Quantifying Liquidity and Default Risks of Corporate Bonds over the Business Cycle^{*}

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Abstract

This paper builds over-the-counter search frictions into a structural model of corporate bonds with time varying macroeconomic and secondary market liquidity conditions. We explain both non-default and default components of corporate bonds by quantitatively studying endogenous liquidity and default risks jointly. Procyclical liquidity and countercyclical risk premium allows the model to match the total credit spread of corporate bonds of different rating classes, as well as their default probabilities and corresponding CDS spreads. We proposed a novel model-based decomposition scheme that captures the interaction between liquidity frictions and corporate default decisions via the rollover channel, and these interactions represent quantitatively important economic forces that were previously overlooked by empirical researchers.

Keywords: Macroeconomic Conditions, Time-Varying Liquidity, Rollover Risk, Over-The-Counter Market, Structural Models, Bid-Ask Spread

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1. Introduction

It is well known that default risk only accounts for a part of the pricing of corporate bonds. For example, Longstaff et al. [2005] estimate that the default component explains about 50% of the spread between the yields of Aaa/Aa-rated bonds and Treasury, or 70% of the credit spreads for Baa-rated bonds. Furthermore, Longstaff et al. [2005] show that the nondefault component of corporate bond spreads is only weakly related to the differential state tax treatment on corporate bonds and Treasury, but is strongly related to measures of corporate bond illiqiduity.

Until recently, the literature on credit risk modeling has mostly focused on understanding the default component of credit spreads. The "credit spread puzzle", first discussed by Huang and Huang (2012), refers to the finding that, when calibrated to match the observed default rates and recovery rates, traditional structural models have difficulty explaining the credit spreads for bonds rated investment grade and above. By introducing time-varying macroeconomic risks into the structual models, Chen, Collin-Dufresne, and Goldstein (2009), Bhamra et al. [2010] and Chen [2010] are able to explain the default components of the credit spreads for investment-grade corporate bonds.¹ However, the significant non-default components in credit spreads still remain to be explained.

Our paper attempts to provide a full resolution of the credit spread puzzle by quantitatively explaining both the default and non-default components of the credit spreads. To achieve this goal, we follow the *endogenous liquidity* approach in He and Milbradt [2012] by introducing the secondary over-the-counter market search friction (a la Duffie et al. [2005]) into the structual credit models with aggregate macroeconomic fluctuations (e.g., Chen [2010]). In our model, bond investors who

¹Chen (2010) explains the default component of the credit spread for BBB rated bonds by relying on the estimates of Longstaff et al. (2005), while Bhamra et al. (2010) focus on the difference between BBB and AAA rated bonds. The difference of spreads between BBB and AAA rated bonds presumably takes out the common liquidity component, which is a widely used practice in the literature. This is valid only if the liquidity components for both bonds are similar, an assumption that we later show is not true.

purchase bonds in the primary market face the risk of idiosyncratic liquidity shocks that drive up their costs for holding the bonds. Market illiquidity arises endogenously because to sell the bonds, these investors have to search for dealers to intermediate transactions with other investors not hit by liquidity shocks. The dealers set bid-ask spreads to capture part of trading surplus when bargaining with the illiquid investors. Default risk affects the liquidity discount on corporate bonds by influencing the bargaining power of the illiquid bond investors.

The endogenous liquidity is further amplified by the *endogenous default* mechanism, first established in Leland and Toft [1996] and emphasized recently by He and Milbradt [2012]. As illustrated in He and Milbradt [2012], a default-liquidity spiral arises: when secondary market liquidity deteriorates, equity holders suffer greater rollover losses in refinancing their maturing bonds and will consequently default earlier. This earlier default in turn worsens secondary bond market liquidity even further, and so on so forth.

In contrast to He and Milbradt [2012], where primitive parameters associted with secondary market illiquidity are assumed to be constant over time, in this paper we explicitly model the cyclical variation in the secondary bond market illiquidity, which interacts with the cyclical variation in the firm's cash flows and aggregate risk prices. Our calibration strategy is as follows. First, we calibrate the parameters governing the pricing kernel to fit key moments of asset prices. The parameters for the cash flow process are calibrated using moments of aggregate corporate profit and equity return volatility. Next, the parameters governing secondary market liquidity are calibrated using data on bond turnovers, dealer's bargaining power and bid-ask spreads. Finally, we design a new strategy to estimate the liquidity discount for defaulted bonds using the emergence returns of defaulted bonds.

We examine the model performance for corporate bonds with four credit ratings: Aaa/Aa, A, Baa, and Ba. There are two contributions relative to the existing literature. First, different from Chen, Collin-Dufresne, and Goldstein (2009), Bhamra et al. [2010] and Chen [2010] who focus on explaining the *default component* of investment grade bonds (specifically, the spread difference between Baa and Aaa), our model aims at explaining the *total credit spreads* across a wide range of ratings including both Aaa and Baa. Second, modelling the bond market liquidity endogenously allows us to directly investigate the model's quantitative performance on bond market liquidity in addition to the the two common measures — cumulative default probabilities and credit spreads — that the previous literature on corporate bonds calibration (e.g., Huang and Huang, 2012) has focused on. More specifically, we investigate the model performance in matching the observed cross-sectional pattern of CDS spreads and bid-ask spreads across different ratings,² as it is widely accepted that CDS spreads mostly price the default component while the bid-ask spread reflects the bond illiquidity.

The advantage of our model is its parsimony thanks to the endogenously linking illiquidity to firm's distance-to-default, so that we only change the distance to default across different credit ratings to match the corresponding historical default rates. Even so, our model is able to match the empirical cross-sectional pattern in bond illiquidity across credit ratings. Relative to previous literature that only focused on explaining the difference between Baa and Aaa bond spreads, we show that our model-predicted credit spreads, which include *both* the default premium and liquidity premium, can quantitatively match the total credit spread we observe in the data.

A common practice in the emprical literature is to decompose credit spreads into liquidity and default components, which natually leads to the interpretation that the iquidity and default components are independent of each other. However, our structual model with endogenous default and endogenous liquidity challenges this view: both liquidity and default components are endogenously linked and may reinforce each other, and thus there can be economically significant

²In a model without bond market illiquidity, there is no bid-ask spread and the CDS spread should be equal to the bond's credit spread.

interaction terms. These dynamic interactions are difficult to capture using reduced-form models with *exogenously imposed* liquidity premia.

Our structual model allows us to quantify the interactions between default and liquidity for corporate bonds, as we propose a finer decomposition that nests the common default-liquidity decomposition. More specifically, similar to the idea of Credit Derivative Swap (CDS) pricing in Longstaff et al. [2005] proxying for default risk, we identify this "default" part by pricing a bond in a counterfactually perfectly liquid market but with the model implied default boundary. We identify the remaining credit spread after subtracting this "default" part as the "liquidity" part. Next, we further decomposes this "default" (or "liquidity") part into a "pure default" (or a "pure liquidity") component and a "liquidity-driven-default" (or a "default-driven-liquidity") component, where the "pure default" (or "pure liquidity") component is defined by the spread implied by a counterfactual model where only the equity holders' endogenous default in a perfectly liquid market (or the over-the-counter secondary market search friction absent default) is at work as in Leland and Toft [1996] (or as in Duffie et al. [2005]). The two interaction terms, i.e., the "liquidity-drivendefault" and the "default-driven-liquidity" components, thus capture the enodgenous positive spiral between default and liquidity. For instance, "liquidity-driven-default" is driven by the rollover risk mechanism in that firms relying on finite-maturity debt financing will default earlier when facing worsening secondary market liquidity.

This finer decomposition proposed by our model not only gives a more complete picture of how the default and liquidity forces affect credit spreads of corporate bonds, but also offers important insight on evaluating hypothetical government policies. For instance, providing subsidized term loans to financial intermediaries who are active in the secondary bond market improves the market liquidity, and our model implies that the first-order impact of such policies aimed at market liquidity is on the pure liquidity part and liquidity-driven-default part. In contrast, the prevailing view in the literature, which masks the interdependence between default and liquidity components, would interpret such a policy as only affecting the liquidity part. Thus, the liquidity-driven-default is mistakenly excluded while the default-driven-liquidity part is mistakenly included.

The paper is strutured as follows. Section 2 introduces the model. Section 3 gives the solutions to debt valuations and default boundaries. Section 4 presents the main calibration. Section 5 introduces the model based decomposition of the results of the calibration. Section 6 presents the analaysis of hypothetical government policies. Section 7 concludes. The appendix provides proofs and the general form of the model.

2. The Model

2.1 Aggregate States and the Firm

The following model elements are similar to Chen [2010] and Bhamra et al. [2010], except that we sudy the case in which firms issue bonds with an average finite maturity a la Leland (1998) so that rollover risk is present.

2.1.1 Aggregate states and stochastic discount factor

The aggregate state of the economy is described by a continuous time Markov chain, with the current Markov state denoted by s_t and the physical transition density between state i and state j denoted by $\zeta_{ij}^{\mathcal{P}}$. We assume an exogenuous stochastic discount factor (SDF):

$$\frac{dm_t}{m_t} = -rdt - \eta(s_t) \, dZ_t^m + \sum_{s_t \neq s_t'} \left(e^{\kappa(s_{t-},s_t)} - 1 \right) dM_t^{(s_{t-},s_t)},\tag{1}$$

where $\eta(\cdot)$ is the state dependent price of risk for Brownian shocks, and $dM_t^{(j,k)}$ is a compensated Poison process capturing switches between states and $\kappa(i,j)$ embeds the jump risk premia such that in the risk neutral measure, the distorted jump intensity between states is $\zeta_{ij}^{\mathcal{Q}} = e^{\kappa(i,j)} \zeta_{ij}^{\mathcal{P}}$.

Note that in Eq. (1), motivated by US data we have assumed a constant (i.e., state-independent) risk-free rate $r_f(s) = r$. In this paper we focus on the case with binary aggregate states, i.e., $s_t \in \{G, B\}$. In Appendix we provide the general setup for the case with n > 2 aggregate states.

2.1.2 Firm cash flows

A firm has assets in place that gerenates cash flows at the rate of y_t , and under the phyical measure \mathcal{P} we have

$$\frac{dy_t}{y_t} = \mu_{\mathcal{P}}\left(s\right)dt + \sigma_m\left(s\right)dZ_t^m + \sigma_f dZ_t^f,\tag{2}$$

with s being an aggregate state that (possibly) influences the cash-flow process. Here, dZ_t^m captures aggregate Brownian risk, while dZ_t^f captures idiosyncratic Brownian risk. Given the stochastic discount factor m_t , risk neutral cash flow dynamics under the risk neutral measure Q follow

$$\begin{aligned} \frac{dy_t}{y_t} &= \mu_{\mathcal{Q}}\left(s\right)dt + \sigma\left(s\right)dZ_t^{\mathcal{Q}},\\ dZ_t^{\mathcal{Q}} &= \frac{\sigma_m(s)}{\sqrt{\sigma_m^2\left(s_t\right) + \sigma_f^2}}dZ_t^m + \sqrt{1 - \frac{\sigma_m^2\left(s\right)}{\sigma_m^2\left(s\right) + \sigma_f^2}}dZ_t^f + \frac{\sigma_m\left(s\right)}{\sigma\left(s\right)}\eta\left(s\right)dt, \end{aligned}$$

where $Z_t^{\mathcal{Q}}$ is a Brownian Motion under the risk-neutral measure \mathcal{Q} . The risk-neutral cash-flow drift is given by

$$\mu_{\mathcal{Q}}(s) \equiv \mu_{\mathcal{P}}(s) - \sigma_m(s) \eta(s), \text{ and } \sigma(s) \equiv \sqrt{\sigma_m^2(s) + \sigma_f^2}$$

For ease of notation, we work with log cash flows $\delta \equiv \log(y)$ throughout. Define

$$\mu\left(s\right) \equiv \mu_{\mathcal{Q}}\left(s\right) - \frac{1}{2}\sigma^{2}\left(s\right) = \mu_{\mathcal{P}}\left(s\right) - \sigma_{m}\left(s\right)\eta\left(s\right) - \frac{1}{2}\left(\sigma_{m}^{2}\left(s\right) + \sigma_{f}^{2}\right)$$

so that we have

$$d\delta_t = \mu(s) dt + \sigma(s) dZ_t^{\mathcal{Q}}.$$
(3)

From now on we work under measure Q unless otherwise stated, so we drop the superscript Q in dZ_t^Q and ζ_{ij}^Q to simply write dZ_t and ζ_{ij} where no confusion can arise.

The unlevered firm value, given the aggregate state s, is denoted by

$$\mathbf{v}_{U}\left(\delta\right) \equiv \left[\mathbf{RR} - \boldsymbol{\mu}\boldsymbol{\mu} - \mathbf{QQ}\right]^{-1} \mathbf{1} \exp\left(\delta\right) \tag{4}$$

where $\mathbf{RR} = r\mathbf{I}_2$ and \mathbf{I}_2 is the 2x2 identity matrix, $\boldsymbol{\mu}\boldsymbol{\mu} \equiv \begin{bmatrix} \mu_G & 0 \\ 0 & \mu_B \end{bmatrix}$, and $\mathbf{QQ} = \begin{bmatrix} -\zeta_G & \zeta_G \\ \zeta_B & -\zeta_B \end{bmatrix}$. Later we will use v_U^s to denote the element of \mathbf{v}_U in state s.

2.1.3 Firm's debt maturity structure and rollover frequency

The firm has bonds in place of measure 1 which are identical except for their time to maturity, and thus the aggregate and individual bond coupon (face value) is c (p). As in Leland (1998), equity holders commits to holding the aggregate coupon and outstanding face-value constant outside default, and thus issues new bonds of the same average maturity as the bonds maturing.

Bonds' maturity is stochastic: each unit of bonds matures "randomly" in a Poisson fashion (i.i.d across individual bonds) with intensity m. This implies an expected average debt maturity of $\frac{1}{m}$. The deeper implication of this assumption is that the firm adopts a "smooth" debt maturity structure with a uniform distribution, and the firm's average refinancing/rollover frequency is m. As shown later, the rollover frequency (at the firm level) is important for the impact of liquidity affecting the firm's endogenous default decisions. Later we will calibrate this number to the actual rollover frequency of US firms.

2.2 Liquidity in Secondary Over-the-Counter Corporate Bond Market

We follow He and Milbradt [2012] to model the over-the-counter corporate bond market. The setting builds on Duffie et al. [2005], in that it seamlessly integrates a search-based dealer intermediation OTC market into the Leand-type structual credit risk frameowork described above. Individual bond holders are subject to liquidity shocks that entail a positive holding cost]. Bond holders hit by liquidity shocks will search dealers in the over-the-counter secondary market, and endogenous transaction prices are determined when they meet a dealer.

More specifically, every bond holder can only hold one unit of the bond. Individual bond holders start in the H state without any holding cost when purchasing corporate bonds in the primary market. As time passes by, H type bond holders are hit independently by liquidity shocks with intensity $\xi(s)$, which lead them to become L types who bear a positive holding cost $\chi(s)$ per unit of time. As there is H investors without the bond waiting on the sideline, it is optimal for L investors to try to sell the bond to H investors. However, there is a trading friction in moving the bond holdings from inefficient L-type bond investors (seller) to efficient H-types, in that trades have to be intermediated by dealers in the over-the-counter market.

