} \frac{\partial J}{\partial W} W_t \mu_{C_t} + \frac{\partial J}{\partial \lambda} (\kappa_\lambda * (\bar{\lambda} - \lambda_t)) - \frac{\partial J}{\partial \mu} (\kappa_\mu * \mu_t) + \frac{1}{2} \frac{\partial^2 J}{\partial W^2} W_t^2 \sigma^2 \\ + \frac{1}{2} \left( \frac{\partial^2 J}{\partial \lambda^2} \right)^\top (\sigma_\lambda^2 * \lambda_t) + \lambda_t^\top \bar{\mathcal{J}}(J(W_t, \mu_t, \lambda_t)) + f(C_t, V) = 0. \quad (\text{B.3}) \end{aligned}$$

By the envelope condition  $\partial f/\partial C = \partial J/\partial W$ , we obtain  $\beta = l^{-1}$ . Given that the consumption-wealth ratio equals  $\beta^{-1}$ , it follows that

$$\begin{aligned} f(C_t, V_t) &= f(W_t l^{-1}, J(W_t, \mu_t, \lambda_t)) \\ &= \beta W_t^{1-\gamma} I(\mu_t, \lambda_t) \left( \log \beta - \frac{\log I(\mu_t, \lambda_t)}{1-\gamma} \right). \end{aligned} \quad (\text{B.4})$$

Substituting (B.4) and (7) into (B.3)

$$\begin{aligned} \mu_{Ct} + (1-\gamma)^{-1} I^{-1} \frac{\partial I}{\partial \lambda} (\kappa_\lambda * (\bar{\lambda} - \lambda_t)) - (1-\gamma)^{-1} I^{-1} \frac{\partial I}{\partial \mu} (\kappa_\mu * \mu_t) - \frac{1}{2} \gamma \sigma^2 \\ + \frac{1}{2} (1-\gamma)^{-1} I^{-1} \left( \frac{\partial^2 I}{\partial \lambda^2} \right)^\top (\sigma_\lambda^2 * \lambda_t) + (1-\gamma)^{-1} \lambda_t^\top \bar{J}(I(\mu_t, \lambda_t)) \\ + \beta \left( \log \beta - \frac{\log I(\mu_t, \lambda_t)}{1-\gamma} \right) = 0. \end{aligned}$$

Note that  $\mu_{Ct} = \bar{\mu}_C + \mu_{1t} + \mu_{2t}$ .

Collecting coefficients on  $\mu_{jt}$  results in the following equation for  $b_{\mu_j}$ :

$$1 - (1-\gamma)^{-1} b_{\mu_j} \kappa_{\mu_j} - \beta (1-\gamma)^{-1} b_\mu = 0,$$

solving this equation yields

$$b_{\mu_j} = \frac{1-\gamma}{\kappa_{\mu_j} + \beta}.$$

Collecting coefficients on  $\lambda_{jt}$  yields

$$b_{\lambda_j} = \frac{\beta + \kappa_{\lambda_j}}{\sigma_{\lambda_j}^2} - \sqrt{\left( \frac{\beta + \kappa_{\lambda_j}}{\sigma_{\lambda_j}^2} \right)^2 - \frac{2E_{\nu_j} [e^{b_{\mu_j} Z_{jt}} - 1]}{\sigma_{\lambda_j}^2}}.$$

Collecting the constant terms:

$$a = \frac{1-\gamma}{\beta} \left( \bar{\mu}_C - \frac{1}{2} \gamma \sigma^2 \right) + (1-\gamma) \log \beta + \sum_j b_{\lambda_j} \frac{\kappa_{\lambda_j}}{\beta} \bar{\lambda}_j.$$

□

**Proof of Corollary 3** The risk-free rate is obtained by taking the derivative of the HJB (B.2) with respect to  $\alpha_t$ , evaluating at  $\alpha_t = 1$  and setting it equal to 0. The result immediately follows. □

### B.3 The state-price density

Duffie and Skiadas (1994) show that the state-price density  $\pi_t$  equals

$$\pi_t = \exp \left\{ \int_0^t \frac{\partial}{\partial V} f(C_s, V_s) ds \right\} \frac{\partial}{\partial C} f(C_t, V_t). \quad (\text{B.5})$$

Note that the exponential term is deterministic. From (6), we obtain

$$\frac{\partial}{\partial C} f(C_t, V_t) = \beta(1 - \gamma) \frac{V_t}{C_t}.$$

The equilibrium condition  $V_t = J(\beta^{-1}C_t, \mu_t, \lambda_t)$ , together with the form of the value function (7), implies

$$\frac{\partial}{\partial C} f(C_t, V_t) = \beta^\gamma C_t^{-\gamma} I(\mu_t, \lambda_t). \quad (\text{B.6})$$

Applying Ito's Lemma to (B.6) implies

$$\frac{d\pi_t}{\pi_{t-}} = \mu_{\pi t} dt + \sigma_{\pi t} dB_t + \sum_j \frac{\mathcal{J}_j(\pi_t)}{\pi_{t-}} dN_{jt}, \quad (\text{B.7})$$

where

$$\sigma_{\pi t} = \left[ -\gamma\sigma, \quad b_{\lambda_1}\sigma_{\lambda_1}\sqrt{\lambda_{1t}}, \quad b_{\lambda_2}\sigma_{\lambda_2}\sqrt{\lambda_{2t}} \right], \quad (\text{B.8})$$

and

$$\mathcal{J}_j(\pi_t) = e^{b_{\mu_j} Z_{jt}} - 1, \quad (\text{B.9})$$

for  $j = 1, 2$ . It also follows from no-arbitrage that

$$\mu_{\pi t} = -r_t - \lambda_t^\top \frac{\bar{\mathcal{J}}(\pi_t)}{\pi_t} \quad (\text{B.10})$$

$$\begin{aligned} &= -r_t - \sum_j \lambda_{jt} E_{\nu_j} [e^{b_{\mu_j} Z_{jt}} - 1] \\ &= -\beta - \mu_{Ct} + \gamma\sigma^2 - \sum_j \lambda_{jt} E_{\nu_j} [e^{b_{\mu_j} Z_{jt}} - 1]. \end{aligned} \quad (\text{B.11})$$

In the event of a disaster, marginal utility (as represented by the state-price density) jumps upward, and in the event of a boom the marginal utility jumps downward, as can be seen by the term multiplying the Poisson process in (B.7). The first element of (B.8) implies that the standard diffusion risk in consumption is priced; more interestingly, changes in  $\lambda_{jt}$  are also priced as reflected by the new element of (B.8).



## B.4 Pricing the general equity claim

We first consider the price of a general form of the dividend stream. The dividend stream on the aggregate market and the dividend stream for value will be special cases. Suppose dividends evolve according to

$$\frac{dD_t}{D_t} = \mu_{Dt} dt + \sigma_D dB_{Ct}, \quad (\text{B.12})$$

where

$$\mu_{Dt} = \bar{\mu}_D + \phi_{D,1}\mu_{1t} + \phi_{D,2}\mu_{2t},$$

$\phi_{D,j}$  denotes the jump multiplier for the type- $j$  jump.

**Lemma B.1.** *Let  $H(D_t, \mu_t, \lambda_t, \tau)$  denote the time  $t$  price of a single future dividend payment at time  $t + \tau$ :*

$$H(D_t, \mu_t, \lambda_t, \tau) = E_t \left[ \frac{\pi_{t+\tau}}{\pi_t} D_{t+\tau} \right].$$

By Ito's Lemma, we can write

$$\frac{dH_t}{H_t} = \mu_{H(\tau),t} dt + \sigma_{H(\tau),t} dB_t + \sum_j \mathcal{J}_j(H_t) dN_{jt}.$$

for a scalar process  $\mu_{H(\tau),t}$  and a vector process  $\sigma_{H(\tau),t}$ , where  $H_t = H(D_t, \mu_t, \lambda_t, \tau)$ . Then no-arbitrage implies that

$$\mu_{\pi,t} + \mu_{H(\tau),t} + \sigma_{\pi,t} \sigma_{H(\tau),t}^\top + \frac{1}{\pi_t H_t} \lambda_t^\top \bar{\mathcal{J}}(\pi_t H_t) = 0. \quad (\text{B.13})$$

**Proof** No-arbitrage implies that  $H(D_s, \lambda_s, \mu_s, 0) = D_s$  and that

$$\pi_t H(D_t, \lambda_t, \mu_t, \tau) = E_t [\pi_s H(D_s, \lambda_s, \mu_s, 0)].$$

For the remainder of the argument, we simplify notation by writing  $H_t = H(D_t, \mu_t, \lambda_t, \tau)$ ,  $\mu_{H,t} = \mu_{H(\tau),t}$  and  $\sigma_{H,t} = \sigma_{H(\tau),t}$ . Ito's Lemma applied to  $\pi_t H_t$  implies

$$\begin{aligned} \pi_t H_t = \pi_0 H_0 + \int_0^t \pi_s H_s (\mu_{H,s} + \mu_{\pi,s} + \sigma_{\pi,s} \sigma_{H,s}^\top) + \int_0^t \pi_s H_s (\sigma_{H,s} + \sigma_{\pi,s}) dB_s \\ + \sum_j \sum_{0 < s_{ij} \leq t} \left( \pi_{s_{ij}} H_{s_{ij}} - \pi_{s_{ij}^-} H_{s_{ij}^-} \right), \quad (\text{B.14}) \end{aligned}$$

where  $s_{ij} = \inf\{s : N_{js} = i\}$  (namely, the time that the  $i$ th type  $j$  jump occurs). Adding and subtracting the jump compensation term from (B.14) yields:

$$\begin{aligned} \pi_t H_t = \pi_0 H_0 &+ \int_0^t \pi_s H_s \left( \mu_{H,s} + \mu_{\pi,s} + \sigma_{\pi,s} \sigma_{H,s}^\top + \sum_j \lambda_j \frac{\bar{\mathcal{J}}_j(\pi_s H_s)}{\pi_s H_s} \right) ds \\ &+ \int_0^t \pi_s H_s (\sigma_{H,s} + \sigma_{\pi,s}) dB_s \\ &+ \sum_j \left( \sum_{0 < s_{ij} \leq t} \left( \pi_{s_{ij}} H_{s_{ij}} - \pi_{s_{ij}^-} H_{s_{ij}^-} \right) - \int_0^t \pi_s H_s \lambda_j \bar{\mathcal{J}}_j(\pi_s H_s) ds \right). \end{aligned} \quad (\text{B.15})$$

Under regularity conditions analogous to those given in Duffie, Pan, and Singleton (2000) the second and the third integrals on the right hand side of (B.15) are martingales. Therefore the first integral on the right hand side of (B.15) must also be a martingale, and it follows that the integrand of this term must equal zero.  $\square$

**Theorem B.2.** *The function  $H$  takes an exponential form:*

$$H(D_t, \mu_t, \lambda_t, \tau) = D_t \exp \left\{ a_\phi(\tau) + b_{\phi\mu}(\tau)^\top \mu_t + b_{\phi\lambda}(\tau)^\top \lambda_t \right\}, \quad (\text{B.16})$$

where  $b_{\phi\mu} = [b_{\phi\mu_1}, b_{\phi\mu_2}]^\top$  and  $b_{\phi\lambda} = [b_{\phi\lambda_1}, b_{\phi\lambda_2}]^\top$  and

$$\frac{db_{\phi\mu_j}}{d\tau} = -\kappa_{\mu_j} b_{\phi\mu_j} + (\phi_{D,j} - 1), \quad (\text{B.17})$$

$$\frac{db_{\phi\lambda_j}}{d\tau} = \frac{1}{2} \sigma_{\lambda_j}^2 b_{\phi\lambda_j}(\tau)^2 + \left( b_{\lambda_j} \sigma_{\lambda_j}^2 - \kappa_{\lambda_j} \right) b_{\phi\lambda_j}(\tau) + E_{\nu_j} \left[ e^{b_{\mu_j} Z_{jt}} \left( e^{b_{\phi\mu_j}(\tau) Z_{jt}} - 1 \right) \right], \quad (\text{B.18})$$

$$\frac{da_\phi}{d\tau} = \bar{\mu}_D - \bar{\mu}_C - \beta + \gamma \sigma (\sigma - \sigma_D) + b_{\phi\lambda}(\tau)^\top (\kappa_\lambda * \bar{\lambda}). \quad (\text{B.19})$$

The boundary conditions are  $b_{\phi\mu_j}(0) = b_{\phi\lambda_j}(0) = a_\phi(0) = 0$ .

**Proof** Let  $H_t = H(D_t, \mu_t, \lambda_t, \tau)$ . It follows from Ito's Lemma that

$$\frac{\bar{\mathcal{J}}_j(\pi_t H_t)}{\pi_t H_t} = E_{\nu_j} \left[ e^{(b_{\mu_j} + b_{\mu_j \phi}(\tau)) Z_{jt}} - 1 \right], \quad (\text{B.20})$$

$$\begin{aligned} \mu_{H(\tau),t} = & \frac{1}{H} \left( \frac{\partial H}{\partial D} \mu_{Dt} + \frac{\partial H}{\partial \lambda} (\kappa_\lambda * (\bar{\lambda} - \lambda_t)) - \frac{\partial H}{\partial \mu} (\kappa_\mu * \mu_t) \right. \\ & \left. - \frac{\partial H}{\partial \tau} + \frac{1}{2} \left( \frac{\partial^2 H}{\partial \lambda^2} \right) (\sigma_\lambda^2 * \lambda_t) \right) \end{aligned} \quad (\text{B.21})$$

$$\begin{aligned} = & \mu_{Dt} + b_{\phi\lambda}(\tau)^\top (\kappa_\lambda * (\bar{\lambda} - \lambda_t)) + b_{\phi\mu}(\tau)^\top (\kappa_\mu * \mu_t) \\ & - \left( \frac{da_\phi}{d\tau} + \lambda_t^\top \frac{db_{\phi\lambda}}{d\tau} + \mu_t^\top \frac{db_{\phi\mu}}{d\tau} \right) + \frac{1}{2} (b_{\phi\lambda}(\tau)^2)^\top (\sigma_\lambda^2 * \lambda_t), \end{aligned} \quad (\text{B.22})$$

and

$$\begin{aligned} \sigma_{H(\tau),t} = & \frac{1}{H} \left( \frac{\partial H}{\partial D} \mu_D [\sigma_D, 0, 0] + \frac{\partial H}{\partial \lambda_1} [0, \sigma_{\lambda_1} \sqrt{\lambda_{1t}}, 0] + \frac{\partial H}{\partial \lambda_2} [0, 0, \sigma_{\lambda_2} \sqrt{\lambda_{2t}}] \right) \\ = & \left[ \sigma_D, b_{\phi\lambda_1}(\tau) \sigma_{\lambda_1} \sqrt{\lambda_{1t}}, b_{\phi\lambda_2}(\tau) \sigma_{\lambda_2} \sqrt{\lambda_{2t}} \right]. \end{aligned} \quad (\text{B.23})$$

Substituting (B.20), (B.22) and (B.23) along with (B.8) and (B.11) into the no-arbitrage condition (B.13) implies

$$\begin{aligned} & \mu_{Dt} + b_{\phi\lambda}(\tau)^\top (\kappa_\lambda * (\bar{\lambda} - \lambda_t)) + b_{\phi\mu}(\tau)^\top (\kappa_\mu * \mu_t) + (b_{\phi\lambda}(\tau)^2)^\top (\sigma_\lambda^2 * \lambda_t) \\ & - \beta - \mu_{Ct} + \gamma \sigma^2 - \gamma \sigma \sigma_D + \sum_j \lambda_{jt} E_{\nu_j} \left[ e^{(b_{\mu_j} + b_{\phi\mu_j}(\tau)) Z_{jt}} - e^{b_{\mu_j} Z_{jt}} \right] \\ & - \left( \frac{da_\phi}{d\tau} + \lambda_t^\top \frac{db_{\phi\lambda}}{d\tau} + \mu_t^\top \frac{db_{\phi\mu}}{d\tau} \right) = 0. \end{aligned}$$

Notice that, by definition,  $\mu_{Dt} - \mu_{Ct} = (\bar{\mu}_D - \bar{\mu}_C) + \sum_j (\phi_{D,j} - 1) \mu_{jt}$ . Matching the terms multiplying  $\mu_j$  implies (B.17), matching the terms multiplying  $\lambda_j$  implies (B.18) and matching the constant terms implies (B.19).  $\square$

Let  $F_t = F(D_t, \mu_t, \lambda_t)$  denote the time  $t$  price of the claim to the dividend stream defined by (B.12).

**Lemma B.3.** *No-arbitrage implies*

$$\mu_{\pi,t} + \mu_{F,t} + \frac{D_t}{F_t} + \sigma_{\pi,t} \sigma_{F,t}^\top + \sum_j \lambda_{jt} \frac{\bar{J}_j(\pi_t F_t)}{\pi_t F_t} = 0, \quad (\text{B.24})$$

where  $\mu_{F,t}$  and  $\sigma_{F,t}$  denote the drift and diffusion term of the  $F_t$  process, respectively.

**Proof** By definition,

$$F(D_t, \mu_t, \lambda_t) = \int_0^\infty H(D_t, \mu_t, \lambda_t, \tau) d\tau.$$

For notational simplicity, we abbreviate  $H(D, \lambda, \mu, \tau)$  as  $H(\tau)$ . It follows from Ito's Lemma applied to  $F(D_t, \mu_t, \lambda_t)$  that

$$F(D_t, \mu_t, \lambda_t)\mu_{F,t} = \int_0^\infty \left( H(\tau)\mu_{Dt} + \sum_j H_{\lambda_j}(\tau)(\bar{\lambda}_j - \lambda_j) + \sum_j H_{\mu_j}(\tau)\mu_j + \frac{1}{2} \sum_j H_{\lambda_j\lambda_j}(\tau) \right) d\tau,$$

where  $H_{\mu_j}$ ,  $H_{\lambda_j}$  and  $H_{\lambda_j\lambda_j}$  denote partial derivatives. It then follows from the equation for  $\mu_{H(\tau),t}$  (B.21) that

$$F(D_t, \mu_t, \lambda_t)\mu_{F,t} = \int_0^\infty \left( H(D_t, \lambda_t, \mu_t, \tau)\mu_{H(\tau),t} - \frac{\partial}{\partial \tau} H(D_t, \mu_t, \lambda_t, \tau) \right) d\tau. \quad (\text{B.25})$$

In short, (B.25) holds because  $H$  is a function of  $\tau$  but  $F$  is not. Because  $\lim_{\tau \rightarrow \infty} H(D_t, \mu_t, \lambda_t, \tau) = 0$ ,

$$- \int_0^\infty \frac{\partial}{\partial \tau} H(D_t, \mu_t, \lambda_t, \tau) d\tau = H(D_t, \mu_t, \lambda_t, 0) = D_t.$$

Ito's Lemma also implies

$$F(D_t, \mu_t, \lambda_t)\sigma_{F,t} = \int_0^\infty H(D_t, \mu_t, \lambda_t, \tau)\sigma_{H(\tau),t} d\tau$$

and

$$\bar{\mathcal{J}}(\pi_t F(D_t, \mu_t, \lambda_t)) = \int_0^\infty \bar{\mathcal{J}}(\pi_t H(D_t, \mu_t, \lambda_t, \tau)) d\tau$$

The result then follows from the no-arbitrage relation for  $H$ , (B.13).  $\square$

Given a stream of cash flows  $D_t$  and its price  $F_t$ , define the expected return on this claim to be

$$r_t^e = \mu_{F,t} + \frac{D_t}{F_t} + \frac{1}{F_t} \lambda_t^\top \bar{\mathcal{J}}(F_t).$$

**Theorem B.4.** *Let  $r_t^e$  denote the instantaneous expected return on the general equity claim.*

*Then*

$$r_t^e - r_t = -\sigma_{\pi,t} \sigma_{F,t}^\top - \sum_j \lambda_{jt} E_{\nu_j} \left[ \frac{\mathcal{J}_j(F_t)}{F_t} \frac{\mathcal{J}_j(\pi_t)}{\pi_t} \right]. \quad (\text{B.26})$$

**Proof** It follows from the definition of  $r_t^e$  (21) that

$$\mu_{F,t} + \frac{D_t}{F_t} = r_t^e - \frac{1}{F_t} \lambda_t^\top \bar{\mathcal{J}}(F_t).$$

Further,  $\mu_{\pi_t}$  can be written in terms of  $r_t$  and a jump term as in (B.10). Finally,

$$E_{\nu_j} \left[ \frac{\mathcal{J}_j(F_t)}{F_t} \frac{\mathcal{J}_j(\pi_t)}{\pi_t} \right] = \bar{\mathcal{J}}_j(F_t \pi_t) - \bar{\mathcal{J}}_j(F_t) - \bar{\mathcal{J}}_j(\pi_t)$$

for  $j = 1, 2$ . The result that follows from rearranging (B.24) in Lemma B.3.  $\square$

## B.5 Further results on equity pricing

The following is an intermediate step in the proof of Corollary 5:

**Lemma B.5.**

$$\lim_{\tau \rightarrow \infty} b_{\phi \lambda_j}(\tau) = -\frac{1}{\sigma_{\lambda_j}^2} \left( \zeta_{\phi_j} - \kappa_{\lambda_j} + b_{\lambda_j} \sigma_{\lambda_j}^2 \right), \quad (\text{B.27})$$

where

$$\zeta_{\phi_j} = \sqrt{(b_{\lambda_j} \sigma_{\lambda_j}^2 - \kappa_{\lambda_j})^2 - 2E_{\nu_j} \left[ e^{\left( b_{\mu_j} + \frac{\phi-1}{\kappa_{\mu_j}} \right) Z_j} - e^{b_{\mu_j} Z_j} \right] \sigma_{\lambda_j}^2}. \quad (\text{B.28})$$

Moreover,  $\lim_{\tau \rightarrow \infty} b_{\phi \lambda_1}(\tau) < 0$  and  $\lim_{\tau \rightarrow \infty} b_{\phi \lambda_2}(\tau) > 0$ .