L type sellers meet dealers with intensity $\lambda(s)$, which we interpret as the intermediation intensity of the financial sector. After L-type investors sell their holdings they exit the market forever. The H-type buyers on the sideline currently not holding the bond also contact dealers with some intensity $\lambda(s)$. As no dealer meets a buyer and a seller at the same time, dealers use the competitive (and instantaneous) interdealer market to lay of or buy a position in bonds. For simplicity, we assume that the flow of H-type buyers contacting dealers is greater than the flows of L-type sellers contacting dealers, so that the secondary market is a *seller's market*.

Fixing any aggregate state s, and denote by D_l^s the individual bond valuation for the investor with type $l \in \{H, L\}$. We follow [duffie2007b] and assume Nash-bargaining weights β of the investor and $1 - \beta$ of the dealer across all dealer-investor pairs. When a contact between a type L investor and a dealer occurs, the dealer can instantaneously sell a bond at a price M to another dealer who is in contact with an H investor. If he does so, the bond travels from an L investor to an Hinvestor via the help of the two dealers who are connected in the inter-dealer market. Denote by B^s the bid price at which the L type is selling his bond, by A^s the ask price at which the H type is purchasing this bond, and by M^s the inter-dealer market price. Similar to DGP05 and He and Milbradt (2013), we have the following proposition.

Proposition 1 Fix valuations D_H^s and D_L^s , and denote the surplus from trade by $\Pi^s = D_H^s - D_L^s > 0$. 0. The ask price A^s and inter-dealer market price M^s are equal to D_H^s , and the bid price is given by $B^s = \beta D_H^s + (1 - \beta) D_L^s$. The dollar bid ask spread is $A^s - B^s = (1 - \beta) (D_H^s - D_L^s) = (1 - \beta) \Pi^s$.

Essentially, Bertrand competition, the holding restriction and the surplus of buyer-dealer pairs in the interedealer market drives the surplus of buyer-dealer pairs to zero. This further implies that the value function of investors not holding the asset is identically zero, which makes the model very tractable.

To compare our model with empirical studies on the bid-ask spread, in the calibration we report the proportional bid-ask spread which is defined as the dollar bid-ask spread divided by the mid price, i.e.,

$$\Delta^{s}\left(\delta,\tau\right) = \frac{2\left(1-\beta\right)\left(D_{H}^{s}-D_{L}^{s}\right)}{\left(1+\beta\right)D_{H}^{s}+\left(1-\beta\right)D_{L}^{s}}$$

The liquidity shock intensity $\xi(s)$, the holding cost $\chi(s)$, and the meeting intensity with dealers $\lambda(s)$, can depend on the macro state s. From the perspective of bond investors, mutual funds that hold corporate bonds may be more likely to experience fund outflows and thus more desperate to selling their bolding holdings in bad state, which gives rise to a higher $\xi(s)$ and $\chi(s)$. From the perspective of market liquidity and intermediation, a rise of meeting intensity $\lambda(s)$ resembles a

shock to the financial system that lowers the intermediation ability of the financial sector.

2.3 Markov Transition Matrix

We will use capitalized bold-faced letters (e.g., \mathbf{X}) to denote matrices, lower case bold face letters (e.g. \mathbf{x}) to denote vectors (the only exceptions are the value functions for debt and equity, \mathbf{D}, \mathbf{E} respectively, which will be vectors, and the matrices for drift and volatility, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$), and nonbold face letters denote scalars (e.g. x). Dimensions for most objects are given underneath the expression. While in this paper we focus on 2-aggregate-state case where $s \in \{G, B\}$, i.e., good and bad states, the Appendix presents general results for any arbitrary number of (Markov) aggregate states.

Denote by \mathbf{Q} the Markov-transition matrix of aggregate and individual states, where each entry $q_{ls \to l's'}$ is the intensity of transitioning from (individual) liquidity state l to l' where $l, l' \in \{H, L\}$ and from aggregate state s to s' where $s, s' \in \{G, B\}$.³ Thus, the transition matrix \mathbf{Q} is

$$\mathbf{Q}_{4\times4} \equiv \begin{bmatrix}
-\sum_{ls\neq HG} q_{HG\rightarrow ls} & q_{HG\rightarrow LG} & q_{HG\rightarrow HB} & 0 \\
q_{LG\rightarrow HG} & -\sum_{ls\neq LG} q_{LG\rightarrow ls} & 0 & q_{LG\rightarrow LB} \\
q_{HB\rightarrow HG} & 0 & -\sum_{ls\neq HB} q_{HB\rightarrow ls} & q_{HB\rightarrow LB} \\
0 & q_{LB\rightarrow LG} & q_{LB\rightarrow HB} & -\sum_{ls\neq LB} q_{LB\rightarrow ls}
\end{bmatrix}$$

$$= \begin{bmatrix}
-\xi (G) - \zeta (G) & \xi (G) & \zeta (G) & 0 \\
\beta\lambda (G) & -\beta\lambda (G) - \zeta (G) & 0 & \zeta (G) \\
\zeta (B) & 0 & -\xi (B) - \zeta (B) & \xi (B) \\
0 & \zeta (B) & \beta\lambda (B) & -\beta\lambda (B) - \zeta (B)
\end{bmatrix}$$
(5)

³Our intensity-based modelling rules out the possibility of coinciding jumps in the aggregate and individual states (i.e., $q_{ls \to l's'} = 0$ if $l \neq l'$ and $s \neq s'$), an assumption that can potentially be relaxed. Economically, this implies that the aggregate shock can bring about more liquidity shocks to individual debt holders given any time interval (but these shocks are still i.i.d across individuals).

The entry $q_{Ls \to Hs}$ in the above transition matrix requires further explanation. In our model, given the aggregate state s, the intensity of switching from H state to L state is $\xi(s)$, and the L-state is absorbing (those L-type investors leave the market forever). Note that L-type's intensitymodulated surplus when meeting the dealer, as shown in Section 2.2, can be rewritten as

$$\lambda(s) \left(B^s - D^s_L \right) = \lambda(s) \beta \left(D^s_H - D^s_L \right).$$

Thus, for the purpose of pricing, the "effective" transition intensity from *L*-type to *H*-type is $q_{Ls \to Hs} = \lambda(s) \beta$ where $\lambda(s)$ is the state-dependent intermediation intensity and β is the investor's bargaining power. To see this, n

2.4 Delayed Bankruptcy Payouts and Effective Recovery Rates

In Leland-type frameworks, when the firm cash flows deteriorates, equity holders are willing to repay the maturing debt holders only when the equity value is still positive so that the option value of keeping the firm alive justifies absorbing rollover losses. The firm defaults when its equity value drops to zero at some endogenous default threshold δ_{def} , which is endogenously chosen by equity holders. As in Chen [2010], we will impose bankruptcy costs as a fraction $1 - \hat{\alpha}(s)$ of the value from unlevered assets $v_U^s(\delta_{def})$ in (4), where the debt holder's bankruptcy recovery $\hat{\alpha}(s)$ may depend on the aggregate state s.

As emphasized in He and Milbradt [2012], because the driving force of liquidity in our model is that agents value receiving cash early, our bankruptcy treatment has to be careful in this regard (and different from typical Leland models). If bankruptcy leads investors to receive the proceeds immediately, then bankruptcy confers a "liquidity" benefit similar to bond maturing. This "expedited payment" benefit runs counter to the fact that in practice bankruptcy leads to the freezing of assets within the company and a delay in the payout of any cash depending on court proceeding.⁴ Moreover, bond investors with defaulted bonds may face a much more illiquid secondary market, and potentially a much higher holding cost once liquidity shocks hit due to regulatory or charter restrictions which prohibit institutions to hold defaulted bonds.

To capture above features, we assume that a bankruptcy court delay leads the bankruptcy cash payout $\hat{\alpha}(s) v_U^s$ to occur at a Poisson arrival time with intensity θ .⁵ The holding cost for *L*-type investors is $\chi_{def} v_U^s$ where $\chi_{def} > 0$, and the secondary market for defaulted bonds is illiquid with contact intensity λ_{def} . Given aggregate state *s* and default boundary δ_{def} , denote the value of defaulted bonds by $D_H^{s,def}$ and $D_L^{s,def}$. Their valuation equations are

$$rD_{H}^{s,def} = \theta \left[\hat{\alpha} \left(s \right) v_{U}^{s} - D_{H}^{s,def} \right] + \xi \left(s \right) \left[D_{L}^{s,def} - D_{H}^{s,def} \right] + \zeta \left(s \right) \left[D_{H}^{s',def} - D_{H}^{s,def} \right]$$
$$rD_{L}^{s,def} = -\chi_{def} \left(s \right) v_{U}^{s} + \theta \left[\hat{\alpha} \left(s \right) v_{U}^{s} - D_{L}^{s,def} \right] + \lambda_{def} \left(s \right) \beta \left[D_{H}^{s,def} - D_{L}^{s,def} \right] + \zeta \left(s \right) \left[D_{L}^{s',def} - D_{L}^{s,def} \right]$$

Take $D_L^{s,def}$ for example: the first term is the illiquidity holding cost, the second term captures the bankruptcy payout, the third term captures trading the defaulted bonds with dealers, and the last term captures the jump of the aggregate state. In Eq. (6) we have assumed that the cashflow rate δ remains constant at δ_{def} during bankrupcy procedures, a simplifying assumption that can be easily relaxed.⁶ Defining $\mathbf{D}^{def} \equiv \left[D_H^{G,def}, D_L^{G,def}, D_H^{B,def}, D_L^{B,def} \right]^{\top}$, it is easy to show that

$$\mathbf{D}^{def}\left(\delta\right) \equiv \left(\mathbf{R} - \mathbf{Q}_{def} + \theta \mathbf{I}\right)^{-1} \operatorname{diag}\left(\mathbf{v}_{U}\left(\delta\right)\right) \left(\theta \hat{\boldsymbol{\alpha}} - \boldsymbol{\chi}_{def}\right) = \boldsymbol{\alpha}^{\top} \cdot \operatorname{diag}\left(\mathbf{v}_{U}\left(\delta\right)\right)$$
(7)

 $^{^{4}}$ The Lehman Brothers bankruptcy in September 2008 is a good case in point. After much legal uncertainty, payouts to the debt holders only started trickling out after about three and a half years.

⁵We could allow for a state-dependent bankruptcy court delay, i.e., $\theta(s)$; but the Moody's Ultimary Recovery Dataset reveals that there is little difference between the recovery time in good time versus bad time.

⁶And we assume that $\alpha V_{FB}^s(\delta_{def}) < p$, i.e., the total debt face value exceeds the payout. The result will be identical if we assume that δ evolves as in (3), and debt holders receive the entire payout (net bankruptcy cost) of αV_{FB} eventually. The defaulted bonds values will be slightly lower if we take into account that equity holders receive some payouts in the event of $\alpha V_{FB} > p$, but one can derive the forumula of $D_H^{s,def}$ and $D_L^{s,def}$ easily.

where $\boldsymbol{\chi}_{def} \equiv [0, \chi_{def}(G), 0, \chi_{def}(B)]^{\top}$ and where \mathbf{Q}_{def} is the post-default counterpart of \mathbf{Q} in in (5).

For easier comparison to existing Leland-type models where debt recovery at bankruptcy is simply $\hat{\alpha}v_U$, we denote the (bold face) vector $\boldsymbol{\alpha} \equiv \left[\alpha_H^G, \alpha_L^G, \alpha_H^B, \alpha_L^B\right]^{\top}$ as the effective bankruptcy recovery rates at the time of default. We will have $\alpha_H^s > \alpha_L^s$ to capture the fact that default is more hurtful for bond holders in the illiquid state. Since we mainly focus on bond spreads before firm default, for the rest of the paper we take $\boldsymbol{\alpha}$ as the primitive parameters, because they are determined by post-default market structures. However, as emphasized in He and Milbradt [2012], because $\mathbf{v}_U(\delta)$ is endogenous as it depends on δ , the dollar bid-ask spread of defaulted bonds is higher if the firm defaults earlier. Thus, the illiquidity of defaulted bonds, relative to that of defaultfree bonds whose dollar bid-ask spreads are proportional to principal p and coupon c, depends on the firm's pre-default parameters through the channel of endogenous default.

These effective bankruptcy recovery factors are the only critical ingredients for us to solve for the pre-default bond valuations, as well as their the market liquidity. In calibration, we choose these effective recovery rates to target the valuation gap between the market price of defaulted bonds observed immedially after default (which are close to L-type valuations) and their ultimate recovery values (which are close to H-type valuations).

3. Debt Valuation and Default Boundaries

Denote by $D_l^{(s)}$ the *l*-type debt value in aggregate state *s*, $E_l^{(s)}$ the equity value in aggregate state *s*, and $\boldsymbol{\delta}_{def} = [\boldsymbol{\delta}_{def}(G), \boldsymbol{\delta}_{def}(B)]^{\top}$ the vector of endogenous default boundaries. We derive the closed-form solution for debt and equity valuations in this section as a function of $\boldsymbol{\delta}_{def}$, along with the characterization of endogenous default boundaries $\boldsymbol{\delta}_{def}$.

3.1 Debt Valuations

Because equity holders will default earlier in bad state, i.e., $\delta_{def}(G) < \delta_{def}(B)$, the domains of debt valuations changes when the aggregate state switches. We deal with this issue by the following treatment; see the Appendix for the generalization of this analysis.

Define two intervals $I_1 = [\delta_{def}(G), \delta_{def}(B)]$ and $I_2 = [\delta_{def}(B), \infty)$, and denote by $D_l^{s,i}$ the restriction of D_l^s to the interval I_i , i.e., $D_l^{s,i}(\delta) = D_l^s(\delta)$ for $\delta \in I_i$. Clearly, $D_l^{B,1}(\delta) = \alpha_l V_U^B(\delta)$ is in the "dead" state, so that the firm immediately defaults in interval I_1 when switching into state B (from state G). In light of this observation, in interval $I_2 = [\delta_{def}(B), \infty)$ all bond valuations denoted by $\mathbf{D}^{(2)} = \left[D_H^{G,2}, D_L^{B,2}, D_H^{B,2}, D_L^{B,2} \right]^{\top}$ are "alive." We have the following system of ODEs for $\mathbf{D}^{(2)}$:

$$[(r+m)\mathbf{I}_{4} - \mathbf{Q}]\mathbf{D}^{(2)} = (c\mathbf{1}_{2i} - \boldsymbol{\chi}^{(2)}) + \boldsymbol{\mu}^{(2)}(\mathbf{D}^{(2)})' + \frac{1}{2}\boldsymbol{\Sigma}^{(2)}(\mathbf{D}^{(2)})'' + m \cdot p\mathbf{1}_{2i} \text{ for } \delta \in I_{2} = [\delta_{def}(B), \infty).$$
(8)
In contrast, in interval $I_{1} = [\delta_{def}(G), \delta_{def}(B)]$ where $D_{l}^{(B,1)}$ is "dead," $\mathbf{D}^{(1)} = \left[D_{H}^{(G,1)}, D_{L}^{(G,1)}\right]^{\top}$

satisfies

$$[(r+m)\mathbf{I}_{4} - \mathbf{Q}]\mathbf{D}^{(1)} = (c\mathbf{1}_{2i} - \boldsymbol{\chi}^{(1)}) + \boldsymbol{\mu}^{(1)}(\mathbf{D}^{(1)})' + \frac{1}{2}\boldsymbol{\Sigma}^{(1)}(\mathbf{D}^{(1)})'' + m \cdot p\mathbf{1}_{2i} \text{ for } \delta \in I_{1} = [\delta_{def}(G), \delta_{def}(B)]$$
(9)

As shown in Appendix, the general solution on interval i is given by

$$\underbrace{\mathbf{D}_{2i\times1}^{(i)}}_{2i\times1} = \underbrace{\mathbf{G}_{2i\times4i}^{(i)}}_{4i\times4i} \cdot \underbrace{\exp\left(\mathbf{\Gamma}^{(i)}\delta\right)}_{4i\times1} \cdot \underbrace{\mathbf{b}_{4i\times1}^{(i)}}_{2i\times1} + \underbrace{\mathbf{k}_{0}^{(i)}}_{2i\times1} + \underbrace{\mathbf{k}_{1}^{(i)}}_{2i\times1} \exp\left(\delta\right) \tag{10}$$

where the constants vector $\mathbf{b}^{(i)}$ will be determined via appropriate boundary conditions. The

boundary conditions at $\delta = \infty$ and $\delta = \delta_{def}(G)$ are standard:

$$\lim_{\delta \to \infty} \left| \mathbf{D}^{(2)} \left(\delta \right) \right| < \infty, \text{ and } \mathbf{D}^{(1)} \left(\delta_{def} \left(G \right) \right) = \begin{bmatrix} \alpha_H^G \\ \alpha_L^G \end{bmatrix} v_U^G \left(\delta_{def} \left(G \right) \right)$$
(11)

For the boundary $\delta_{def}(B)$, we must have value matching conditions for all functions across $\delta_{def}(B)$:

$$\mathbf{D}^{(2)}\left(\delta_{def}\left(B\right)\right) = \begin{bmatrix} \mathbf{D}^{(1)}\left(\delta_{def}\left(B\right)\right) \\ \begin{bmatrix} \alpha_{H}^{B} \\ \alpha_{L}^{B} \end{bmatrix} v_{U}^{B}\left(\delta_{def}\left(B\right)\right) \end{bmatrix}$$
(12)

and smooth pasting conditions for functions that are alive across $\delta_{def}(B)$ ($\mathbf{x}_{[1,2]}$ selects the first 2 rows of vector \mathbf{x}):

$$\left(\mathbf{D}^{(2)}\right)'\left(\delta_{def}\left(B\right)\right)_{[1,2]} = \left(\mathbf{D}^{(1)}\right)'\left(\delta_{def}\left(B\right)\right).$$
(13)

3.2 Equity Valuations and Default Boundaries

When the firm refinances the maturing bonds, we assume that it can place newly issued bonds with H investors in a competitive primary market. This implies that there is a rollover cash flow (inflow or outflow) of $m \left[\mathbf{S}^{(i)} \cdot \mathbf{D}^{(i)}(\delta) - p \mathbf{1}_i \right]$ at each instant as a mass $m \cdot dt$ of debt holders matures on [t, t + dt], where $\mathbf{S}^{(i)}$ is a $i \times 2i$ matrix that selects D_H . For instance, for $\delta \in I_2 = [\delta_{def}(B), \infty)$, we have $\mathbf{D}^{(2)} = \begin{bmatrix} D_H^{G,2}, D_L^{G,2}, D_H^{B,2}, D_L^{B,2} \end{bmatrix}^{\mathsf{T}}$ and $\mathbf{S}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

For ease of exposition, we will denote by double letters (e.g. $\mathbf{x}\mathbf{x}$) a constant for equity that takes a similar place as a single letter (i.e. \mathbf{x}) constant for debt. We can write down the HJB

⁷This formulation can also accommodate the situation where some maturing bonds are rolled over to L investors by adjusting $\mathbf{S}^{(i)}$.

equation for equity on interval I_i similar to (8) and (9). For instance, on interval I_2 we have

$$\left(r\mathbf{I}_{2} - \mathbf{Q}\mathbf{Q}^{(2)}\right)\mathbf{E}^{(2)}(\delta) = \mu\mu^{(2)}\left(\mathbf{E}^{(2)}\right)'(\delta) + \frac{1}{2}\Sigma\Sigma^{(2)}\left(\mathbf{E}^{(2)}\right)''(\delta) \\ + \underbrace{\mathbf{1}_{2}\exp\left(\delta\right)}_{Cashflow} - \underbrace{\left(1 - \pi\right)c\mathbf{1}_{2}}_{Coupon} + \underbrace{m\left[\mathbf{S}^{(2)}\cdot\mathbf{D}^{(2)}(\delta) - p\mathbf{1}_{2}\right]}_{Rollover}$$
(14)

where

$$\boldsymbol{\mu}\boldsymbol{\mu}^{(2)} = \operatorname{diag}\left(\left[\mu\left(G\right), \mu\left(B\right)\right]\right), \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{(2)} = \operatorname{diag}\left(\left[\sigma^{2}\left(G\right), \sigma^{2}\left(B\right)\right]\right), \mathbf{Q}\mathbf{Q}^{(2)} = \begin{bmatrix} -\zeta\left(G\right) & \zeta\left(G\right) \\ & (15) \\ \zeta\left(B\right) & -\zeta\left(B\right) \end{bmatrix}\right)$$

The general solution to equity value is (please put dimensions in the following equation)

$$\underbrace{\mathbf{E}^{(i)}_{i\times 1}(\delta)}_{i\times 1} = \underbrace{\mathbf{G}\mathbf{G}^{(i)}_{i\times 2i}}_{i\times 2i} \cdot \underbrace{\exp\left(\mathbf{\Gamma}\mathbf{\Gamma}^{(i)}\delta\right)}_{2i\times 2i} \cdot \underbrace{\mathbf{b}\mathbf{b}^{(i)}_{2i\times 1}}_{2i\times 1} + \underbrace{\mathbf{K}\mathbf{K}^{(i)}_{i\times 4i}}_{i\times 4i} \underbrace{\exp\left(\mathbf{\Gamma}^{(i)}\delta\right)}_{4i\times 4i} \underbrace{\mathbf{b}^{(i)}_{i\times 1}}_{i\times 1} + \underbrace{\mathbf{k}\mathbf{k}^{(i)}_{1}}_{i\times 1} \exp\left(\delta\right) \text{ for } \delta \in I_i$$

and the particular solution is

$$\underbrace{\mathbf{E}^{(2)}(\delta)}_{2\times 1} = \underbrace{\mathbf{G}\mathbf{G}^{(2)}}_{2\times 4} \underbrace{\exp\left(\mathbf{\Gamma}\mathbf{\Gamma}^{(2)}\delta\right)}_{4\times 4} \cdot \underbrace{\mathbf{bb}^{(2)}}_{4\times 1} + \underbrace{\mathbf{KK}^{(2)}}_{2\times 8} \underbrace{\exp\left(\mathbf{\Gamma}^{(2)}\delta\right)}_{8\times 8} \underbrace{\mathbf{bb}^{(2)}}_{4\times 2} + \underbrace{\mathbf{kk}^{(2)}}_{8\times 8} + \underbrace{\mathbf{kk}^{(2)}}_{2\times 1} + \underbrace{\mathbf{kk}^{(2)}}_{2\times 1} \exp\left(\delta\right) \text{ for } \delta \in I_2$$

$$\underbrace{\mathbf{E}^{(1)}(\delta)}_{1\times 1} = \underbrace{\mathbf{G}\mathbf{G}^{(1)}}_{1\times 2} \underbrace{\exp\left(\mathbf{\Gamma}\mathbf{\Gamma}^{(1)}\delta\right)}_{2\times 2} \cdot \underbrace{\mathbf{bb}^{(1)}}_{2\times 1} + \underbrace{\mathbf{KK}^{(1)}}_{1\times 4} \underbrace{\exp\left(\mathbf{\Gamma}^{(1)}\delta\right)}_{4\times 4} \underbrace{\mathbf{bb}^{(1)}}_{4\times 4} + \underbrace{\mathbf{kk}^{(1)}}_{1\times 1} + \underbrace{\mathbf{kk}^{(1)}}_{1\times 1} \exp\left(\delta\right) \text{ for } \delta \in I_1$$

where $\mathbf{GG}^{(i)}, \mathbf{\Gamma\Gamma}^{(i)}, \mathbf{bb}^{(i)}, \mathbf{KK}^{(i)}, \mathbf{kk}_0^{(i)}$ and $\mathbf{kk}_1^{(i)}$ for $i \in \{1, 2\}$ are given in Appendix. In particular, the constant vector $\mathbf{bb}^{(i)}$ is determined by boundary conditions similar to those in Section 3.1.

Finally, the endogenous bankruptcy boundaries $\boldsymbol{\delta}_{def} = [\delta_{def}(G), \delta_{def}(B)]^{\top}$ are given by the standard optimization / smooth pasting condition:

$$\left(\mathbf{E}^{(1)}\right)' \left(\delta_{def}\left(G\right)\right)_{[1]} = 0, \text{ and } \left(\mathbf{E}^{(2)}\right)' \left(\delta_{def}\left(B\right)\right)_{[2]} = 0.$$
 (16)

Symbol	Interpretation	State G	State B
$\zeta^{\mathbb{P}}$	Transition Density	0.10	0.50
κ	Jump Risk Premium	2.50	0.40
$\mu_{\mathbb{P}}$	Cash Flow Growth	0.04	0.02
η	Risk Price	0.18	0.25
σ_m	Systematic Vol	0.11	0.15
σ_i	Idiosyncratic Vol	0.22	0.22
m	Average Maturity Intensity	0.20	0.20
χ	Holding Cost	-2.15	-2.65
ξ	Liquidity Shock Intensity	0.70	0.70

Table 1: Benchmark Parameters. This table reports the parameters values used in the benchmark calibration. Unreported parameters are tax rate $\pi = 0.35$, bond holders' bargaining power $\beta = 5\%$, and risk free rate $r_f = 0.02$.

4. Calibration

We calibrate the parameters governing firm fundamentals and pricing kernels to key moments of aggregate economy and asset pricing. Parameters governing time-varying liquidity conditions are calibrated to their data counterparts on bond turnover, dealer's bargaining power and observed bid-ask spread. Since the credit spread of the randomly-maturing bonds in the model are not directly compared to the data, we use simulation methods to compute the fixed maturity bond whose holders are subject to the same liquidity shocks as modeled before.

4.1 Benchmark Parameters

4.1.1 Cash flows and liquidity parameters

We follow Chen et al. [2012] in calibrating firm fundamentals and investors' pricing kernel. Table 1 reports the benchmark parameters we use. For average maturity, we consider a firm with a continuum of bonds that matures (randomly) with intensity m = 0.2 so that the average debt maturity is about 1/m = 5 years. This is close to the empirical median debt maturity (including bank loans and public bonds) reported in Chen et al. [2012]. The set of parameters governing the risk prices and firm's cash flows are standard. For liquidity parameters, we assume a bondholder will be hit by liquidity shock of intensity $\xi = 0.7$ and take this number to be state independent. When hit by a liquidity shock, it takes a bond holder on average 3 days ($\lambda(G) = 100$) in good state and 1 week ($\lambda(B) = 50$) in bad state to find an intermediary to sell the holding. Taken together, the model implies an average annual turnover of about 70%, which is close to the value-weighted turnover as we see in the data. Pro-cyclicality of λ captures time-varying liquidity conditions in the secondary market, and is strongly supported in the data. We interpret lower λ as a weakining of the financial system and its ability to intermediate markets. In our model, adverse macroeconomic condition (risk prices) coincides and interacts with weaker firm fundamentals and worsened secondary market liquidity to generate quantitatively important implications for the pricing of defaultable bonds. Finally, we set bond holders bargaining power to be fixed at $\beta = 0.05$ and it does not vary across states. This number is taken from empirical work that estimates search frictions in the bond market (Feldhütter [2012]).

4.1.2 Effective recovery rates

The parameters that are specific to our model is the type and state-dependent recovery rates α_l^s for $l \in \{L, H\}$ and $s \in \{G, B\}$. We first borrow from the existing literature (say, Chen [2010]) who treats the traded prices right after default as recovery rates, and the estimates for recovery is $57.55\% \cdot v_u^G$ at normal time and $30.60\% \cdot v_u^B$ at recession. Assuming that post-default prices are bid prices that investors are selling, then we have from Proposition 1 that

$$0.5755 = \alpha_L^G + \beta(\alpha_H^G - \alpha_L^G), \text{ and } 0.3060 = \alpha_L^B + \beta(\alpha_H^B - \alpha_L^B).$$
(17)

We need two more pieces of bid-ask information for defaulted bonds to rocover α_l^s 's. Edwars Harris Piwowar (2006) (henceforth EHP) report that in normal times, the transaction cost of defaulted bonds for median-sized trades is about 200 *bps*. To gauge the bid-ask spread for defaulted

Default Time	# of Defaulted Bond	Mean Annualized Net PME	Mean Annualized Net Return
Non-Recession	512	0.3126	0.3922
Recession	130	0.5537	0.4672
Full Sample	642	0.3613	0.4074

Table 2: Summary Statistics for Annualized Net PME on Defaulted Bond by Default Time

bonds during recession, we take the following approach. Using TRACE data, we first follow Bao, Pan and Wang (2012) (henceforth BPW) to calculate the implied bid-ask spreads for low rated bonds (C and below) for both normal and recession times. We find that relative to normal times, during recessions the implied bid-ask spread is around 2.6 times higher. Given a bid-ask spread of 200 *bps* for defaulted bonds, this multipler implies that the bid-ask spread for defaulted bonds during recession is about 520 *bps*. Hence we have

$$2\% = \frac{2\left(\alpha_H^G - \alpha_L^G\right)}{\alpha_L^G + \beta(\alpha_H^G - \alpha_L^G) + \alpha_H^G}, \text{ and } 5.2\% = \frac{2\left(\alpha_H^B - \alpha_L^B\right)}{\alpha_L^B + \beta(\alpha_H^B - \alpha_L^B) + \alpha_H^B}.$$
 (18)

Solving (17) and (18) gives us the estimates of $\boldsymbol{\alpha} = \left[\alpha_{H}^{G}, \alpha_{L}^{G}, \alpha_{H}^{B}, \alpha_{L}^{B}\right].$

4.1.3 Ultimate recovery rates

For later decomposition, we need information on ultimate recovery rates $\hat{\alpha}_s$ in different states. We extract information on these recovery rates from Moody's default and recovery database that covers defaulted corporate bond between 1987 and 2011. We track the price path for each defaulted bond from the default date to the settlement (or emergence) date. For each bond, Moody's calculates the emergence price using three methods: trading price, settlement price or liquidity price and indicates which one is preferred. We follow Moody's preferred method. The majority of our sample are bankrupcy cases. The average time from credit event to ultimate resolution is 501 days, which varies little across recession and non-recession periods.