**Proof** Let  $\bar{b}_{\phi \lambda_j}$  denote the limit, should it exist. In the limit, small changes in  $\tau$  do not change  $b_{\phi \lambda_j}(\tau)$ . Taking the limit of both sides of (17) implies that  $\bar{b}_{\phi \lambda_j}$  must satisfy the quadratic equation

$$0 = \frac{1}{2} \sigma_{\lambda_j}^2 \bar{b}_{\phi \lambda_j}^2 + (b_{\lambda_j} \sigma_{\lambda_j}^2 - \kappa_{\lambda_j}) \bar{b}_{\phi \lambda_j} + E_{\nu_j} \left[ e^{\left( b_{\mu_j} + \frac{\phi-1}{\kappa_{\mu_j}} \right) Z_{jt}} - e^{b_{\mu_j} Z_{jt}} \right]$$

This equation has two solutions; as for the value function, the solution corresponding to the negative root has the more reasonable economic properties and is given in (B.27).<sup>30</sup>

To prove that the limits have the signs given in the Lemma, note that  $Z_1 < 0$  implies that

$$E_{\nu_1} \left[ e^{\left( b_{\mu_1} + \frac{\phi-1}{\kappa_{\mu_1}} \right) Z_1} - e^{b_{\mu_1} Z_1} \right] < 0.$$

---

<sup>30</sup>We have verified that (B.27) does indeed correspond to the limit when the ordinary differential equation (17) is solved numerically.

Therefore,

$$\zeta_{\phi_1} > |b_{\lambda_1} \sigma_{\lambda_1}^2 - \kappa_{\lambda_1}|.$$

Now, note that  $Z_2 > 0$  implies that

$$E_{\nu_1} \left[ e^{(b_{\mu_1} + \frac{\phi-1}{\kappa_{\mu_1}})Z_1} - e^{b_{\mu_1}Z_1} \right] > 0.$$

The parameter assumptions imply that  $\zeta_{\phi_2}$  is real-valued. As shown in Corollary 2,  $b_{\lambda_2} < 0$ , and that

$$\zeta_{\phi_2} < |b_{\lambda_2} \sigma_{\lambda_2}^2 - \kappa_{\lambda_2}|$$

In both cases the result on the sign follows.  $\square$

**Proof of Corollary 5** The result for  $\mu_{jt}$  follows immediately from the form of  $b_{\phi\mu_j}(\tau)$ . For  $\lambda_{1t}$ , first note that  $b_{\phi\lambda_1}(0) = 0$  and  $\lim_{\tau \rightarrow \infty} b_{\phi\lambda_1}(\tau) < 0$  by Lemma B.5. Therefore, it suffices to show that  $b_{\phi\lambda_1}(\tau)$  is a monotonic function of  $\tau$ .

Assume, by contradiction that  $db_{\phi\lambda_1}(\tau)/d\tau = 0$  for some  $\tau, \tau^*$ . Then, by (17),

$$b_{\phi\lambda_1}(\tau^*) = \frac{1}{\sigma_{\lambda_1}^2} \left( \sqrt{(b_{\lambda_1} \sigma_{\lambda_1}^2 - \kappa_{\lambda_1})^2 - 2E_{\nu_1} \left[ e^{(b_{\mu_1} + b_{\phi\mu_1}(\tau^*))Z_1} - e^{b_{\mu_1}Z_1} \right] \sigma_{\lambda_1}^2} - \kappa_{\lambda_1} + b_{\lambda_1} \sigma_{\lambda_1}^2 \right) \quad (\text{B.29})$$

However, differentiating (B.29) with respect to  $\tau$  implies  $db_{\phi\lambda_1}(\tau^*)/d\tau \neq 0$ . Therefore,  $db_{\phi\lambda_1}(\tau)/d\tau$  must be nonzero for all finite  $\tau$ , and, because (17) implies that the derivative is a continuous function, it must be either (weakly) positive or negative. It follows that  $b_{\phi\lambda_1}(\tau)$  is monotonic, and, by the argument given above, it must be negative and decreasing in  $\tau$ . Analogous reasoning holds for  $j = 2$ .  $\square$

**Proof of Corollary 6** It follows from Ito's Lemma and the definition of  $G$  that

$$\sigma_{F,t} = \left[ \phi \sigma_D, \frac{1}{G} \frac{\partial G}{\partial \lambda_1} \sigma_{\lambda_1} \sqrt{\lambda_{1t}}, \frac{1}{G} \frac{\partial G}{\partial \lambda_2} \sigma_{\lambda_2} \sqrt{\lambda_{2t}} \right].$$

Because dividends are not subject to jumps

$$\frac{\mathcal{J}_j(F_t)}{F_t} = \frac{\mathcal{J}_j(G_t)}{G_t}$$

for  $j = 1, 2$ . The result follows from substituting these expressions and the corresponding expressions for the state-price density  $\pi_t$  (given in (B.8) and (B.9)) into (B.26) of Theorem B.4.  $\square$

Note that the proof of Corollary 12 follows along similar lines.

## C Return simulation

For each asset, the realized return between time  $t$  and  $t + \Delta t$  is defined as

$$R_{t+\Delta t} = \frac{F_{t+\Delta t} + \int_t^{t+\Delta t} D_s ds}{F_t}$$

see Duffie (2001, Chapter 6.L). For assets that pay a dividend in each period, namely the aggregate market and the value sector, this return can be computed based on the series of price-dividend ratios and payouts. Using the approximation  $D_{t+\Delta t}\Delta t \approx \int_t^{t+\Delta t} D_s ds$ , it follows that

$$\begin{aligned} R_{t,t+\Delta t} &\approx \frac{F_{t+\Delta t} + D_{t+\Delta t}\Delta t}{F_t} \\ &= \frac{\frac{F_{t+\Delta t}}{D_{t+\Delta t}} + \Delta t}{\frac{F_t}{D_t}} \frac{D_{t+\Delta t}}{D_t} \\ &= \frac{G(\mu_{t+\Delta t}, \lambda_{t+\Delta t}) + \Delta t}{G(\lambda_t)} \frac{D_{t+\Delta t}}{D_t}. \end{aligned}$$

Computing the return on the growth sector requires a different approach. For  $u \geq s \geq t$ , let  $R_{t,s,u}^g$  denote the return between  $s$  and  $u$  on the growth sector formed at time  $t$ . Because value and growth must add up to the aggregate market,

$$R_{t,t+\Delta t}^m = \frac{F_{t,t}^v}{F_t} R_{t,t+\Delta t}^v + \left(1 - \frac{F_{t,t}^v}{F_t}\right) R_{t,t+\Delta t}^g.$$

Rearranging, it follows that one-period returns on the growth sector equal

$$R_{t,t,t+\Delta t}^g = \frac{1}{1 - \frac{F_{t,t}^v}{F_t}} \left( R_{t,t,t+\Delta t}^m - \frac{F_{t,t}^v}{F_t} R_{t,t,t+\Delta t}^v \right). \quad (\text{C.1})$$

Because the price of the value sector formed at time  $t$  relative to the aggregate market is given by

$$\frac{F_{t,t}^v}{F_t} = \frac{G^v(\mu_t, \lambda_t)}{G(\mu_t, \lambda_t)},$$

it is straightforward to compute the return (C.1) on the growth sector.