We borrow from the empirical literature on vecture capital / private equity (eg. Kaplan and

Schoar [2005])to adjust for risk by discounting the return for each defaulted bond by a public market reference return over the same horizon (from default date to emergence date). We use SP500 total return (including dividend) as the relevant benchmark. The resulting excess returns are called "Public Market Equivalent" (PME). A PME greater than 1 implies the return earned by investing in defaulted bond beat the public market over the same horizon. In table 2 we report the mean annualized net PME.

To account for state dependence in risk premium, we sort our sample into two groups based whether the default month is classified as recession by NBER. We map the G state and B state in the model to recession and non-recession state in this way. Out of our full sample of 642 bonds, 130 defaulted in recession months. Table 2provides summary statistics on our excess return matric and figure 3 plots its empirical distribution.

Following the above procedure, we find that over the bankruptcy resolution period of 501 days, the average adjusted buy-and-hold return when default occurs in recession is about 212%, and when default occurs in non-recession time is about 153%. Since at aggregate state s the trading price right after default is $[\alpha_L^s + \beta(\alpha_H^s - \alpha_L^s)] v_U^s$ while the ultimate recovery is $\hat{\alpha}_s v_U^s$, we reach the estimates for $\hat{\alpha}_s$'s as (recall (17)):⁸

 $\hat{\alpha}_G = 1.53 \times 0.5755 = 88\%$, and $\hat{\alpha}_B = 2.12 \times 0.3060 = 65\%$.

⁸This calculation implicitely assumes that there is no transitioning of aggregate states when waiting for ultimate recoveries, as we only have average buy-and-hold return for bonds that defaulted at a given aggregate state. One can potentially calculate the adjusted buy-and-hold return not only conditional on the state in which the firm defaults (which is our treatment), but also conditional on the state in which the ultimate recovery occurs. This will significantly complicate the analysis, and it is unclear which direction of bias that this treatment introduces.

Symbol	Interpretation	State G	State B
$lpha_H$	Recovery Rate of H Type	59.78%	33.78%
α_L	Recovery Rate of L Type	57.43%	30.43%
\hat{lpha}	Ultimate Recovery Rate	88.05%	64.87%

Table 3: Implied Recovery Values

4.2 Calibration Results on Credit Spreads and Default Probabilities

Table 4 presents our calibration results on aggregate default probability and total credit spread for bonds of four rating classes: Aaa/Aa rated, A, Baa, and Ba; the first three ratings are investment grade, while Ba is speculative grade. We combine Aaa and Aa together because there are few observations for Aaa firms. The data counterpart is from Huang and Huang (2012).

Given a firm's default boundary, we compute the default probability and total credit spread of bonds at 5 and 10 year maturity using Monte-Carlo methods.⁹ As typical in structural corporate bond pricing models, we find that the model implied default probability and total credit spread are highly nonlinear in market leverage (see Figure 1). As inDavid [2008], the non-linearity inherent in the model implies that the average credit spreads are higher than the spreads at average market leverage. We thus follow David [2008] in computing model implied aggregate moments. Specifically, we compute the market leverage of all Compustat firms for which we have ratings data between 1985 and 2012, and classify each quarterly observation as either in "G State" or "B State" based on whether the specific quarter is classified as NBER recession.¹⁰ We then match each firm-quarter observed in Compustat to its model counterpart and compute the average across aggregate states, and repeat the procedure for each rating class and each maturity (5 or 10 years).

⁹Specifically, we simulate the cash flow of the firm and aggregate state for 50,000 times for a fixed duration of 5 or 10 years and count the times where the cash flow cross the state dependent default boundary and also record the cash flow received by bond holders of either H or L type. Following the literature, we adjust the principle of the bond to make it issue at par.

¹⁰ For empirical distribution of market leverage for each rating, see Figure 2. Market leverage is defined as the ratio of book debt over book debt plus market equity (sometimes it is called quasi market leverage). We dropped financial and utility firms.

The resulting summary statistics are presented in Table 4. Our model matches a variety of empirical moments commonly considered to be a puzzle in the literature, it also generates flatter and more realistic term structure of default probability and credit spread. For example, for Aaa/Aa (superior grade) bonds, we generate a 10 year credit spread of 84.89 *bps* (85.21 *bps* in data) with a default probability of 2.09% (2.06 in data), and for Baa the 10 year credit spread is 219.29 *bps* (194 *bps* in data) with a default probability of 8.82% (7.02% in data). Regarding term structure implications on bond spreads, our model undershoot a bit for the 5 year aggregate moments in the data.

We emphasize that relative to the existing literature, our calibration aims at explaining total credit spread across ratings, rather than differences between ratings. For instance, Bhamra et al. [2010] and Chen [2010] only focus on explaining the difference between Baa and Aaa rated bonds, which is considered as the default component of Baa rated bonds under the assumption that the observed spreads for AAA rated bonds are mostly driven by liquidity premium. Because our framework endogenously models bond liquidity, we are able to match the *total credit spread* that we observe in the data across ratings, especially across the superior ratings (Aaa/Aa) and the lower end of investment rating bonds (say Baa).

4.3 Model Performance on Default and Non-Default Measures

Our model features an illiquid secondary market for corporate bonds, which implies that the equilibrium credit spread must compensate the bond investors for bearing not only default risk but also liquidity risk. This richness allows us to investigate the model's quantitative performance on dimensions specific to bond market liquidity, in addition to the two common measures — cumulative default probabilities and credit spreads — that the previous literature on corporate bond

	Panel A: Aaa/Aa Bonds			
	Maturity	= 5 Years	Maturity	= 10 Years
	Data	Model	Data	Model
Default Probability $(\%)$	0.66	0.49	2.06	2.09
Total Credit Spread (bps)	62.93	51.11	85.21	84.89
CDS Spread (bps)	30.72	16.69	44.78	43.93
		Panel B:	A Bonds	
	Maturit	y = 5 Years	Maturity	= 10 Years
	Data	Model	Data	Model
Default Probability $(\%)$	1.31	1.23	3.44	4.55
Total Credit Spread (bps)	96.00	80.89	123.00	137.87
CDS Spread (bps)	45.10	40.20	64.32	88.16
		Panel C: I	Baa Bonds	
	Maturit	y = 5 Years	Maturity	= 10 Years
	Data	Model	Data	Model
Default Probability (%)	3.08	3.25	7.02	8.82
Total Credit Spread (bps)	158.00	148.84	194.00	219.29
CDS Spread (bps)	91.19	90.66	120.15	151.37
		Panel D:	Ba Bonds	
	Maturit	Maturity $= 5$ Years		= 10 Years
	Data	Model	Data	Model
Default Probability (%)	9.81	8.32	19.01	16.66
Total Credit Spread (bps)	320.00	306.17	320.00	365.72
CDS Spread (bps)	237.82	204.14	281.77	264.62

 Table 4: Calibration Results on Aggregate Moments Across Ratings

calibration (e.g., Huang and Huang, 2012) has focused on.

4.3.1 Credit Default Swap (CDS) Spread

Longstaff et al. [2005] argue that because the market for CDS contract is much more liquid than secondary market for corporate bonds, the CDS spread should mainly price the default risk of a bond, while the total credit spread also includes liquidity premium to compensate for the illiquidity in the bond market. This is exactly what our model is trying to capture. To assess the model's quantitative performance, we compute the model implied CDS spread, under the assumption that the CDS market is indeed perfectly liquid. Let τ (in years) be the first time that the cash flow δ hits default boundary δ_B . The required flow payment f corresponding to a T-year CDS is the solution to the following equation

$$\mathbb{E}^{Q}\left[\int_{0}^{\min[\tau,T]} \exp\left(-r_{f}t\right) f dt\right] = \mathbb{E}^{Q}\left[\exp\left(-r_{f}\tau\right) LGD\left(s\right)\right]$$
(19)

where LGD(s) is the loss-given-default when the default occurs in state s. If there is no default, no LGD is paid out by the CDS seller. The loss-given-default is defined as the bond face value minus its recovery value, which is usually defined as the transaction price right after default (thus is subject to liquidity frictions). We calculate the required flow payment f that solves (19) using a simulation method. Then the CDS spread $CDS_{spread} = f/p$ is defined as the ratio between the flow payment f and the bond's face value p. Thus, the model implied CDS spread is not equal to credit spread. Finally, we calculate model implied CDS spread for each firm-quarter observation in Compustat based on its leverage (see Section 4.2).

The last row in each panel of Table 4 reports the model-implied CDS spreads, together with data counterparts that are only available from 2003 onwards. Overall, the quantitative matching

is not as good as total credit spread, but is reasonably close. For example, for Aaa/Aa bonds our model implies a 10 year CDS spread of 43.94 bps (44.78 bps in data), and 151.37 bps (120.15 bps in data) for Baa rated bonds. The steeper model implied term structure relative to data is also reflected in CDS spreads; for instance, for Baa rated bonds the 5 year CDS spread is close to data (90.66 bps in the modelvs 91.99 bps in the data), while overshoot on 10 year CDS spread (151.37 in the model vs 120.15 in the data).

4.3.2 Bid-Ask Spread

For measures of non-default component in corporate bond yields, we focus on the bond's endogenous bid-ask spread. In our model, the trade between L type holders and dealers gives rise to what we observe as bid-ask spread in bond trades. Previous empirical studies has uncovered rich patterns of bid-ask spreads, and we investigate whether our model is able to match these patterns quantitatively.

We combine Edwards et al. [2007] and Bao et al. [2011] to construct the data counterparts for bid-ask spread, because Edwards et al. [2007] only reports the average bid-ask spread across ratings in normal time (2003-2005). The ratings considered in Edwards et al. [2007] are supeior grade (Aaa/Aa) with an bid-ask spread of 40 bps, investment grade (A/Baa) with an bid-ask spread of 50 bps, and junk grade (below Ba) with a bid-ask spread of 70 bps.¹¹ For each grade, we then compute the Roll's measure of liquidity as in Bao et al. [2011] and used them to back out the ratio of *B* state bid-ask spread to the *G*-state bid-ask spreads. We multiply this ratio by the level of bid-ask spread estimated by Edwards et al. [2007] to arrive at bid-ask spread in *B* state. These empirical estimates are reported in Table 5.

On the model side, again we rely on empirical leverage distribution in Compustat across ratings

¹¹We take the median size trade around 240K. Edwards et al. [2007] show that trade size is an important determinants for transaction costs of corporate bonds. But, for tractability reasons, we have abstracted away from the trade size.

Table 5: Calibration Results on Bid-Ask Spread (in bps). The normal time bid-ask spread are taken from Edward et. al. (2007) for a median size trade. The recession time numbers are normal time numbers multiple the ratio of bid-ask spread implied by Roll's measure of illiquidity and constructed following Bao, Pan and Wang (2010). The model counterpart are computed for a bond with average time to maturity of 8.3 years, which is the mean time-to-maturity of frequently traded bonds (where we can compute a Roll's measure) in the TRACE sample.

	State G		State B		
	Data	Data Model		Model	
Superior Grade	40.00	37.23	71.86	79.14	
Investment Grade	50.00	47.99	108.33	109.74	
Junk Grade	70.00	75.94	144.2	171.48	

and aggregate states to calculate the average of model implied bid-ask spreads. We calibrate two state-dependent holding cost parameters (χ_G and χ_B) to match bid-ask spread of investment grade bonds across two aggregate states. Since the average maturity in TRACE data is around 8.3 years, the model implied bid-ask spread is calculated as the weighted average between the bid-ask spread of a 5-year bond and a 10-year bond. We then ask whether the model is able to generate aggregate state-, rating class- and horizon-dependent patterns in bid-ask spread that quantitatively matches the data.

The model implied bid-ask spreads are reported in Table 5. The model is able to generate several patters that quantitatively matches their data counterpart. First, in normal time, the average bid-ask spread is 37.23 bps for superior grade bonds, 47.99 bps for investment grade bonds and 75.94 bps for junk grade bonds, which are close to those estimated by Edwards et al. [2007]. Second, these bid-ask spreads doubles when the economy switches from G state to B state. Finally, although not reported here, the model implies bid-ask spread of longer-maturity bonds are higher than shorter-maturity bonds and this is also consistent with previous empirical studies (eg. Edwards et al. [2007]; Bao et al. [2011]).

Our model provides a coherent economic explanation for the patterns documented in the data. As the bond is closer to default (compare junk bonds to superior grade bonds), the valuation gap between H and L type widens a consequence of the heterogeneity in their recovery value when the bond does default. When an intermediary meets an L type holder, it extracts part of the trading surplus. When the bond become riskier, the trade surplus goes up, giving rise to a larger bid-ask spread. The same logic applies when the economy switches to the B state since bonds are riskier in the B state. Lower intermediary intensity in the B state further reduces the outside option of L holders, driving up bid-ask spreads further. This worsened liquidity in turn leads to earlier default by equity holders – a liquity-default loop arises. Although the economy spends considerably longer time in the good state than in the bad state, and the fact that therefore most transactions happen in a good state with low bid ask spreads does not remove the risk of holding such bonds. An investor is most likely to get stuck with the illiquid bond *precisely* in B state, with high risk prices, low recovery value and longer waiting time before he/she can sell the holdings. Our quantitative results show this state-dependant liquidity risk contributes significantly to the overall risk profile of a defaultable bond and thus goes a long way in explaining its credit spread.

5. Model Based Decomposition

Our structual model of corporate bonds with search friction in secondary market features a full interaction between default and liquidity in determining the credit spread of corporate bonds. It has been a common practice to decompose the credit spread into liquidity and default components in an additive way, such as in Longstaff et al. [2005]. From the perspective of our model, this decomposition — though intuitively appealing — over-simplifies the role of liquidity in determining the credit spread. More importantly, the additive structure often leads to the interpretation that liquidity or default is the cause of the corresponding component, and each component will be the resulting credit spread when we shut down the other channel. We emphasize that this interpretation may give rise to misleading answers in certain policy related questions. For instance, as our decomposition indicates, part of the default risk comes from the illiquidity in the secondary market. Thus, when the government is considering providing liquidity to the marke, it is not only improving liquidity directly but will also lower the default risk by via lowering the liquidity-driven default. In contrast, this additional beneft can be easily missed in the traditional additive view, as it imposes the assumption that default risk will not be affected when the bond market liquidity is improved.

5.1 Decomposition Scheme

We propose a more detailed model-based decomposition, which nests the additive default-liquidity decomposition that is common in the literature. Specifically, we further decompose the default part into pure-default part and liquidity-driven-default part, and similarly decompose the liquidity part into pure-liquidity and default-driven-liquidity parts:

$$\hat{y} = \underbrace{\begin{array}{l} \text{Default Component } \hat{y}_{DEF}}_{\hat{y}_{pureDEF} + \hat{y}_{LIQ \to DEF} + \hat{y}_{pureLIQ} + \hat{y}_{DEF \to LIQ} \end{array}} \quad \text{Liquidity Component } \hat{y}_{LIQ}$$

In this way, we separate *cause* and from *effect*, and emphasize that liquidity (default) can lead to the rise of spread through default (liquidity), which is important in evaluating the economic consequence of improving market liquidity or allieviating default issues (through direct bailouts).