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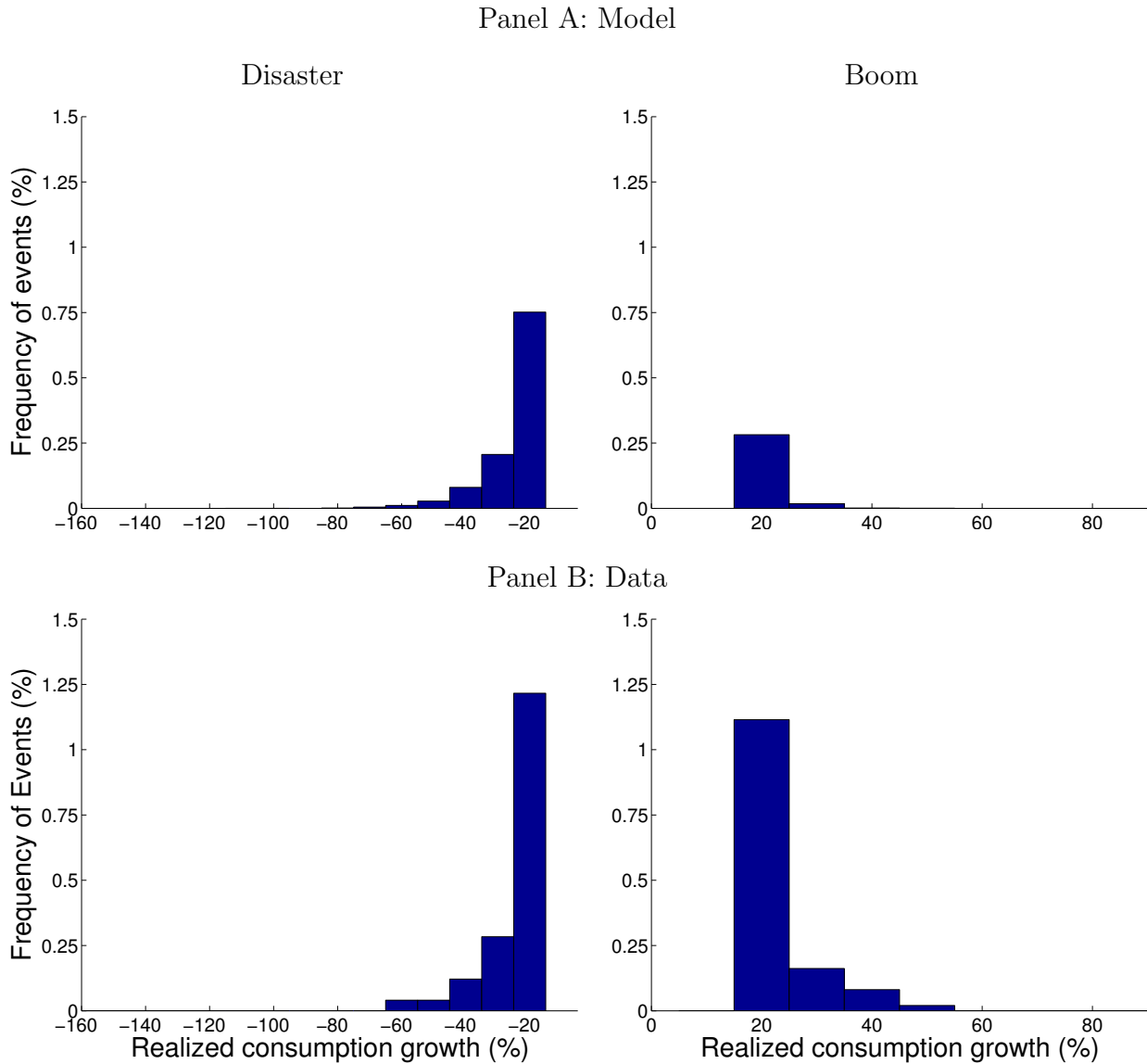
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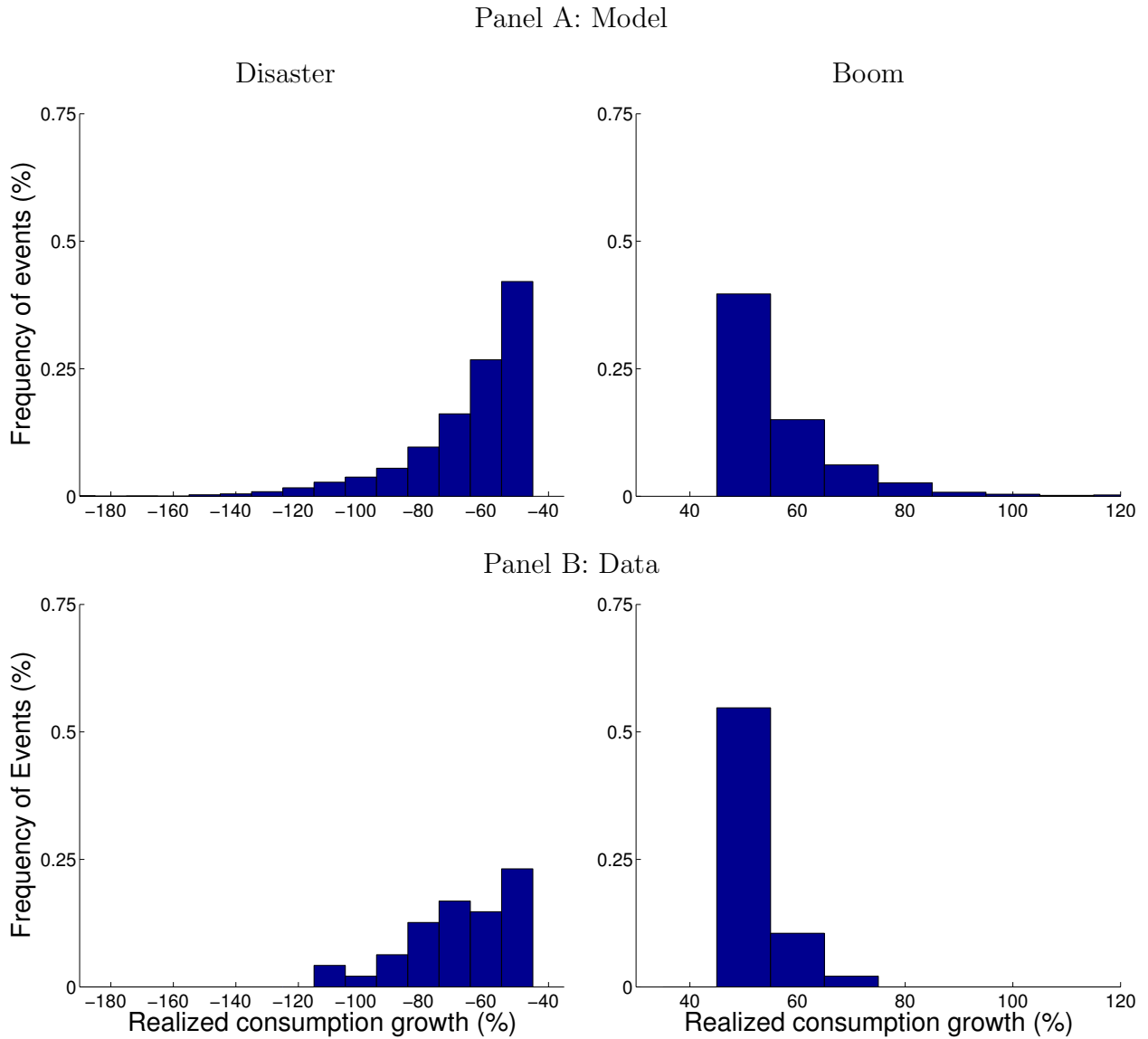
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Figure 1: Tails of the one-year consumption growth rate distribution



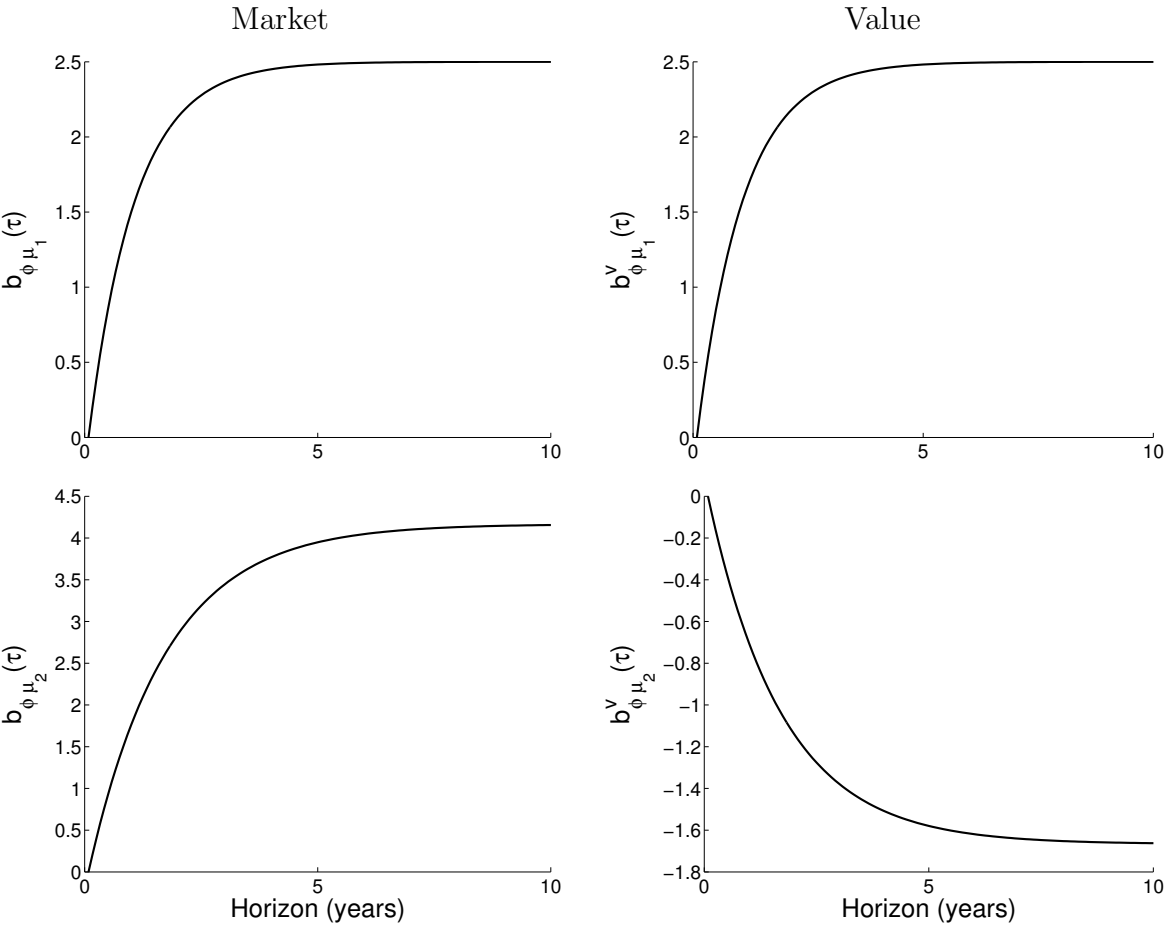
Note: This figure shows histograms of one-year consumption growth rates. The right panel considers growth rates above 15%. The left panel considers growth rates below -15%. The frequency is calculated by the number of observations within a range, divided by the total number of observations in the sample. Panel A shows results from simulated data from the model. Panel B shows results from the data. Data are from Barro and Ursua (2008). For the consumption booms, we exclude observations between 1944 and 1953.

Figure 2: Tails of the five-year consumption growth rate distribution



Notes: This figure shows histograms of five-year consumption growth rates. The right panel considers growth rates above 45%. The left panel considers growth rates below -45%. Panel A shows results from simulated data from the model. Panel B shows results from the data. Data are from Barro and Ursua (2008). For the consumption booms, we exclude five-year periods beginning between 1940 and 1948.