We start with the default component of the bond. Longstaff et al. [2005] proposes using CDS price to measure the default component, because CDS contract prices the default event but is not (or much less) subject to secondary market liquidity problems. We stress that one can turn off the liquidity channel (on the derivative CDS market) without affecting the equity holders' default policy of δ_B^* (who refinance their debt in the primary corporate bond market). This motivates us to consider a hypothetical identical bond with the same default boundary, but for which bond investors are not subject to liquidity problems, both pre default and post default. The spread of this hypothetical bond over treasury, denoted by \hat{y}_{DEF} , is the default component of our bond.

Our strucutal model suggests that this default component \hat{y}_{DEF} is generally greater than \hat{y}_{LT} , which is the bond spread without any liquidity/search frictions as in Leland and Toft [1996] in which we allow the equity holders to reoptimize with respect to the default boundary.¹² We call this the pure-default spread, because \hat{y}_{LT} is the spread contribution solely comes from the default by equity holders who default at the reoptimized boundary δ_B^{LT} . The difference $\hat{y}_{DEF} - \hat{y}_{LT}$ arises because the illiquidity of bond market leads equity holders to face heavier rollover losses, and they hence default earlier at $\delta_B^* > \delta_B^{LT}$. We label this difference as the liquidity-driven default part, which quantifies the effect that bond illiquidity makes default more likely.

Now we move on to the liquidity component. As liquidity and default are the only two factors that affect bond prices in our model, we define the liquidity component as the difference between credit spread and default compoent, i.e., $\hat{y}_{LIQ} = \hat{y} - \hat{y}_{DEF}$. In a similar vein, we further decompose $\hat{y} - \hat{y}_{DEF}$ into a pure liquidity part and a default-driven liquidity part. We first calculate \hat{y}_{DGP} , which is the spread of a bond that is only subject to search/liquidity friction as in Duffie et al. [2005] but does not have any default risk. This spread captures the pure liquidity part and is given by $\delta \to \infty$ to make it default free. The remaining residual after controlling for the pure liquidity component, i.e., $\hat{y}_{LIQ} - \hat{y}_{DGP}$, is what we term the default-driven liquidity part of our credit spread. Intuitively, this part of liquidity component is driven by bond default, because default leads to a

$$r\alpha_{LT}^{G} = \theta \left(\hat{\alpha}_{G} - \alpha_{LT}^{G} \right) + \zeta_{G} \left(\alpha_{LT}^{B} \frac{x_{B}}{x_{G}} - \alpha_{LT}^{G} \right)$$
$$r\alpha_{LT}^{B} = \theta \left(\hat{\alpha}_{B} - \alpha_{LT}^{B} \right) + \zeta_{B} \left(\alpha_{LT}^{G} \frac{x_{G}}{x_{B}} - \alpha_{LT}^{G} \right)$$
$$\begin{bmatrix} \alpha_{LT}^{G} \\ \alpha_{LT}^{B} \end{bmatrix} = \begin{bmatrix} r + \theta + \zeta_{G} & -\zeta_{G} \frac{x_{B}}{x_{G}} \\ -\zeta_{B} \frac{x_{G}}{x_{B}} & r + \theta + \zeta_{B} \end{bmatrix}^{-1} \begin{bmatrix} \theta \hat{\alpha}_{G} \\ \theta \hat{\alpha}_{B} \end{bmatrix}$$

 \mathbf{SO}

¹²Because of delayed bankruptcy payout, the recovery rates for LT96 model where investors are not subject to liquidity problems are different from the ultimate recovery rates in Table XX. Under our calibration, given the bankruptcy resolution time of 501 days, the recovery rates for an investor who is not subject to liquidity problem are $\alpha_{LT}^G = XX$ and $\alpha_{LT}^B = XX$. How to calculate LT96 recovery rates given delay? let $[x_G, x_B]^T \equiv [\mathbf{RR} - \mu\mu - \mathbf{QQ}]^{-1}\mathbf{1}$, then

more illiquid post-default secondary market.

In sum, we calculate the spreads for following hypothetical bonds:

- \hat{y}_{LT} : the hypothetical bond spread with perfect liquidy secondary bond market as in Leland and Toft [1996], i.e., $\lambda = \infty$; the equity holders' default policy adjusts endogenously.
- \hat{y}_{DEF} : the hypothetical bond spread with perfect liquid secondary bond market, i.e., $\lambda = \infty$; but the equity holders' default policy remains at δ_B^* in the economy with liquidity friction.¹³
- y_{DGP} : the hypothetical risk-less bond spread with illiquid secondary bond market as in Duffie-Garlenu-Pederson, i.e., $\delta = \infty$.

We then decompose credit spread in the following four parts:

$$\hat{y} = \underbrace{\hat{y}_{pureDEF} + \hat{y}_{LIQ \to DEF}}_{\hat{y}_{pureLIQ} + \hat{y}_{DEF} \to LIQ} + \underbrace{\hat{y}_{pureLIQ} + \hat{y}_{DEF \to LIQ}}_{\hat{y}_{pureLIQ} + (\hat{y}_{DEF} - \hat{y}_{LT}) + \hat{y}_{DGP} + [(\hat{y} - \hat{y}_{DEF}) - \hat{y}_{DGP}]$$

We call the four components "Pure Default", "Liquidity-Driven Default", "Pure Liquidity" and "Default-Driven Liquidity" respectively.

5.2 Decomposition Results

We performed the above decomposition of three bonds whose total credit spreads resemble those of a typical 10 year superior, investment and junk grade bonds. Table 6 to table 8 present our results. The standard decomposition scheme, as in Longstaff et al. [2005] measures the ratio of CDS spread to total credit spread. While our model implies a ratio that is quantitatively close

¹³Keep in mind that although the firm defaults at V_B (so the selling price is pV_B), the hypothetical bond is valued by investors who are not subject to liquidity shocks and thus recover more than pV_B .

to those reported in Longstaff et al. [2005] for different rating classes, it is clear from our model that part of CDS spread is to compensate investors holding for "non-default" risks of corporate bonds, and part of the total credit spread - CDS gap is to compensate investors for endogenuos liquidity risks driven by cash flow risks ("default component"). Specifically, the interaction between liquidity and default risks, first captured by "liquidity driven default" is quantitatively large even for bonds with high credit ratings. The "liquidity driven default" term captures how corporate optimal default decisions are affected by secondary market liquidity frictions via the rollover channel. Since this term is quantitatively important, the indirect effect of improved secondary market liquidity on reducing the borrowing cost of corporations need to be taken into account when considering improving liquidity of markets. Second, "Default driven liquidity" captures how secondary market liquidity endogenuously worsens when a bond is closer to default. This is due to the reduced outside option of L type holders when bargaining with a dealer. Not surprisingly, this term becomes larger when the bond falls into lower rating class, but it remains an non-negligible term even for superior grade bonds.

Our decomposition offers a fresh perspective for a question that has interested empirical researchers for a long time, that is, how much of the soaring credit spread when the economy switches from boom to recession is due to increased credit risk and how much is due to worsened secondary market liquidity (see, eg. Dick-Nielsen et al. [2011]; Friewald et al. [2012]). As our model endogenizes both liquidity and credit risks, it provides a model-based answer to this question. As evident from column 3 of table 6 to table 8, increased credit risks constitutes a large fraction of the jump in credit spreads. Moreover, when the bond becomes riskier, its liquidity worsens in the model, giving rise to larger "default driven liquidity", which consists of 9.36% (superior) to 16.51% (junk) of the increased credit spread when the economy encounters a recession. These interaction terms represent quantitatively relevant economic forces that are new to this literature.

	State G	State B	Change $G \to B$
Total Credit Spread	76.68	102.03	25.35
Pure Default	28.73	41.56	12.83
$\% \ of \ Total$	37.46	40.73	50.62
Liquidity Driven Default	9.13	13.36	4.23
$\% \ of \ Total$	11.91	13.10	16.70
Pure Liquidity	34.05	39.96	5.91
$\% \ of \ Total$	44.40	39.16	23.32
Default Driven Liquidity	4.78	7.15	2.37
$\% \ of \ Total$	6.23	7.01	9.36
Implied CDS Spread	46.14	67.20	
% of Total Credit Spread	60.17	65.87	

Table 6: Model Based Decomposition: 10 year Superior Grade Bonds

	State G	State B	Change $G \to B$
Total Credit Spread	140.61	185.20	44.59
Pure Default	70.09	96.18	26.09
$\% \ of \ Total$	49.85	51.93	58.51
Liquidity Driven Default	16.56	22.82	6.27
% of Total	11.77	12.32	14.05
Pure Liquidity	38.78	45.53	6.75
% of Total	27.58	24.58	15.14
Default Driven Liquidity	15.19	20.67	5.49
$\% \ of \ Total$	10.80	11.16	12.30
Implied CDS Spread	108.52	148.97	
% of Total Credit Spread	77.18	80.44	

Table 7: Model Based Decomposition: 10 year Investment Grade Bonds

The decomposition is highly informative for evaluating the effect of policies that targets on lowering the borrowing cost of corporations in recession times by injecting liquidity in the secondary market. As argued before, a full analysis of the effectiveness of such policy must take into account of how corporate's default policy responds to liquidity conditions and how liquidity conditions respond to the default risks.

6. Policy Experiments

(to be completed)

	State G	State B	Change $G \to B$
Total Credit Spread	273.16	353.95	80.78
Pure Default	156.38	208.83	52.45
% of Total	57.25	59.00	64.92
Liquidity Driven Default	34.69	41.11	6.41
% of Total	12.70	11.61	7.94
Pure Liquidity	49.07	57.66	8.59
% of Total	17.96	16.29	10.63
Default Driven Liquidity	33.02	46.35	13.33
% of Total	12.09	13.10	16.51
Implied CDS Spread	227.88	305.60	
% of Total Credit Spread	83.42	86.34	

Table 8: Model Based Decomposition: 10 year Junk Grade Bonds

7. Concluding Remarks

(to be completed)

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Figure 1: Model Implied Relationship Between Market Leverage, Default Rates and Total Credit Spread



A Appendix

where $\operatorname{diag}(\cdot)$ is the diagonalization operator mapping a vector into a diagonal matrix.

We follow the Markov-modulated dynamics approach of Jobert and Rogers (2006).

We note that there are multiple possible bankruptcy boundaries, $\delta_B(s)$, for each aggregate state s one boundary. Order states s such that s > s' implies that $\delta_B(s) > \delta_B(s')$ and denote the intervals $I_s = [\delta_B(s), \delta_B(s+1)]$ where $\delta_B(n+1) = \infty$, so that $I_s \cap I_{s+1} = \delta_B(s+1)$. Finally, let $\delta_B = [\delta_B(1), ..., \delta_B(n)]^\top$ be the vector of bankruptcy boundaries.

It is important to have a clean notational arrangement to handle the proliferation of states. Let $D_l^{(s)}$ denote the value of debt for an creditor in individual liquidity state l and with aggregate state s. We will use the following notation: $D_l^{(s,i)} \equiv D_l^{(s)}, \delta \in I_i$, that is $D_l^{(s,i)}$ is the restriction of $D_l^{(s)}$ to the interval I_i . It is now clear that $D_l^{(s,i)} = 0$ for any i < s, as it would imply that the company immediately defaults in interval I_i for state s. Let us, for future reference, call debt in states i < s dead and in states $i \ge s$ alive. Finally, let us stack the alive functions along states sbut still restricted to interval i so that $\mathbf{D}^{(i)} = \left[D_H^{(1,i)}, D_L^{(1,i)}, ..., D_H^{(i,i)}, D_L^{(i,i)} \right]^{\top}$ where $D_l^{(s,i)}$ has s denoting the state, i denotes the interval and l denotes the individual liquidity state. The separation of s and i will clarify the pasting arguments that apply when δ crosses from one interval to the next. Let

$$\underbrace{\mathbf{I}}_{i\times i} = \begin{bmatrix} 1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1 \end{bmatrix}$$
(20)

i.e. a 2x2 diagonal identity matrix, and let

$$\underbrace{\mathbf{1}_i}_{i\times 1} = \begin{bmatrix} 1, \dots, 1 \end{bmatrix}^\top \tag{21}$$

be a column vector of just ones.



Figure 2: Empirical Distribution of Market Leverage for Compustat Firms by Aggregate State and Rating classes.

Figure 3: Distribution of Annualized Net Return (left) and Public Market-Ajusted Return (right) of Defaulted Bonds





Fundamental parameters. For a 2x2 case, we have a transition matrix Q that looks like

$$\mathbf{Q}_{2n\times 2n} = \begin{bmatrix}
-\sum_{ls\neq H1} \xi_{H1\to ls} & \xi_{H1\to L1} & \xi_{H1\to H2} & \xi_{H1\to L2} \\
\xi_{L1\to H1} & -\sum_{ls\neq L1} \xi_{L1\to ls} & \xi_{L1\to H2} & \xi_{L1\to L2} \\
\xi_{H2\to H1} & \xi_{H2\to L1} & -\sum_{ls\neq H2} \xi_{H2\to ls} & \xi_{H2\to L2} \\
\xi_{L2\to H1} & \xi_{L2\to L1} & \xi_{L2\to H2} & -\sum_{ls\neq L2} \xi_{L2\to ls}
\end{bmatrix}$$
(22)

Further, define the possibly state-dependent discount rates

$$\underbrace{\mathbf{R}}_{2n\times 2n} = \begin{bmatrix} \operatorname{diag}\left(\begin{bmatrix} r_{H}\left(1\right)\\ r_{L}\left(1\right) \end{bmatrix} \right) & \cdots & \mathbf{0}_{2} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{2} & \cdots & \operatorname{diag}\left(\begin{bmatrix} r_{H}\left(n\right)\\ r_{L}\left(n\right) \end{bmatrix} \right) \end{bmatrix} + m\mathbf{I}_{2n}$$
(23)

where we are including the intensity of the random maturity in the definition of \mathbf{R} for notational convenience and brevity.

Building blocks for interval I_i . We now decompose the matrix \mathbf{Q} . Let $\mathbf{Q}^{(i)}$ be the transition matrix of jumping into an alive state $s' \leq i$ when currently in interval *i* and in an alive state $s \leq i$. Let $\tilde{\mathbf{Q}}^{(i)}$ be the transition matrix of jumping into a default state s' > i when currently in interval *i* and in an alive state $s \leq i$.

Let $\mathbf{v}^{(i)}$ be the recovery or salvage value of the firm when default is declared in states s > i when currently in interval *i*, where $v_l^{(s,i)} \exp(\delta) = \alpha_{(s,i)} \frac{\exp(\delta)}{r_H}$. Thus, $\mathbf{v}^{(i)}$ is a vector containing recovery values for states $(i + 1, ..., n) \times (H, L)$ (i.e., it is of dimension $2(n - i) \times 1$).

Let $\chi^{(i)}$ be a vector of holding costs in states $(1, ..., i) \times (H, L)$ (i.e., it is of dimension $2i \times 1$). The holding costs are all positive, and are deducted from the coupon payment. Higher holding costs indicate more severe liquidity states L for the agent.

First, let us start with the interval i = n. On this interval, all debt $D_l^{(s,n)}$ is alive. Let

$$\underbrace{\boldsymbol{\mu}_{2n\times 2n}^{(n)}}_{2n\times 2n} = \begin{bmatrix} \mu(1) \mathbf{I}_2 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mu(n) \mathbf{I}_2 \end{bmatrix}$$
(24)

and similarly let

$$\underbrace{\boldsymbol{\Sigma}}_{2n\times 2n}^{(n)} = \begin{bmatrix} \sigma^2(1) \mathbf{I}_2 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \sigma^2(n) \mathbf{I}_2 \end{bmatrix}$$
(25)

and let

$$\mathbf{Q}^{(n)} = \mathbf{Q} \tag{26}$$

$$\mathbf{R}^{(n)} = \mathbf{R} \tag{27}$$

$$\tilde{\mathbf{Q}}^{(n)} = 0 \tag{28}$$

Next, for the interval i = n - 1 we drop the last two rows and columns (i.e. rows and columns 2n and 2n - 1) (because they account for different liquidity states) of $\boldsymbol{\mu}^{(n)}, \boldsymbol{\Sigma}^{(n)}, \mathbf{Q}^{(n)}, \mathbf{R}^{(n)}$ to form $\boldsymbol{\mu}^{(n-1)}, \boldsymbol{\Sigma}^{(n-1)}, \mathbf{Q}^{(n-1)}, \mathbf{R}^{(n-1)}$ which are all $2(n-1) \times 2(n-1)$ matrices. In contrast, we form $\tilde{\mathbf{Q}}^{(n-1)}$ by dropping the last two rows and the first 2(n-1) columns of $\mathbf{Q}^{(n)}$ to form a $2(n-1) \times 2$ matrix.

We repeat this procedure, dropping rows and columns and thus shrinking the matrices, step by step all the all the way down to i = 1.

Debt valuation within an interval I_i . Debt valuation follows the following differential equation on interval I_i :

$$\left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)}\right)\mathbf{D}^{(i)} = \left(c\mathbf{1}_{2i} - \boldsymbol{\chi}^{(i)}\right) + \boldsymbol{\mu}^{(i)}\left(\mathbf{D}^{(i)}\right)' + \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\left(\mathbf{D}^{(i)}\right)'' + \tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}\exp\left(\delta\right) + m \cdot p\mathbf{1}_{2i}$$
(29)

where $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}\exp(\delta)$ represents the intensity of jumping into default times the recovery in the default state and $m \cdot p \mathbf{1}_{2i}$ represents the intensity of randomly maturing times the payoff in the maturity state. Next, let us conjecture a solution of the kind $\mathbf{g} \exp(\gamma \delta) + \mathbf{k}_0^{(i)} + \mathbf{k}_1^{(i)} \exp(\delta)$ where \mathbf{g} is a vector and γ is a scalar. The particular part stemming from $\mathbf{c}^{(i)}$ is solved by a term $\mathbf{k}_0^{(i)}$ with

$$\underbrace{\mathbf{k}_{0}^{(i)}}_{2i\times 1} = \underbrace{\left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)}\right)^{-1}}_{2i\times 2i} \underbrace{\left(c + m \cdot p\right) \mathbf{1}_{2i} - \boldsymbol{\chi}^{(i)}}_{2i\times 1}$$
(30)

and the particular part stemming from $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}$ is solved by a term $\mathbf{k}_{1}^{(i)}\exp\left(\delta\right)$ with

$$\underbrace{\mathbf{k}_{1}^{(i)}}_{2i\times 1} = \underbrace{\left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)} - \mu^{(i)} - \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\right)^{-1}}_{2i\times 2i} \underbrace{\tilde{\mathbf{Q}}^{(i)}}_{2i\times 2(n-i)^{2(n-i)\times 1}} \underbrace{\mathbf{v}^{(i)}}_{2i\times 2(n-i)\times 1} \tag{31}$$

It should be clear that $\mathbf{k}_1^{(n)} = \mathbf{0}$ as on I_n there is no jump in the aggregate state that would result in immediate default. Plugging in, dropping the $\mathbf{c}^{(i)}$ and $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)} \exp(\delta)$ terms, canceling out $\exp(\gamma\delta) > 0$, we have

$$\mathbf{0}_{2i} = \left(\mathbf{Q}^{(i)} - \mathbf{R}^{(i)}\right)\mathbf{g} + \boldsymbol{\mu}^{(i)}\gamma\mathbf{g} + \frac{1}{2}\boldsymbol{\Sigma}^{(i)}\gamma^{2}\mathbf{g}$$
(32)

Following JR06, we premultiply by $2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}$ and define $\mathbf{h} = \gamma \mathbf{g}$ to get

$$\gamma \mathbf{g} = \mathbf{h}$$

$$\gamma \mathbf{h} = -2 \left(\boldsymbol{\Sigma}^{(i)} \right)^{-1} \boldsymbol{\mu}^{(i)} \mathbf{h} + 2 \left(\boldsymbol{\Sigma}^{(i)} \right)^{-1} \left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)} \right) \mathbf{g}$$
(33)
(34)

Stacking the vectors $\mathbf{j} = \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix}$ we have $\gamma \mathbf{j} = \begin{bmatrix} \mathbf{0}_{2i} & \mathbf{I}_{2i} \\ 2\left(\mathbf{\Sigma}^{(i)}\right)^{-1} \left(\mathbf{R}^{(i)} - \mathbf{Q}^{(i)}\right) & -2\left(\mathbf{\Sigma}^{(i)}\right)^{-1} \boldsymbol{\mu}^{(i)} \end{bmatrix} \mathbf{j} = \underbrace{\mathbf{A}^{(i)}_{4i \times 4i}}_{4i \times 4i}$ (35)

where **I** is of appropriate dimensions. The problem is now a simple eigenvalue-eigenvector problem and each solution j is a pair $\left(\underbrace{\gamma_j^{(i)}}_{1\times 1}, \underbrace{\mathbf{j}_j^{(i)}}_{4i\times 1}\right)$ (or rather $\left(\underbrace{\gamma_j^{(i)}}_{1\times 1}, \underbrace{\mathbf{g}_j^{(i)}}_{2i\times 1}\right)$, as the vector $\mathbf{j}_j^{(i)}$ contains the same information as $\mathbf{g}_j^{(i)}$ when we know $\gamma_j^{(i)}$, so we discard the lower half of $\mathbf{j}_j^{(i)}$).¹⁴ The number of solutions j to this eigenvector-eigenvalue problem is

¹⁴Note that if a program like MATLAB or Mathematica is used to calculate eigenvectors, it usually norms the eigenvectors \mathbf{j} so that they have unit length, i.e. $\left|\left|\mathbf{j}_{j}^{(i)}\right|\right| = 1$ where $||\cdot||$ denotes the Euclidian norm. The norm of $\mathbf{g}_{j}^{(i)}$ is $\left|\left|\mathbf{g}_{j}^{(i)}\right|\right| = \left(\sqrt{1 + \left(\gamma_{j}^{(i)}\right)^{2}}\right)^{-1} \left|\left|\mathbf{j}_{j}^{(i)}\right|\right|$. This can be easily derived: First, note that $\gamma \mathbf{h} = \mathbf{g}$. Second, writing out

the Euclidian norm, we have

$$\begin{aligned} ||\mathbf{j}|| &= \sqrt{\underbrace{j_1^2 + \ldots + j_{2i}^2}_{\mathbf{g}} + \underbrace{j_{2i+1}^2 + \ldots + j_{4i}^2}_{\mathbf{h} = \gamma \mathbf{g}}} \\ &= \sqrt{j_1^2 + \ldots + j_{2i}^2 + \gamma^2 j_1^2 + \ldots + \gamma^2 j_{2i}^2} \\ &= \sqrt{(1 + \gamma^2) (j_1^2 + \ldots + j_{2i}^2)} \\ &= \sqrt{(1 + \gamma^2) ||\mathbf{g}||} \end{aligned}$$

so that we would have to re-norm \mathbf{g} by a factor $\sqrt{1 + (\gamma_j^{(i)})^2}$ to recover a unit length vector if $\left\| \mathbf{j}_j^{(i)} \right\| = 1$. Thus, we can easily re-norm $\mathbf{G}^{(i)}$ by the matrix operation $\mathbf{G}^{(i)}\sqrt{\mathbf{I}_{4i} + (\mathbf{\Gamma}^{(i)})^2}$ where special care should be taken to select the matrix power function when programming.

4i. Let

$$\mathbf{G}^{(i)} \equiv \begin{bmatrix} \mathbf{g}_1^{(i)}, ..., \mathbf{g}_{2 \times 2 \times i}^{(i)} \end{bmatrix}$$
(36)

be the matrix of eigenvectors, and let

$$\boldsymbol{\gamma}^{(i)} \equiv \left[\gamma_1^{(i)}, \dots, \gamma_{2\times 2\times i}^{(i)}\right]' \tag{37}$$

$$\boldsymbol{\Gamma}^{(i)} \equiv \operatorname{diag}\left[\boldsymbol{\gamma}^{(i)}\right] \tag{38}$$

be the corresponding vector and diagonal matrix, respectively, of eigenvalues.

The general solution on interval i is thus

$$\underbrace{\mathbf{D}_{2i\times1}^{(i)}}_{2i\times4i} = \underbrace{\mathbf{G}_{i\times4i}^{(i)}}_{4i\times4i} \cdot \underbrace{\exp\left(\mathbf{\Gamma}^{(i)}\delta\right)}_{4i\times4i} \cdot \underbrace{\mathbf{c}_{i\times1}^{(i)}}_{4i\times1} + \underbrace{\mathbf{k}_{0}^{(i)}}_{2i\times1} + \underbrace{\mathbf{k}_{1}^{(i)}}_{2i\times1} \exp\left(\delta\right)$$
(39)

where the constants $\mathbf{c}^{(i)} = \left[c_1^{(i)}, ..., c_{4i}^{(i)}\right]^\top$ will have to be determined via conditions at the boundaries of interval I_i (**NOTE:** $c_j^{(i)} \neq c$ where c is the coupon payment).

Boundary conditions. The different value functions $\mathbf{D}^{(i)}$ for $i \in \{1, ..., n\}$ are linked at the boundaries of their domains I_i . Note that $I_i \cap I_{i+1} = \{\delta_B (i+1)\}$ for i < n.

For i = n, we can immediately rule out all positive solutions to γ as debt has to be finite and bounded as $\delta \to \infty$, so that the entries of $\mathbf{C}^{(n)}$ corresponding to positive eigenvalues will be zero:¹⁵

$$\lim_{\delta \to \infty} \left| \mathbf{D}^{(n)} \left(\delta \right) \right| < \infty \tag{40}$$

For i < n, we must have value matching of the value functions that are alive across the boundary, and we must have value matching of the value functions that die across the boundary:

$$\mathbf{D}^{(i+1)}\left(\delta_B\left(i+1\right)\right) = \begin{bmatrix} \mathbf{D}^{(i)}\left(\delta_B\left(i+1\right)\right) \\ \begin{bmatrix} v_{H}^{i+1} \\ v_{L}^{i+1} \end{bmatrix} \exp\left(\delta_B\left(i+1\right)\right) \end{bmatrix}$$
(41)

For i < n, we must have mechanical (i.e. non-optimal) smooth pasting of the value functions that are alive across the boundary:

$$\left(\mathbf{D}^{(i+1)}\right)' \left(\delta_B \left(i+1\right)\right)_{[1...2i]} = \left(\mathbf{D}^{(i)}\right)' \left(\delta_B \left(i+1\right)\right)$$
(42)

where $\mathbf{x}_{[1...2i]}$ selects the first 2i rows of vector \mathbf{x} .

Lastly, for i = 1, we must have

$$\mathbf{D}^{(1)}\left(\delta_{B}\left(1\right)\right) = \begin{bmatrix} v_{H}^{1} \\ v_{L}^{1} \end{bmatrix} \exp\left(\delta_{B}\left(1\right)\right)$$
(43)

Full solution. We can now state the full solution to the debt valuation given cut-off strategies: Proposition 2 The debt value functions D for a given default vector δ_B are

$$\mathbf{D}\left(\delta\right) = \begin{cases} \underbrace{\mathbf{D}^{(n)}\left(\delta\right)}_{2n\times1} = \mathbf{G}^{(n)} \cdot \exp\left(\mathbf{\Gamma}^{(n)}\delta\right) \cdot \mathbf{c}^{(n)} + \mathbf{k}_{0}^{(n)} & \delta \in I_{n} \\ \vdots & \vdots \\ \underbrace{\mathbf{D}^{(i)}\left(\delta\right)}_{2i\times1} = \mathbf{G}^{(i)} \cdot \exp\left(\mathbf{\Gamma}^{(i)}\delta\right) \cdot \mathbf{c}^{(i)} + \mathbf{k}_{0}^{(i)} + \mathbf{k}_{1}^{(i)} \exp\left(\delta\right) & \delta \in I_{i} \\ \vdots & \vdots \\ \underbrace{\mathbf{D}^{(1)}\left(\delta\right)}_{2\times1} = \mathbf{G}^{(1)} \cdot \exp\left(\mathbf{\Gamma}^{(1)}\delta\right) \cdot \mathbf{c}^{(1)} + \mathbf{k}_{0}^{(1)} + \mathbf{k}_{1}^{(1)} \exp\left(\delta\right) & \delta \in I_{1} \end{cases}$$

¹⁵ According to JR06, there are exactly $2 \times |S| = 2n$ eigenvalues of **A** in the left open half plane (i.e. negative) and 2n eigenvalues in the right open half plane (i.e. positive) (actually, they only argue that this holds if $\mu = \mathbf{R} - \frac{1}{2}\boldsymbol{\Sigma}$, but maybe not for general $\boldsymbol{\mu}$).

with the following boundary conditions to pin down vectors $\mathbf{c}^{(i)}$:

$$\lim_{\delta \to \infty} \left| \underbrace{\mathbf{D}^{(n)}(\delta)}_{2n \times 1} \right| < \infty$$
(44)

$$\underbrace{\mathbf{D}^{(i+1)}\left(\delta_{B}\left(i+1\right)\right)}_{2(i+1)\times 1} = \underbrace{\left[\begin{array}{c} \mathbf{D}^{(i)}\left(\delta_{B}\left(i+1\right)\right)\\ \begin{bmatrix} v_{H}^{i+1}\\ v_{L}^{i+1} \end{bmatrix} \exp\left(\delta_{B}\left(i+1\right)\right)}_{2(i+1)\times 1}\right]}_{2(i+1)\times 1}$$
(45)

$$\underbrace{\left(\mathbf{D}^{(i+1)}\right)'(\delta_B(i+1))_{[1...2i]}}_{2i\times 1} = \underbrace{\left(\mathbf{D}^{(i)}\right)'(\delta_B(i+1))}_{2i\times 1}$$
(46)

$$\underbrace{\mathbf{D}^{(1)}\left(\delta_{B}\left(1\right)\right)}_{2\times1} = \underbrace{\left[\begin{array}{c}v_{H}^{1}\\v_{L}^{1}\end{array}\right]\exp\left(\delta_{B}\left(1\right)\right)}_{2\times1}$$
(47)

where $\mathbf{x}_{[1..2i]}$ selects the first 2*i* rows of vector \mathbf{x} .