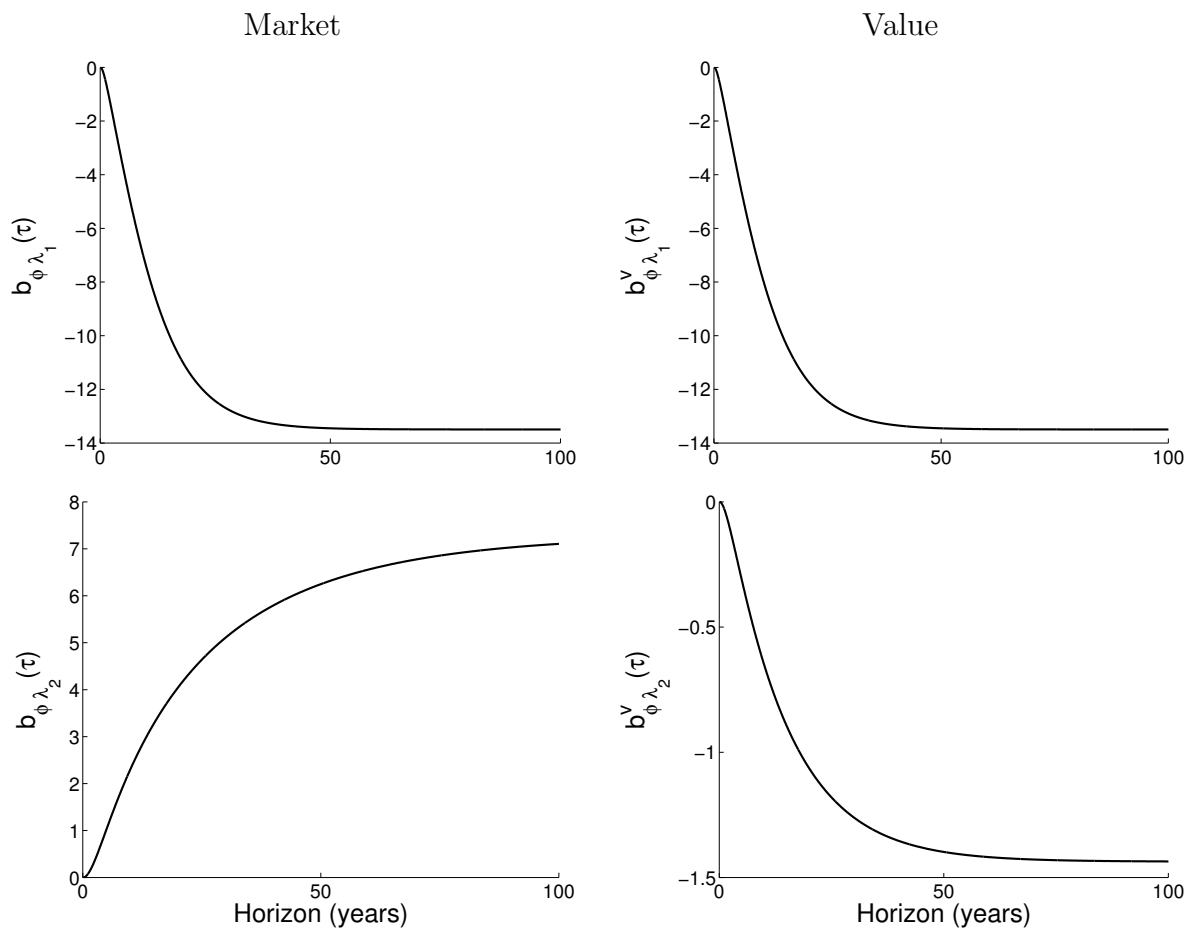
Figure 3: Solution for the price-dividend ratio: Coefficients on terms in the expected growth rate



Notes: The left panel shows the coefficients multiplying  $\mu_{1t}$  and  $\mu_{2t}$  in the price-dividend ratio for the market. The right panel shows the analogous coefficients for the value claim. The scales on the right and left for  $b_{\phi\mu_2}$  differ.

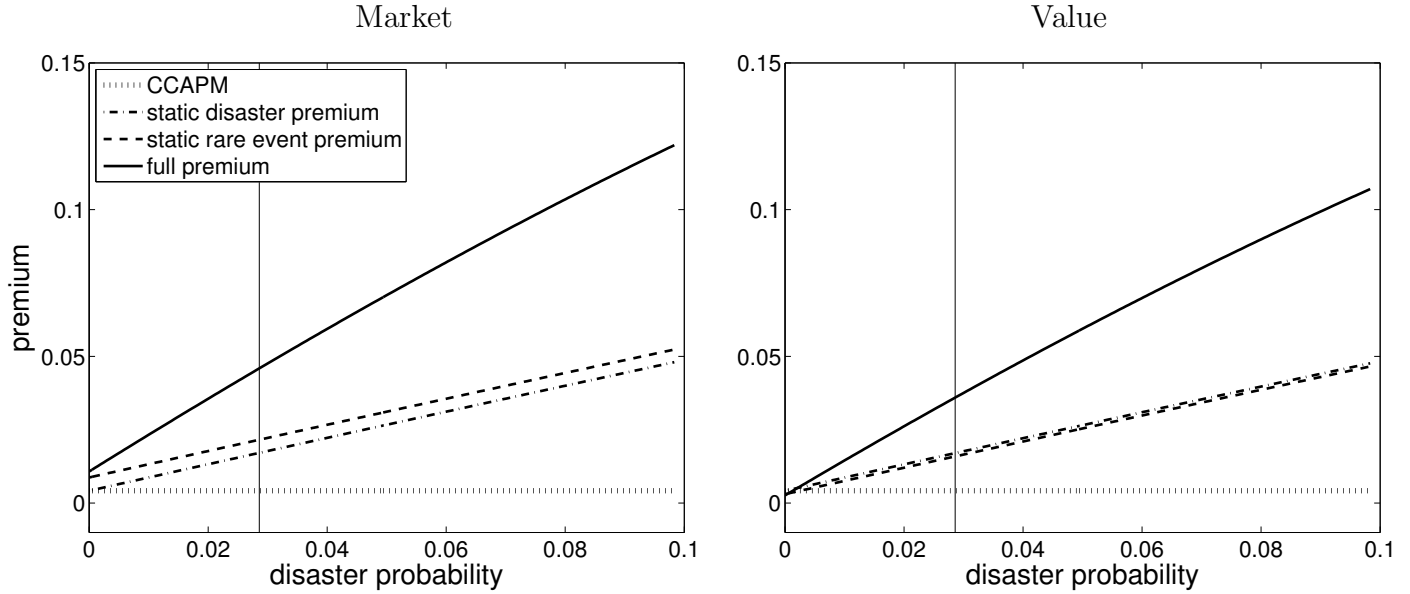


Figure 4: Solution for the price-dividend ratio: Coefficients on the jump probabilities



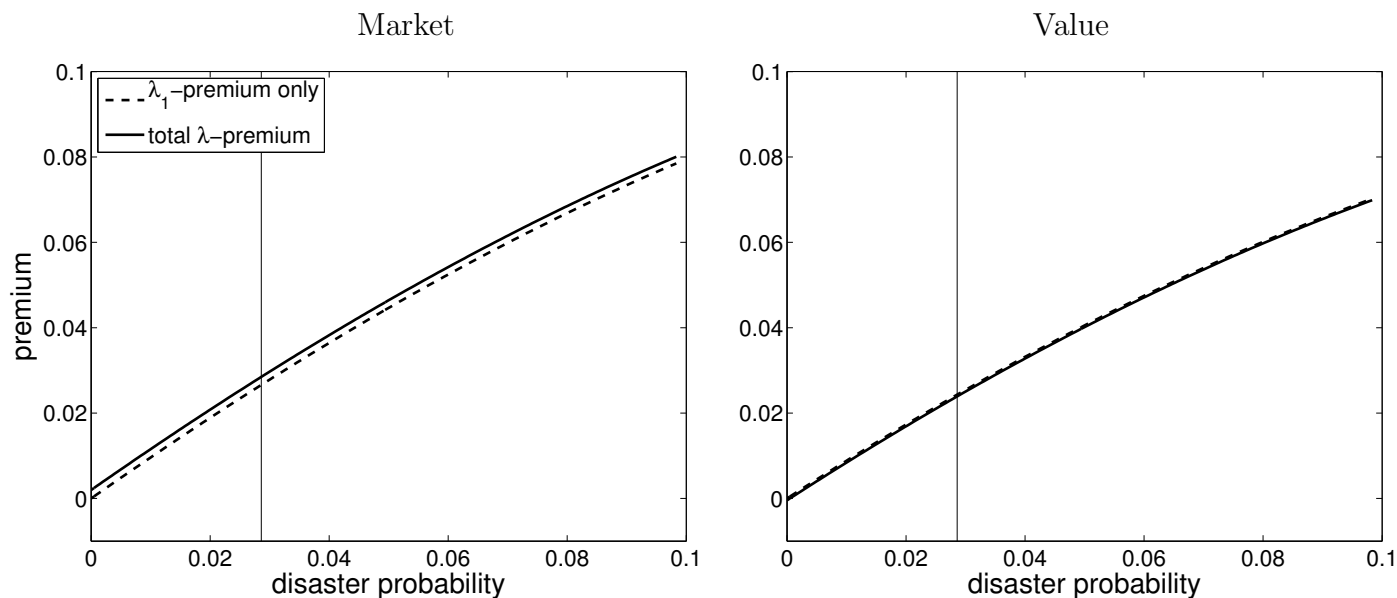
Notes: The left panel shows the coefficients multiplying  $\lambda_{1t}$  (the probability of a disaster) and  $\lambda_{2t}$  (the probability of a growth miracle) in the price-dividend ratio for the market. The right panel shows the analogous coefficients for the value claim. The scales on the right and the left for  $b_{\phi\lambda_2}(\tau)$  differ.

Figure 5: Risk premiums as functions of the probability of disaster



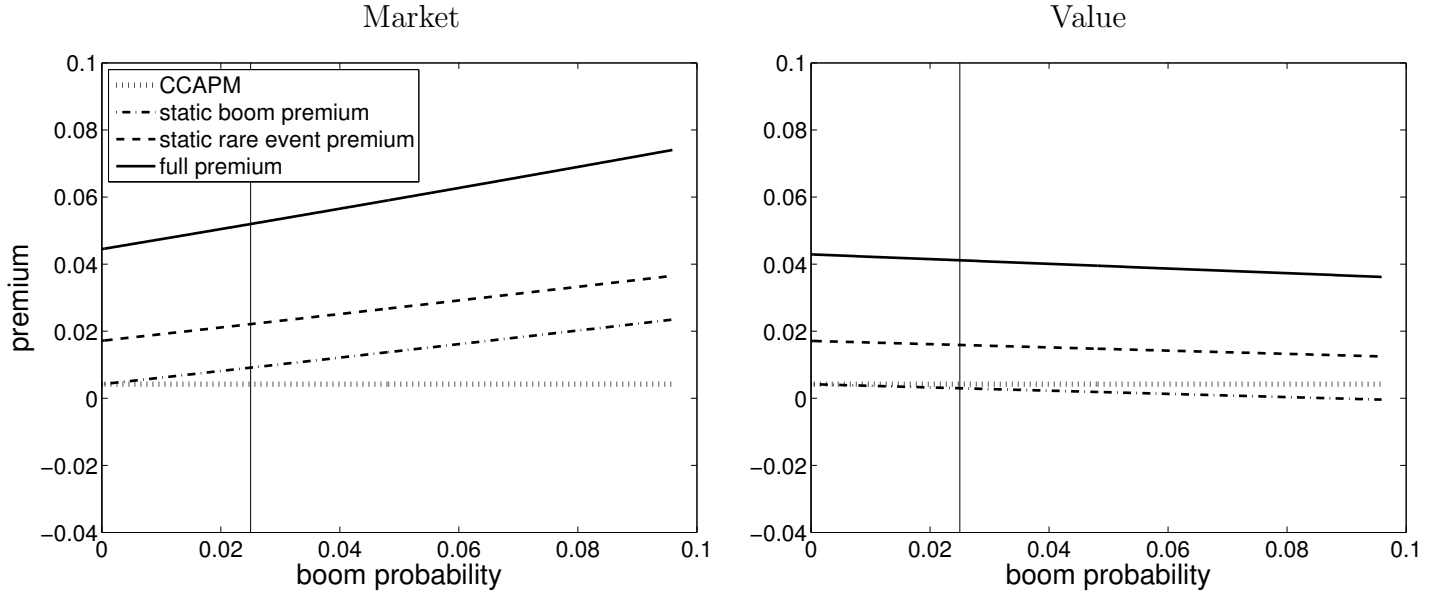
Notes: The figure shows components of the equity premium (left figure) and of the risk premium on the value claim (right figure). The solid line represents the full premium, the dotted line the CCAPM premium, the dashed-dotted line the static disaster premium and the dashed line the static rare event premium (namely, the static disaster premium plus the static boom premium). Premiums are shown as a function of the disaster probability,  $\lambda_1$ , while the boom probability,  $\lambda_2$ , is fixed at its mean of 2.5%. The vertical line represents the mean of the disaster probability. Premiums are defined relative to the riskfree rate and are in annual terms.

Figure 6:  $\lambda$ -premiums (compensation for changing rare event probabilities) as functions of the probability of disaster.



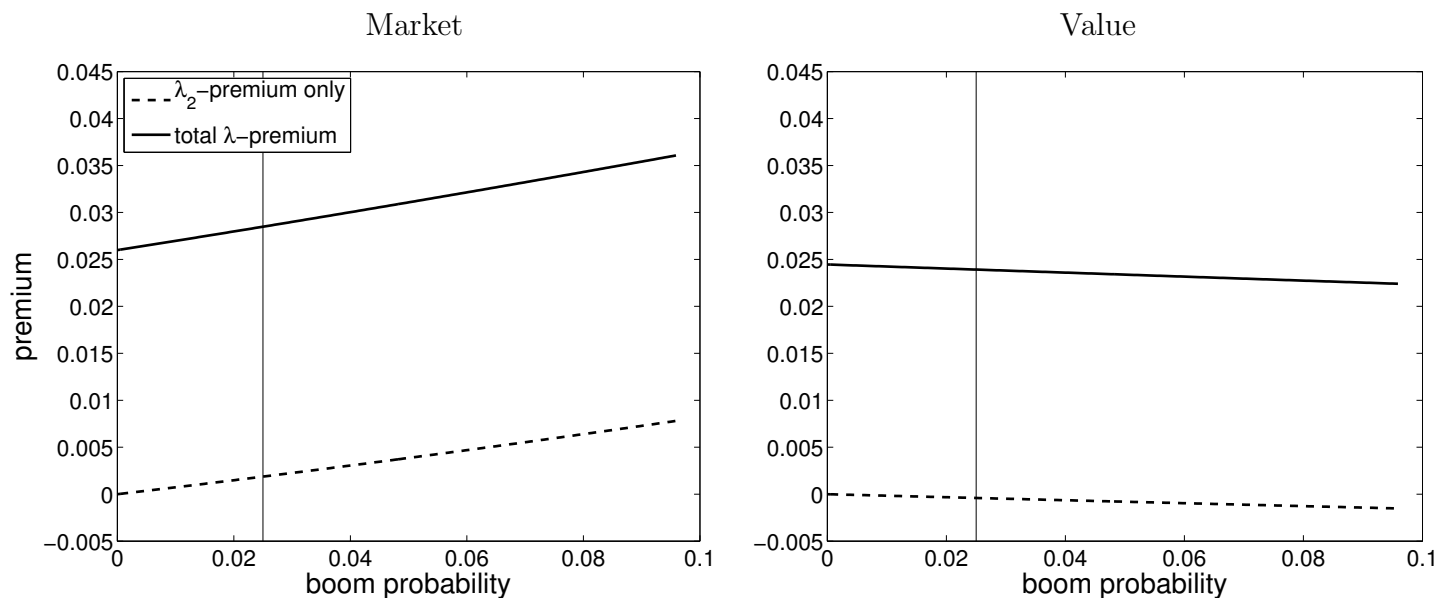
Notes: The solid line shows the component of the equity premium (left figure) and of the risk premium on the value claim (right figure) that compensates for the risk of changing rare event probabilities. This term, referred to as the  $\lambda$ -premium, can be divided into the compensation for disaster probabilities ( $\lambda_1$ -premium; shown by the dashed line) and the compensation for boom probabilities ( $\lambda_2$ -premium). Premiums are shown as a function of the disaster probability,  $\lambda_1$ , while the boom probability,  $\lambda_2$ , is fixed at its mean of 2.5%. The vertical line represents the mean of the disaster probability. Premiums are defined relative to the riskfree rate and are in annual terms.

Figure 7: Risk premiums as functions of the probability of a boom



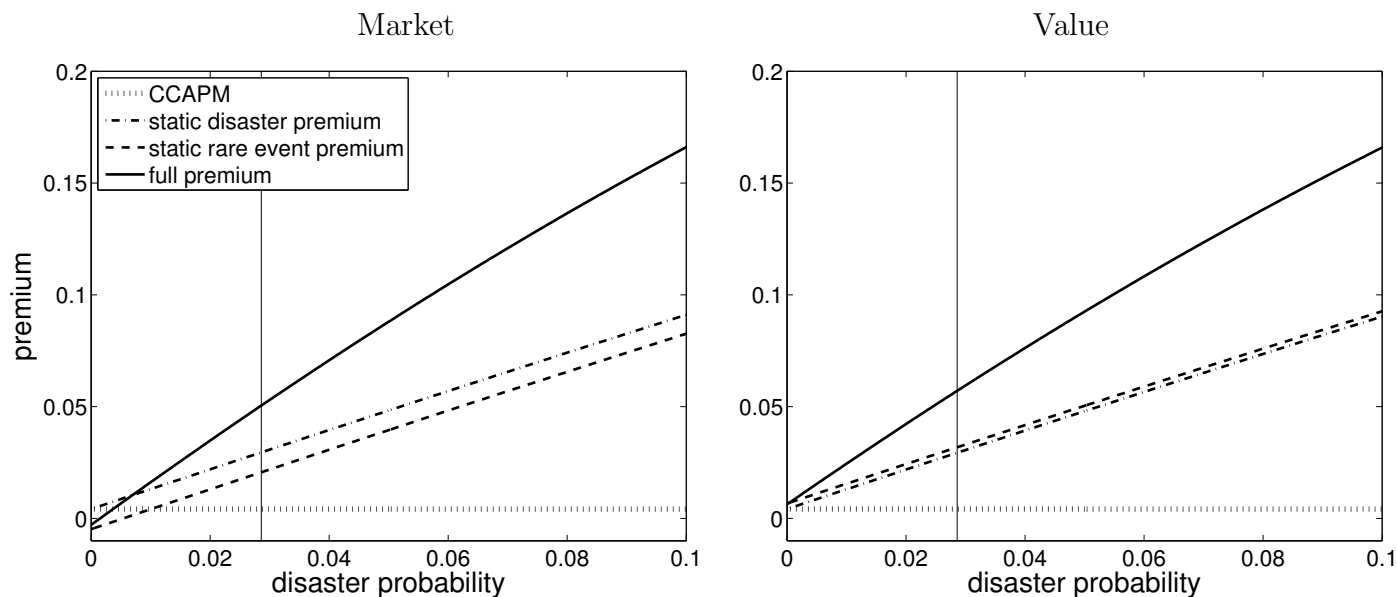
Notes: The figure shows components of the equity premium (left figure) and of the risk premium on the claim (right figure). The solid line represents the full premium, the dotted line the CCAPM premium, the dashed-dotted line the static boom premium and the dashed line the static rare event premium (namely, the static disaster premium plus the static boom premium). Premiums are shown as a function of the boom probability,  $\lambda_2$ , while the disaster probability,  $\lambda_1$ , is fixed at its mean of 2.86%. The vertical line represents the mean of the boom probability. Premiums are defined relative to the riskfree rate and are in annual terms.

Figure 8:  $\lambda$ -premiums (compensation for changing rare event probabilities) as a function of the probability of a boom.



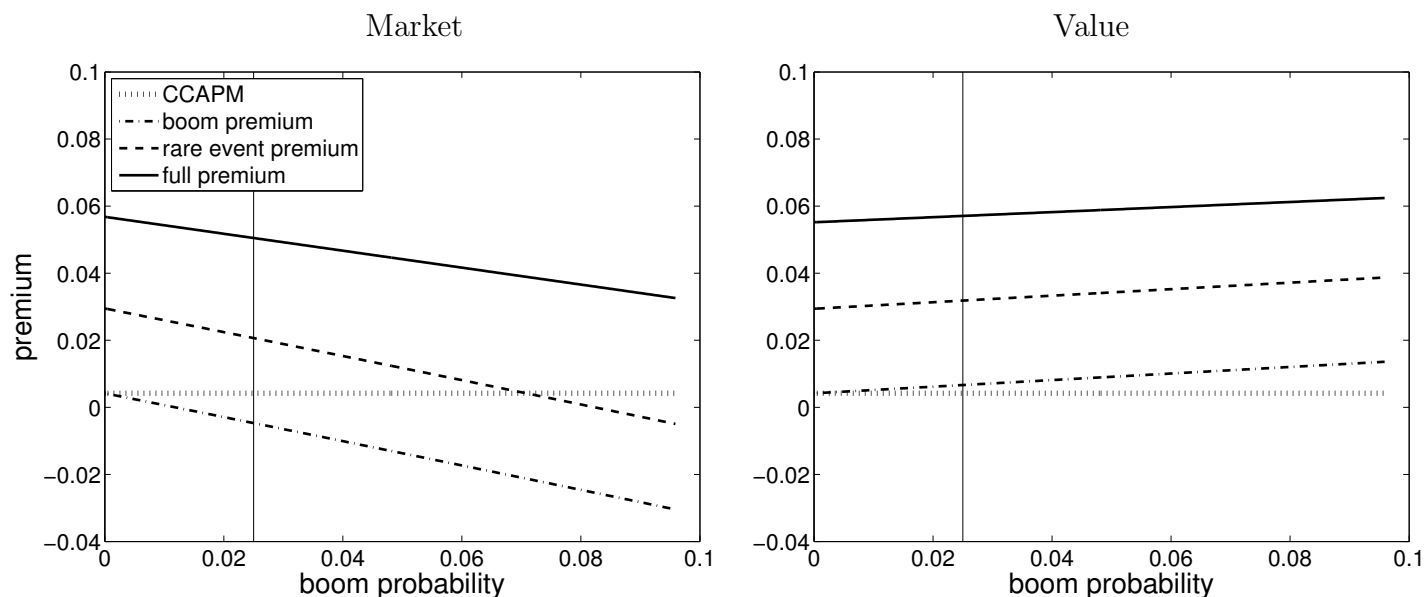
Notes: The solid line shows the component of the equity premium (left figure) and of the risk premium on the value claim (right figure) that compensates for the risk of changing rare event probabilities. This term, referred to as the  $\lambda$ -premium, can be divided into the compensation for disaster probabilities ( $\lambda_1$ -premium) and the compensation for boom probabilities ( $\lambda_2$ -premium; shown by the dashed line). Premiums are shown as a function of the boom probability,  $\lambda_2$ , while the disaster probability,  $\lambda_1$ , is fixed at its mean of 2.86%. The vertical line represents the mean of the boom probability. Premiums are defined relative to the riskfree rate and are in annual terms.