Note that the derivative of the debt value vector is

$$\underbrace{\left(\mathbf{D}^{(i)}\right)'(\delta)}_{2i\times 1} = \mathbf{G}^{(i)}\mathbf{\Gamma}^{(i)} \cdot \exp\left(\mathbf{\Gamma}^{(i)}\delta\right) \cdot \mathbf{c}^{(i)} + \mathbf{k}_{1}^{(i)}\exp\left(\delta\right)$$
(48)

where we note that $\Gamma^{(i)} \cdot \exp\left(\Gamma^{(i)}\delta\right) = \exp\left(\Gamma^{(i)}\delta\right) \cdot \Gamma^{(i)}$ as both are diagonal matrices (although this interchangeability only is important when s = 1 as it then helps collapse some equations).

The first boundary condition (44) essentially implies that we can discard any positive entries of $\gamma^{(n)}$ by setting the appropriate coefficients of $\mathbf{C}^{(n)}$ to 0. The second boundary condition (45) implies that we have value matching at any boundary $\delta_B(i+1)$ for i < n, be it to a continuation state or a bankruptcy state. The third boundary condition (46) implies that we also have smooth pasting at the boundary $\delta_B(i+1)$ for those states in which the firm stays alive on both sides of the boundary. Finally, the fourth boundary condition (47) implies value matching at the boundary $\delta_B(1)$, but of course only for those states in which the firm is still alive.

Thus, let us summarize the solution steps:

- 1. Order states so that the most restrictive/illiquid states are with the highest indices, such that $\delta_B(i) < \delta_B(j)$ implies i < j (i.e. they appear in the lowest rows/columns in the following matrices).
- 2. Define the suitable matrices \mathbf{R}, \mathbf{Q} for the transitions, and of course $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ for drift and variance. These apply on the highest interval I_n .
- 3. Set up the eigenvalue-eigenvector problem and solve for (the matrix of) eigenvectors $\mathbf{G}^{(n)}$ and (the vector of) eigenvalues $\boldsymbol{\gamma}^{(n)}$. Solve for the constant $\mathbf{k}_0^{(n)}$ on this interval.
- 4. For intervals I_{n-i} we drop for each increment *i* the last pair of rows and columns of the appropriate matrices, with the following exception. We define $\mathbf{Q}^{(n-i)}$ as the matrix that arises out of \mathbf{Q} when we drop the last *i* pair of rows **and** columns, i.e. rows 1-2 and columns 1-2 survive in the 4x4 case. We similarly define $\mathbf{R}^{(n-i)}, \boldsymbol{\mu}^{(n-i)}, \boldsymbol{\Sigma}^{(n-i)}$. We define $\tilde{\mathbf{Q}}^{(n-i)}$ as the matrix that arises out of \mathbf{Q} when we drop the last *i* pair of rows **and** the first n-i pairs of columns, i.e. rows 1-2 and columns 3-4 survive in the 4x4 case.
- 5. Set up the eigenvalue-eigenvector problem for interval I_{n-i} and solve for (the matrix of) eigenvectors $\mathbf{G}^{(n-1)}$ and (the vector of) eigenvalues $\boldsymbol{\gamma}^{(n-1)}$. Solve for the constant $\mathbf{k}_0^{(n-1)}$ on this interval and also for the particular part $\mathbf{k}_1^{(n-1)} \exp(\delta)$.
- 6. Build the system of boundary conditions via the matrix definitions of the debt to solve for the linear coefficients $\mathbf{c}^{(i)}$. To impose boundary condition (44), it is probably easiest to just use those entries of $\boldsymbol{\gamma}^{(n)}$ that are negative. Thus, the appropriate $\mathbf{C}^{(n)}$ for I_n is only a $2n \times 1$ vector, and not a $4n \times 1$ vector.

1.1 Deterministic maturity PDE

Debt valuation follows the following differential equation on interval I_i :

$$(\mathbf{R} - \mathbf{Q})\mathbf{D} = (c\mathbf{1}_{2i} - \boldsymbol{\chi}) + \boldsymbol{\mu}(\mathbf{D})' + \frac{1}{2}\boldsymbol{\Sigma}(\mathbf{D})'' - \dot{\mathbf{D}}$$
(49)

where $\dot{\mathbf{D}}$ is derivative wrt time-to-maturity and R is different

$$\underbrace{\mathbf{R}}_{2n\times 2n} = \begin{bmatrix} \operatorname{diag}\left(\begin{bmatrix} r_{H}(1) \\ r_{L}(1) \end{bmatrix} \right) & \cdots & \mathbf{0}_{2} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{2} & \cdots & \operatorname{diag}\left(\begin{bmatrix} r_{H}(n) \\ r_{L}(n) \end{bmatrix} \right) \end{bmatrix}$$
(50)

The above differential equation holds only for $\delta > \delta_B^B$, any $\tau \in [0, T]$, i.e. alive even in bad time. In states $\delta \in [\delta_B^G, \delta_B^B]$, the PDE should be written in a way that in good time, the states is still moving so an equation like (37) applies but with

$$(\mathbf{R} - \mathbf{Q})\mathbf{D} = (c\mathbf{1}_{2i} - \boldsymbol{\chi}) + \boldsymbol{\mu}(\mathbf{D})' + \frac{1}{2}\boldsymbol{\Sigma}(\mathbf{D})'' - \dot{\mathbf{D}} + \tilde{\mathbf{Q}}\mathbf{v}\exp(\delta)$$

; but in bad states the state does not move, and D_i^B in **D** should be a constant function.

There are four debt values in (37). It remains to specify boundary conditions. When δ is sufficiently high, all four debt contracts are riskfree. When $\tau = 0$, $\mathbf{D}(\delta, \tau = 0) = \mathbf{D}_{\tau=0}(\delta)$, so that $D_{i,\tau=0}^G = p$ for $\delta \in [\delta_B^G, \infty]$, while in state B,

 $D^B_{i,\tau=0} = p \text{ for} \delta \in \left[\delta^G_B, \infty\right], \text{ and } D^B_{i,\tau=0} = \mathbf{v} \exp\left(\delta\right) \text{ for} \delta \in \left[\delta^G_B, \delta^B_B\right]$

Finally, the boundary condition at δ_B^G , we have

$$\mathbf{D}_{\delta=\delta_{\mathbf{B}}^{\mathbf{G}}} = \tilde{\mathbf{Q}}\mathbf{v}\exp\left(\delta\right)$$

note that since we force our D_i^B in **D** to be constant over the state space of $\delta \in [\delta_B^G, \delta_B^B]$ and $\tau \in [0, T]$, the boundary at δ_B^G is sufficient.

B Equity

The equity holders are unaffected by the individual liquidity shocks the debt holders are exposed to. The only shocks the equity holders are directly exposed to are the shifts in $\mu(s)$ and $\sigma(s)$, i.e. shifts to the cash-flow process.

However, as debt has maturity and is rolled over, equity holders are indirectly affected by liquidity shocks in the market through the effect it has on debt prices. Thus, when debt matures, it is either rolled over if the debt holders are of type H, or it is reissued to different debt holders in the case that the former debt holder is of type L. Either way, there is a cash flow (inflow or outflow) of $m \left[\mathbf{S}^{(i)} \cdot \mathbf{D}^{(i)} (\delta) - p \mathbf{1}_i \right]$ at each instant as a mass $m \cdot dt$ of debt holders matures on [t, t + dt].

For notational ease, we will denote by double letters (e.g. $\mathbf{x}\mathbf{x}$) a constant for equity that takes a similar place as a single letter (i.e. \mathbf{x}) constant for debt. Then, the HJB for equity on interval I_i is given by

$$\left(\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)}\right) \mathbf{E}^{(i)}(\delta) = \mu \mu^{(i)} \left(\mathbf{E}^{(i)}\right)'(\delta) + \frac{1}{2} \Sigma \Sigma^{(i)} \left(\mathbf{E}^{(i)}\right)''(\delta) \\ + \underbrace{\mathbf{1}_{i} \exp\left(\delta\right)}_{Cashflow} - \underbrace{(1 - \pi) c \mathbf{1}_{i}}_{Coupon} + \underbrace{m \left[\mathbf{S}^{(i)} \cdot \mathbf{D}^{(i)}(\delta) - p \mathbf{1}_{i}\right]}_{Rollover}$$
(51)

where

$$\mathbf{RR}^{(i)} = \operatorname{diag}\left([r_H(1), ..., r_H(i)]\right)$$
(52)

$$\mu \mu^{(i)} = \operatorname{diag}\left(\left[\mu\left(1\right), ..., \mu\left(i\right)\right]\right)$$
(53)

$$\Sigma\Sigma^{(i)} = \operatorname{diag}\left(\left[\sigma^{2}\left(1\right), ..., \sigma^{2}\left(i\right)\right]\right)$$
(54)

are $i \times i$ square matrices, $\mathbf{QQ}^{(i)}$ is the transition matrix only between aggregate states that is also an $i \times i$ square matrix, and $\mathbf{S}^{(i)}$ is a $i \times 2i$ matrix that selects which debt values the firm is able to issue (each row has to sum to 1), and m is a scalar (**NOTE**: In contrast to **R**, the matrix **RR** does not contain the maturity intensity m). For example, for i = 2, if the company is able to place debt only to H types, then $\mathbf{S}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. It is important that for reach row i only entries 2i - 1 and 2i are possibly nonzero, whereas all other entries are identically zero (otherwise, one would issue bonds belonging to a different state).

Writing out $\mathbf{D}^{(i)}(\delta) = \mathbf{G}^{(i)} \exp\left(\mathbf{\Gamma}^{(i)}\delta\right) \mathbf{c}^{(i)}$ and conjecturing a solution to the particular, non-constant part

	Debt Parameters		Equity Parameters			
Symbol	Interpretation	Dimension	Symbol	Interpretation	Dimension	
$\mathbf{D}^{\left(i ight)}\left(\delta ight)$	Debt Value Function	$2i \times 1$	$\mathbf{E}^{(i)}\left(\delta ight)$	Equity Value Function	$i \times 1$	
$oldsymbol{\mu}^{(i)}$	(Log)-Drifts	$2i \times 2i$	$oldsymbol{\mu}oldsymbol{\mu}^{(i)}$	(Log-)Drifts	i imes i	
$\mathbf{\Sigma}^{(i)}$	Volatilities	$2i \times 2i$	$\mathbf{\Sigma}\mathbf{\Sigma}^{(i)}$	Volatilities	i imes i	
$\mathbf{R}^{(i)}$	Discount rates and maturity	$2i \times 2i$	$\mathbf{RR}^{(i)}$	Discount rates	i imes i	
$oldsymbol{\chi}^{(i)}$	Holding costs	$2i \times 1$	c	Coupon	1×1	
$\mathbf{Q}^{(i)}$	Transition to cont. states	$2i \times 2i$	$\mathbf{Q}\mathbf{Q}^{(i)}$	Transition to cont. states	i imes i	
$ ilde{\mathbf{Q}}^{(i)}$	Transition to default states	$2i \times 2(n-i)$	$\mathbf{A}\mathbf{A}^{(i)}$	Matrix to be decomposed	$2i \times 2i$	
$\mathbf{v}^{(i)}$	Vector of recovery values	$2(n-i) \times 1$	$\mathbf{\Gamma}\mathbf{\Gamma}^{(i)}$	Diag matrix of eigenvalues	$2i \times 2i$	
$\mathbf{A}^{(i)}$	Matrix to be decomposed	$4i \times 4i$	$\mathbf{GG}^{(i)}$	Matrix of eigenvectors	$i \times 2i$	
$\Gamma^{(i)}$	Diag matrix of eigenvalues	$4i \times 4i$	$\mathbf{k}\mathbf{k}_{0}^{(i)},\mathbf{k}\mathbf{k}_{1}^{(i)}$	Coeff. of particular sol.	$i \times 1$	
$\mathbf{G}^{(i)}$	Matrix of eigenvectors	$2i \times 4i$	$\mathbf{S}^{(i)}$	Issuance matrix	$i \times 2i$	
$\mathbf{k}_{0}^{(i)},\mathbf{k}_{1}^{(i)}$	Coeff. of particular sol.	$2i \times 1$	$\mathbf{K}\mathbf{K}^{(i)}$	Coeff. of particular sol.	$i \times 4i$	
$\mathbf{c}^{(i)}$	Vector of constants	$4i \times 1$	$\mathbf{cc}^{(i)}$	Vector of constants	$2i \times 1$	

Table 9: Matrix & Vector Dimensions.

$$\underbrace{\mathbf{K}\mathbf{K}^{(i)}}_{i\times 4i} \underbrace{\exp\left(\mathbf{\Gamma}^{(i)}\delta\right)}_{4i\times 4i} \underbrace{\mathbf{c}^{(i)}}_{4i\times 1}, \text{ we have }$$

$$\left(\mathbf{R}\mathbf{R}^{(i)} - \mathbf{Q}\mathbf{Q}^{(i)} \right) \mathbf{K}\mathbf{K}^{(i)} \exp\left(\mathbf{\Gamma}^{(i)}\delta\right) \mathbf{c}^{(i)}$$

$$= \left[\mu\mu^{(i)} \cdot \mathbf{K}\mathbf{K}^{(i)} \cdot \mathbf{\Gamma}^{(i)} + \frac{1}{2}\mathbf{\Sigma}\mathbf{\Sigma}^{(i)}\mathbf{K}\mathbf{K}^{(i)} \cdot \left(\mathbf{\Gamma}^{(i)}\right)^{2} + m \cdot \mathbf{S}^{(i)} \cdot \mathbf{G}^{(i)} \right] \exp\left(\mathbf{\Gamma}^{(i)}\delta\right) \mathbf{c}^{(i)}$$
(55)

We can solve this by considering each $\gamma_j^{(i)}$ separately — recall that $\mathbf{c}^{(i)}$ is a vector and $\exp\left(\mathbf{\Gamma}^{(i)}\delta\right)$ is a diagonal matrix and in total there are 4i different roots. Consider the part of the particular part $\mathbf{S}^{(i)} \cdot \mathbf{g}_j^{(i)} \exp\left(\gamma_j^{(i)}\delta\right) \cdot c_j^{(i)}$ and our conjecture gives $\underbrace{\mathbf{KK}_j^{(i)} \exp\left(\gamma_j^{(i)}\delta\right)}_{i \times 1} \cdot \underbrace{c_j^{(i)}}_{1 \times 1} + \underbrace{c_j^{(i)}}_{1 \times 1}$ for each root $j \in [1, ..., 4i]$. Plugging in and multiplying out the scalar $\exp\left(\gamma_j^{(i)}\delta\right) c_j^{(i)}$, we find that

$$\left(\mathbf{RR}^{(i)} - \mathbf{QQ}^{(i)}\right)\mathbf{KK}_{j}^{(i)} = \boldsymbol{\mu}\boldsymbol{\mu}^{(i)} \cdot \mathbf{KK}_{j}^{(i)} \cdot \boldsymbol{\gamma}_{j}^{(i)} + \frac{1}{2}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{(i)}\mathbf{KK}_{j}^{(i)} \cdot \left(\boldsymbol{\gamma}_{j}^{(i)}\right)^{2} + \boldsymbol{m} \cdot \mathbf{S}^{(i)} \cdot \mathbf{g}_{j}^{(i)}$$
(56)