Figure 9: Observed expected excess returns in a sample without jumps as a function of disaster probability



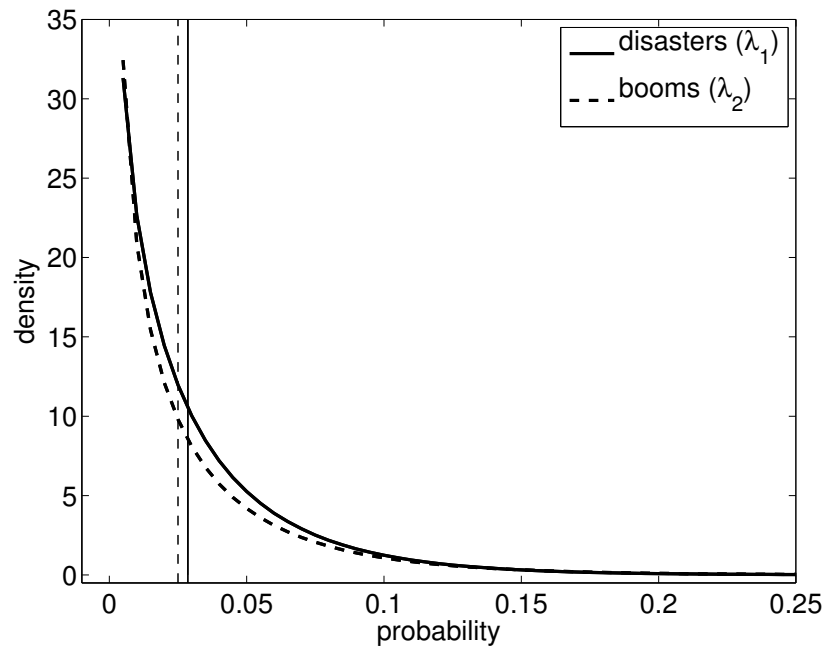
Notes: This figure shows expected realized returns in excess of the riskfree rate in a sample without jumps. The left panel shows expected excess returns on the market, while the right panel shows expected excess returns on the value claim. The solid line represents the full premium, the dotted line the CCAPM premium, the dashed-dotted line the static disaster premium (observed in a sample without jumps) and the dashed line the static rare events premium (also observed in a sample without jumps; this is the sum of the static disaster premium and the static boom premium). Premiums are shown as a function of the disaster probability,  $\lambda_1$ , while the boom probability,  $\lambda_2$ , is fixed at its mean of 2.5%. The vertical line represents the mean of the disaster probability. Premiums are defined relative to the riskfree rate and are in annual terms.

Figure 10: Observed expected excess returns in a sample without jumps as a function of boom probability



Notes: This figure shows expected realized returns in excess of the riskfree rate in a sample without jumps. The left panel shows expected excess returns on the market, while the right panel shows expected excess returns on the value claim. The solid line represents the full premium, the dotted line the CCAPM premium, the dashed-dotted line the static disaster premium (observed in a sample without jumps) and the dashed line the static rare events premium (also observed in a sample without jumps; this is the sum of the static disaster premium and the static boom premium). premiums are shown as a function of the boom probability,  $\lambda_2$ , while the disaster probability,  $\lambda_1$ , is fixed at its mean of 2.86%. The vertical line represents the mean of the boom probability. Premiums are defined relative to the riskfree rate and are in annual terms.

Figure 11: Stationary distributions of rare event probabilities



Notes: The figure shows the probability density function of the disaster probability  $\lambda_1$  and the boom probability  $\lambda_2$ . The probabilities are in annual terms. The vertical solid line shows the location of the mean of the disaster probability while the vertical dashed line shows the location of the mean of the boom probability.



Table 1: Parameter values

Panel A: Basic parameters	
Average growth in consumption (normal times) $\bar{\mu}_C$ (%)	2.02
Average growth in dividend (normal times) $\bar{\mu}_D$ (%)	3.55
Volatility of consumption growth (normal times) $\sigma$ (%)	2.00
Leverage $\phi$	3.5
Rate of time preference $\beta$	0.012
Relative risk aversion $\gamma$	3.0
Panel B: Disaster parameters	
Average probability of disaster $\bar{\lambda}_1$ (%)	2.86
Mean reversion in disaster probability $\kappa_{\lambda_1}$	0.11
Volatility parameter for disasters $\sigma_{\lambda_1}$	0.094
Mean reversion in expected consumption growth $\kappa_{\mu_1}$	1.00
Minimum consumption disaster (%)	10
Power law parameter for consumption disaster	7
Panel C: Boom parameters	
Average probability of boom $\bar{\lambda}_2$ (%)	2.50
Mean reversion in boom probability $\kappa_{\lambda_2}$	0.05
Volatility parameter for booms $\sigma_{\lambda_2}$	0.070
Mean reversion in expected consumption growth $\kappa_{\mu_2}$	0.60
Minimum consumption boom (%)	5
Power law parameter for consumption booms	20

Notes: Parameter values for the main calibration, expressed in annual terms.

Table 2: Extreme negative consumption events in the model and in the data

Panel A: 1-year rates of decline					
Growth rate	5 – 15	15 – 25	25 – 35	35 – 45	> 45
Data	6.89	1.22	0.28	0.12	0.08
Model 1	2.71	0.68	0.20	0.08	0.05
Model 2	0.92	0.80	0.44	0.27	0.33
Panel B: 5-year rates of decline					
Growth rate	35 – 45	45 – 55	55 – 65	65 – 75	> 75
Data	0.44	0.23	0.15	0.17	0.25
Model 1	0.65	0.41	0.27	0.17	0.25
Model 2	0.83	0.52	0.33	0.20	0.34

Notes: This table reports frequencies of rates of decline in consumption in the Barro and Ursua (2008) data and in data simulated from the model, for periods of lengths 1 and 5 years. Model 1 refers to the model presented in the text, with jumps in expected consumption growth. Model 2 refers to a model with jumps of the same size in realized consumption, but that is otherwise identical. We compute  $(C_t - C_{t+h})/C_t$ , where  $C$  is consumption and  $h$  is the relevant horizon. In both the model and in the data, growth rates are computed using overlapping annual observations. Frequencies are calculated by taking the number of observations within the given range divided by the total number of observations. Frequencies are expressed in percentage terms; for example, 1.22 refers to 1.22% of the observations.

Table 3: Extreme positive consumption events in the model and in the data

Panel A: 1-year growth rates					
Growth rate	5 – 15	15 – 25	25 – 35	35 – 45	> 45
Data	18.14	1.11	0.16	0.08	0.02
Model	10.44	0.28	0.02	0.00	0.00
Panel B: 5-year growth rates					
Growth rate	35 – 45	45 – 55	55 – 65	65 – 75	> 75
Data	2.13	0.55	0.11	0.02	0.00
Model	1.10	0.39	0.15	0.06	0.04

Notes: This table reports frequencies of growth rates in consumption in the Barro and Ursua (2008) data and in data simulated from the model, for periods of lengths 1 and 5 years. Namely, we compute  $(C_{t+h} - C_t)/C_t$ , where  $C$  is consumption and  $h$  is the relevant horizon. In both the model and in the data, growth rates are computed using overlapping annual observations. Frequencies are calculated by taking the number of observations within the given range divided by the total number of observations. Frequencies are expressed in percentage terms; for example, 1.11 refers to 1.11% of the observations. For the data, we exclude years following World War II as described in Figures 1 and 2.

Table 4: Log consumption and dividend growth moments

Panel A: Consumption growth								
		No-Jump Simulations			All Simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
mean	1.95	1.59	2.00	2.44	-0.10	1.78	3.15	1.70
standard deviation	1.45	1.69	1.99	2.28	1.94	3.32	6.96	4.19
skewness	-0.37	-0.49	0.01	0.48	-3.81	-0.91	1.42	-3.30
kurtosis	3.22	2.17	2.81	3.96	2.62	6.96	21.71	36.57

Panel B: Dividend growth								
		No-Jump Simulations			All Simulations			
	Data	0.05	0.50	0.95	0.05	0.50	0.95	Population
mean	1.67	1.85	3.30	4.83	-4.04	2.54	7.32	2.25
standard deviation	6.46	5.93	6.97	7.99	6.79	11.63	24.37	14.67
skewness	0.10	-0.49	0.01	0.48	-3.81	-0.91	1.42	-3.30
kurtosis	4.66	2.17	2.81	3.96	2.62	6.96	21.71	36.57

Notes: Data moments are calculated using annual data from 1947 to 2010. Population moments are calculated from simulating data from the model at a monthly frequency for 600,000 years and then aggregating monthly growth rates to an annual frequency. We also simulate 10,000 60-year samples and report the 5th-, 50th- and 95th-percentile for each statistic both from the full set of simulations and for the subset of samples for which no jumps occur.

Table 5: Aggregate market moments

	Data	No-Jump Simulations			All Simulations			Population
		0.05	0.50	0.95	0.05	0.50	0.95	
$E[R^b]$	1.25	2.67	2.93	3.06	0.90	2.70	3.97	2.60
$\sigma(R^b)$	2.75	0.10	0.22	0.43	0.24	2.40	5.76	3.28
$E[R^m - R^b]$	7.25	2.57	4.83	7.36	2.11	5.48	11.37	5.95
$\sigma(R^m)$	17.8	11.1	15.1	21.7	13.5	21.3	37.6	25.6
Sharpe ratio	0.41	0.19	0.32	0.45	0.10	0.26	0.41	0.23
$\exp(E[p - d])$	32.5	26.6	32.1	35.9	21.3	31.0	40.9	30.5
$\sigma(p - d)$	0.43	0.09	0.18	0.32	0.13	0.28	0.54	0.39
AR1( $p - d$ )	0.92	0.55	0.78	0.90	0.55	0.79	0.92	0.87

Notes: Data moments are calculated using annual data from 1947 to 2010. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 10,000 60-year samples and report the 5th-, 50th- and 95th-percentile for each statistic both from the full set of simulations and for the subset of samples for which no jumps occur.  $R^b$  denotes the government bond return,  $R^m$  denotes the return on the aggregate market and  $p - d$  denotes the log price-dividend ratio.

Table 6: Cross-sectional moments

	Data	No-Jump Simulations			All Simulations			Population
		0.05	0.50	0.95	0.05	0.50	0.95	
$E[R^v - R^b]$	7.95	3.18	5.36	7.90	1.67	4.57	7.73	4.59
$E[R^g - R^b]$	6.62	0.34	3.32	7.07	0.43	7.54	25.94	9.67
$E[R^v - R^g]$	1.34	-0.36	2.16	3.90	-21.58	-2.70	3.35	-5.07
$\sigma(R^v)$	17.0	10.4	14.0	19.9	11.8	17.9	26.3	18.8
$\sigma(R^g)$	21.0	18.2	25.5	37.0	23.1	42.8	120.1	66.9
$\sigma(R^v - R^g)$	11.7	12.6	18.4	26.2	15.2	36.3	120.3	64.0
Sharpe ratio, value	0.48	0.25	0.38	0.53	0.09	0.27	0.44	0.24
Sharpe ratio, growth	0.32	0.02	0.13	0.23	0.02	0.17	0.30	0.14
Sharpe ratio, value-growth	0.11	-0.01	0.12	0.27	-0.22	-0.07	0.20	-0.08
alpha, value	1.26	0.77	1.25	2.38	0.15	1.30	6.05	1.57
alpha, growth	-1.26	-6.97	-4.91	-3.03	-13.68	-3.93	0.70	-2.97
alpha, value-growth	2.53	4.01	6.16	8.86	-0.35	5.37	18.66	4.54
beta, value	0.92	0.77	0.91	0.97	0.14	0.79	0.96	0.51
beta, growth	1.09	1.18	1.44	1.73	1.21	1.63	3.34	2.12
beta, value-growth	-0.16	-0.94	-0.54	-0.22	-0.35	5.37	18.66	-1.62

Data moments are calculated using annual data from 1947 to 2010. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 10,000 60-year samples and report the 5th-, 50th- and 95th-percentile for each statistic both from the full set of simulations and for the subset of samples for which no jumps occur.  $R^v$  denotes the gross return on the value sector,  $R^g$  denotes the gross return on the growth sector, alpha denotes the loading of the constant term of the CAPM regression and beta denotes the loading on the market equity excess return of the CAPM regression. In the data, the growth portfolio is the lowest book-to-market decile. The remaining nine deciles comprise the value portfolio.

Table 7: Value spread moments

	No-Jump Simulations				All Simulations			Population
	Data	0.05	0.50	0.95	0.05	0.50	0.95	
$\exp(E[\log(\text{value spread})])$	1.23	1.16	1.20	1.32	1.16	1.26	1.71	1.32
$\sigma(\log(\text{value spread}))$	0.08	0.02	0.05	0.14	0.03	0.11	0.34	0.23
Value spread autocorrelation	0.79	0.57	0.80	0.93	0.55	0.78	0.92	0.89

Notes: Data moments are calculated using annual data from 1947 to 2010. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 10,000 60-year samples and report the 5th-, 50th- and 95th-percentile for each statistic both from the full set of simulations and for the subset of samples for which no jumps occur. The value spread is defined as the log of the book-to-market ratio for the value sector minus the book-to-market ratio for the aggregate market in the data, and as log price-dividend ratio for the aggregate market minus the log price-dividend ratio for the value sector in the model. In the data, the growth portfolio is the lowest book-to-market decile. The remaining nine deciles comprise the value portfolio.

Table 8: Long-horizon regressions of aggregate market returns on the price-dividend ratio

Panel A: 1-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	-0.12	[-2.41]	-0.56	-0.30	-0.16	-0.43	-0.16	0.03	-0.08
$R^2$	0.09		0.06	0.15	0.27	0.00	0.05	0.19	0.01
Panel B: 3-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	-0.29	[-3.37]	-1.17	-0.73	-0.40	-1.00	-0.43	0.10	-0.20
$R^2$	0.22		0.10	0.33	0.55	0.00	0.11	0.43	0.04
Panel C: 5-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	-0.41	[-3.37]	-1.54	-0.99	-0.52	-1.35	-0.63	0.16	-0.31
$R^2$	0.27		0.11	0.42	0.69	0.00	0.16	0.55	0.05

Notes: The table reports coefficients and  $R^2$ -statistics from predictive regressions of continuously compounded aggregate market returns in excess of the continuously compounded government bill rate. The predictor variable is the log of the price-dividend ratio on the market. Coef. refers to the coefficient on the predictor variable. Data are annual, from 1947 to 2010. For the data coefficients, we report *t*-statistics constructed using Newey-West standard errors. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 10,000 60-year samples and report the 5th-, 50th- and 95th-percentile for each statistic both from the full set of simulations and for the subset of samples for which no jumps occur.



Table 9: Long-horizon regressions of aggregate market returns on the value spread

Panel A: 1-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	-0.50	[-1.86]	-1.55	-0.36	0.04	-1.24	-0.12	0.22	$-3 \times 10^{-3}$
$R^2$	0.05		0.00	0.03	0.12	0.00	0.01	0.09	$7 \times 10^{-6}$
Panel B: 3-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	-1.18	[-2.28]	-3.85	-0.93	0.21	-3.15	-0.33	0.62	$-5 \times 10^{-3}$
$R^2$	0.12		0.00	0.07	0.29	0.00	0.03	0.22	$8 \times 10^{-6}$
Panel C: 5-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	-1.28	[-3.13]	-5.53	-1.31	0.44	-4.76	-0.50	1.03	$-4 \times 10^{-3}$
$R^2$	0.09		0.00	0.09	0.39	0.00	0.04	0.31	$4 \times 10^{-6}$

Notes: The table reports coefficients and  $R^2$ -statistics from predictive regressions of continuously compounded aggregate market returns in excess of the continuously compounded government bill rate. The predictor variable is the value spread, defined in the model as the log price-dividend ratio of the aggregate market minus log price-dividend ratio of the value sector and in the data as the log book-to-market of the value sector minus log book-to-market of the aggregate market. Coef. refers to the coefficient on the predictor variable. Data are annual, from 1947 to 2010. For the data coefficients, we report  $t$ -statistics constructed using Newey-West standard errors. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 10,000 60-year samples and report the 5th-, 50th- and 95th-percentile for each statistic both from the full set of simulations and for the subset of samples for which no jumps occur.

Table 10: Long-horizon regressions of value-minus-growth returns on the price-dividend ratio

Panel A: 1-year horizon									
			No-Jump Simulations			All Simulations			Population
	Data	$t$ -stat	0.05	0.50	0.95	0.05	0.50	0.95	
Coef.	0.01	[0.37]	-0.04	0.14	0.48	-0.12	0.06	0.38	$-5 \times 10^{-3}$
$R^2$	0.00		0.00	0.02	0.11	0.00	0.01	0.09	$4 \times 10^{-5}$
Panel B: 3-year horizon									
			No-Jump Simulations			All Simulations			Population
	Data	$t$ -stat	0.05	0.50	0.95	0.05	0.50	0.95	
Coef.	0.05	[0.51]	-0.11	0.36	1.10	-0.32	0.16	0.96	$-1 \times 10^{-2}$
$R^2$	0.01		0.00	0.06	0.27	0.00	0.03	0.21	$1 \times 10^{-4}$
Panel C: 5-year horizon									
			No-Jump Simulations			All Simulations			Population
	Data	$t$ -stat	0.05	0.50	0.95	0.05	0.50	0.95	
Coef.	0.09	[0.76]	-0.14	0.53	1.62	-0.50	0.26	1.40	$-2 \times 10^{-2}$
$R^2$	0.02		0.00	0.09	0.38	0.00	0.04	0.30	$2 \times 10^{-4}$

Notes: The table reports coefficients and  $R^2$ -statistics from predictive regressions of continuously compounded returns on the value portfolio in excess of continuously compounded returns on the growth portfolio. The predictor variable is the log of the price-dividend ratio on the market. Coef. refers to the coefficient on the predictor variable. Data are annual, from 1947 to 2010. For the data coefficients, we report  $t$ -statistics constructed using Newey-West standard errors. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 10,000 60-year samples and report the 5th-, 50th- and 95th-percentile for each statistic both from the full set of simulations and for the subset of samples for which no jumps occur.

Table 11: Long-horizon regressions of value-minus-growth returns on the value spread

Panel A: 1-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	0.46	[2.52]	0.19	0.86	2.41	-0.13	0.26	1.70	$-1 \times 10^{-2}$
$R^2$	0.10		0.01	0.09	0.22	0.00	0.02	0.15	$2 \times 10^{-4}$
Panel B: 3-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	1.13	[2.44]	0.56	2.23	5.18	-0.37	0.76	3.98	$-4 \times 10^{-2}$
$R^2$	0.19		0.05	0.23	0.47	0.00	0.05	0.36	$4 \times 10^{-4}$
Panel C: 5-year horizon									
		No-Jump Simulations			All Simulations				
	Data	<i>t</i> -stat	0.05	0.50	0.95	0.05	0.50	0.95	Population
Coef.	1.48	[2.37]	1.02	3.39	6.47	-0.61	1.24	5.44	$-6 \times 10^{-2}$
$R^2$	0.21		0.07	0.34	0.60	0.00	0.09	0.49	$6 \times 10^{-4}$

Notes: The table reports coefficients and  $R^2$ -statistics from predictive regressions of continuously compounded returns on the value portfolio in excess of continuously compounded returns on the growth portfolio. The predictor variable is the value spread, defined in the model as the log price-dividend ratio of the aggregate market minus log price-dividend ratio of the value sector and in the data as the log book-to-market of the value sector minus log book-to-market of the aggregate market. Coef. refers to the coefficient on the predictor variable. Data are annual, from 1947 to 2010. For the data coefficients, we report  $t$ -statistics constructed using Newey-West standard errors. Population moments are calculated from simulating monthly data from the model for 600,000 years and then aggregating to an annual frequency. We also simulate 10,000 60-year samples and report the 5th-, 50th- and 95th-percentile for each statistic both from the full set of simulations and for the subset of samples for which no jumps occur.