Solving for $\mathbf{K}\mathbf{K}_{j}^{(i)}$, we have

$$\underbrace{\mathbf{K}\mathbf{K}_{j}^{(i)}}_{i\times 1} = \underbrace{\left[\mathbf{R}\mathbf{R}^{(i)} - \mathbf{Q}\mathbf{Q}^{(i)} - \mu\mu^{(i)} \cdot \gamma_{j}^{(i)} - \frac{1}{2}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{(i)} \cdot \left(\gamma_{j}^{(i)}\right)^{2}\right]^{-1}}_{i\times i} m \cdot \underbrace{\mathbf{S}_{i\times 2i}^{(i)}}_{2i\times 1} \mathbf{g}_{j}^{(i)} \tag{57}$$

Finally, for the homogenous part we use the same approach as above, but now we have less states as the individual liquidity state drops out. Thus, we conjecture $\mathbf{gg} \exp(\gamma \gamma \delta)$ to get

$$\mathbf{0}_{i} = \left(\mathbf{Q}\mathbf{Q}^{(i)} - \mathbf{R}\mathbf{R}^{(i)}\right)\mathbf{g}\mathbf{g} + \boldsymbol{\mu}\boldsymbol{\mu}^{(i)}\gamma\gamma\mathbf{g}\mathbf{g} + \frac{1}{2}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{(i)}\gamma\gamma\mathbf{g}\mathbf{g}$$
(58)

so that, again, we have the following eigenvector eigenvalue problem

$$\gamma\gamma\mathbf{j}\mathbf{j} = \begin{bmatrix} \mathbf{0}_{i} & \mathbf{I}_{i} \\ 2\left(\mathbf{\Sigma}\mathbf{\Sigma}^{(i)}\right)^{-1} \left(\mathbf{R}\mathbf{R}^{(i)} - \mathbf{Q}\mathbf{Q}^{(i)}\right) & -2\left(\mathbf{\Sigma}\mathbf{\Sigma}^{(i)}\right)^{-1}\mu\mu^{(i)} \end{bmatrix} \mathbf{j}\mathbf{j} = \underbrace{\mathbf{A}\mathbf{A}^{(i)}}_{2i\times 2i}\mathbf{j}\mathbf{j}$$
(59)

which gives $\left(\gamma \gamma_{j}^{(i)}, \mathbf{gg}_{j}^{(i)}\right)$ for $j \in [1, ..., 2i]$ solutions. We stack these into a matrix of eigenvectors $\mathbf{GG}^{(i)}$ and a vector of eigenvalues $\gamma \gamma^{(i)}$, from which we define the diagonal matrix of eigenvalues $\mathbf{\Gamma}\mathbf{\Gamma}^{(i)} \equiv \operatorname{diag}\left(\gamma \gamma^{(i)}\right)$. What remains is

to solve for $\mathbf{kk}_{0}^{(i)}$ and $\mathbf{kk}_{1}^{(i)}$. We have

$$\mathbf{k}\mathbf{k}_{0}^{(i)} = \left[\mathbf{R}\mathbf{R}^{(i)} - \mathbf{Q}\mathbf{Q}^{(i)}\right]^{-1} \left[-(1-\pi)c\mathbf{1}_{i} + m\left(\mathbf{S}^{(i)}\mathbf{k}_{0}^{(i)} - p\mathbf{1}_{i}\right)\right]$$
(60)

and

$$\mathbf{k}\mathbf{k}_{1}^{(i)} = \left[\mathbf{R}\mathbf{R}^{(i)} - \mathbf{Q}\mathbf{Q}^{(i)} - \boldsymbol{\mu}\boldsymbol{\mu}^{(i)} - \frac{1}{2}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{(i)}\right]^{-1} \left(\mathbf{1}_{i} + m \cdot \mathbf{S}^{(i)}\mathbf{k}_{1}^{(i)}\right)$$
(61)

with $\mathbf{k}_1^{(n)} = \mathbf{0}$.

We are left with the following proposition.

Proposition 3 The equity value functions **E** for a given default vector $\boldsymbol{\delta}_B$ are

$$\underbrace{\underbrace{\mathbf{E}^{(n)}(\delta)}_{n\times 1} = \mathbf{G}\mathbf{G}^{(n)} \cdot \exp\left(\mathbf{\Gamma}\mathbf{\Gamma}^{(n)}\delta\right) \cdot \mathbf{c}\mathbf{c}^{(n)} + \mathbf{K}\mathbf{K}^{(n)}\exp\left(\mathbf{\Gamma}^{(n)}\delta\right)\mathbf{c}^{(n)} + \mathbf{k}\mathbf{k}_{0}^{(n)} + \mathbf{k}\mathbf{k}_{1}^{(n)}\exp\left(\delta\right) \quad \delta \in I_{r}$$
:

$$\mathbf{E}\left(\delta\right) = \begin{cases} \vdots \\ \mathbf{E}_{i\times 1}^{(i)}\left(\delta\right) = \mathbf{G}\mathbf{G}^{(i)} \cdot \exp\left(\mathbf{\Gamma}\mathbf{\Gamma}^{(i)}\delta\right) \cdot \mathbf{c}\mathbf{c}^{(i)} + \mathbf{K}\mathbf{K}^{(i)}\exp\left(\mathbf{\Gamma}^{(i)}\delta\right)\mathbf{c}^{(i)} + \mathbf{k}\mathbf{k}_{0}^{(i)} + \mathbf{k}\mathbf{k}_{1}^{(i)}\exp\left(\delta\right) & \delta \in I_{i} \end{cases}$$

$$\underbrace{ \underbrace{\mathbf{E}^{(1)}(\delta)}_{1\times 1} = \mathbf{G}\mathbf{G}^{(1)} \cdot \exp\left(\mathbf{\Gamma}\mathbf{\Gamma}^{(1)}\delta\right) \cdot \mathbf{c}\mathbf{c}^{(1)} + \mathbf{K}\mathbf{K}^{(1)}\exp\left(\mathbf{\Gamma}^{(1)}\delta\right)\mathbf{c}^{(1)} + \mathbf{k}\mathbf{k}_{0}^{(1)} + \mathbf{k}\mathbf{k}_{1}^{(1)}\exp\left(\delta\right) \qquad \delta \in I_{1}$$

with the following boundary conditions to pin down the vector $\mathbf{cc}^{(i)}$:

$$\lim_{\delta \to \infty} \left| \underbrace{\mathbf{E}^{(n)}(\delta) \exp\left(-\delta\right)}_{n \times 1} \right| < \infty$$
(62)

$$\underbrace{\left(\mathbf{E}^{(i+1)}\right)'\left(\delta_B\left(i+1\right)\right)_{[1\dots i]}}_{i\times 1} = \underbrace{\left(\mathbf{E}^{(i)}\right)'\left(\delta_B\left(i+1\right)\right)}_{i\times 1}$$
(64)

$$\underbrace{\mathbf{E}^{(i)}\left(\delta_B\left(1\right)\right)}_{i\times 1} = 0 \tag{65}$$

where $\mathbf{x}_{[1...i]}$ selects the first *i* rows of vector \mathbf{x} .

Note first the dimensionalities: $\underbrace{\Gamma\Gamma^{(i)}}_{2i\times 2i}, \underbrace{\mathbf{GG}^{(i)}}_{i\times 2i}$ and $\underbrace{\Gamma^{(i)}}_{4i\times 4i}, \underbrace{\mathbf{G}^{(i)}}_{2i\times 4i}$. Note second the derivative of the equity value vector

is

$$\underbrace{\left(\mathbf{E}^{(i)}\right)'(\delta)}_{i\times 1} = \mathbf{G}\mathbf{G}^{(i)}\mathbf{\Gamma}\mathbf{\Gamma}^{(i)} \cdot \exp\left(\mathbf{\Gamma}\mathbf{\Gamma}^{(i)}\delta\right) \cdot \mathbf{c}\mathbf{c}^{(i)} + \mathbf{K}\mathbf{K}^{(i)}\mathbf{\Gamma}^{(i)}\exp\left(\mathbf{\Gamma}^{(i)}\delta\right)\mathbf{c}^{(i)} + \mathbf{k}\mathbf{k}_{1}^{(i)}\exp\left(\delta\right)$$
(66)

where we note that $\Gamma^{(i)} \cdot \exp(\Gamma^{(i)}\delta) = \exp(\Gamma^{(i)}\delta) \cdot \Gamma^{(i)}$ and $\Gamma\Gamma^{(i)} \cdot \exp(\Gamma\Gamma^{(i)}\delta) = \exp(\Gamma\Gamma^{(i)}\delta) \cdot \Gamma\Gamma^{(i)}$ as both are diagonal matrices (although this interchangeability only is important when s = 1 as it then helps collapse some equations).

The optimality conditions for bankruptcy boundaries $\{\delta_B(i)\}_i$ are given by

$$\left(\mathbf{E}^{(i)}\right)' (\delta_B(i))_{[i]} = 0$$
 (67)

i.e., a smooth pasting condition at the boundaries at which default is declared.

2.1 The explicit matrices for the 4x2 case

• 2 individual states (H and L, i.e. normal and liquidity shocked)

• 4 aggregate states (1, 2, 3, 4) so that n = 4

Let us make the following assumptions:

- The bargaining power of the creditors is constant at β
- The firm is able to place new debt exclusively with H types
- The aggregate state switching intensities for switching from state s to state s' are $\zeta_{ss'} = \zeta_{s \to s'}$. Let us write out the transition matrix for the aggregate state only, where rows and columns are ordered according to [1, 2, 3, 4]:

$$\mathbf{QQ} = \begin{bmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} & \zeta_{14} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} & \zeta_{24} \\ \zeta_{31} & \zeta_{32} & \zeta_{33} & \zeta_{34} \\ \zeta_{41} & \zeta_{42} & \zeta_{43} & \zeta_{44} \end{bmatrix}$$

where $\zeta_{ss} = -\sum_{s' \neq s} \zeta_{ss'}$ is the 'intensity' of staying in state s and we assume that $\zeta_{ss'} \ge 0$ for $s \neq s'$.

• The transition matrices for equity holders in ${\cal I}_i$ are

$$\begin{aligned} \mathbf{QQ}^{(4)} &= \mathbf{QQ} \\ \mathbf{QQ}^{(3)} &= \begin{bmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} \\ \zeta_{31} & \zeta_{32} & \zeta_{33} \end{bmatrix} \\ \mathbf{QQ}^{(2)} &= \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{bmatrix} \\ \mathbf{QQ}^{(1)} &= [\zeta_{11}] \end{aligned}$$

- There is a (possibly state dependent) individual liquidity shock intensity $\xi(s) = \xi_{H_s \to L_s}$. We will assume this constant s.t. $\xi = \xi(s), \forall s$
- There is a (possibly state dependent) intermediation intensity $\lambda(s)$
- Aggregate and liquidity shocks are independent
- Let us now write out the creditor transition matrix, where states are ordered according to the ordering [H1, L1, H2, L2, H3, L3, H4, L4]:

$$\mathbf{Q} = \begin{bmatrix} -\sum \dots & \xi(1) & \zeta_{12} & 0 & \zeta_{13} & 0 & \zeta_{14} & 0 \\ \lambda(1)\beta & -\sum \dots & 0 & \zeta_{12} & 0 & \zeta_{13} & 0 & \zeta_{14} \\ \zeta_{21} & 0 & -\sum \dots & \xi(2) & \zeta_{23} & 0 & \zeta_{24} & 0 \\ 0 & \zeta_{21} & \lambda(2)\beta & -\sum \dots & 0 & \zeta_{23} & 0 & \zeta_{24} \\ \zeta_{31} & 0 & \zeta_{32} & 0 & -\sum \dots & \xi(3) & \zeta_{34} & 0 \\ 0 & \zeta_{31} & 0 & \zeta_{32} & \lambda(3)\beta & -\sum \dots & 0 & \zeta_{34} \\ \zeta_{41} & 0 & \zeta_{42} & 0 & \zeta_{43} & 0 & -\sum \dots & \xi(4) \\ 0 & \zeta_{41} & 0 & \zeta_{42} & 0 & \zeta_{43} & \lambda(4)\beta & -\sum \dots \end{bmatrix}$$

where $-\sum \ldots$ is the term that makes each row of **Q** sum to zero.

• For interval I_4 , we have

$$\mathbf{Q}^{(4)} = \mathbf{Q} \tilde{\mathbf{Q}}^{(4)} = 0$$

• For interval I_3 , we have

$$\mathbf{Q}^{(3)} = \begin{bmatrix} -\sum \dots & \xi(1) & \zeta_{12} & 0 & \zeta_{13} & 0 \\ \lambda(1)\beta & -\sum \dots & 0 & \zeta_{12} & 0 & \zeta_{13} \\ \zeta_{21} & 0 & -\sum \dots & \xi(2) & \zeta_{23} & 0 \\ 0 & \zeta_{21} & \lambda(2)\beta & -\sum \dots & 0 & \zeta_{23} \\ \zeta_{31} & 0 & \zeta_{32} & 0 & -\sum \dots & \xi(3) \\ 0 & \zeta_{31} & 0 & \zeta_{32} & \lambda(3)\beta & -\sum \dots \end{bmatrix}$$
$$\tilde{\mathbf{Q}}^{(3)} = \begin{bmatrix} \zeta_{14} & 0 \\ 0 & \zeta_{14} \\ \zeta_{24} & 0 \\ 0 & \zeta_{24} \\ \zeta_{34} & 0 \\ 0 & \zeta_{34} \end{bmatrix}$$

• For interval I_2 , we have

$$\mathbf{Q}^{(2)} = \begin{bmatrix} -\sum \dots & \xi(1) & \zeta_{12} & 0 \\ \lambda(1)\beta & -\sum \dots & 0 & \zeta_{12} \\ \zeta_{21} & 0 & -\sum \dots & \xi(2) \\ 0 & \zeta_{21} & \lambda(2)\beta & -\sum \dots \end{bmatrix}$$
$$\tilde{\mathbf{Q}}^{(2)} = \begin{bmatrix} \zeta_{13} & 0 & \zeta_{14} & 0 \\ 0 & \zeta_{13} & 0 & \zeta_{14} \\ \zeta_{23} & 0 & \zeta_{24} & 0 \\ 0 & \zeta_{23} & 0 & \zeta_{24} \end{bmatrix}$$

• For interval I_1 , we have

$$\begin{aligned} \mathbf{Q}^{(1)} &= \begin{bmatrix} -\sum \dots & \xi & (1) \\ \lambda & (1) & \beta & -\sum \dots \end{bmatrix} \\ \tilde{\mathbf{Q}}^{(1)} &= \begin{bmatrix} \zeta_{12} & 0 & \zeta_{13} & 0 & \zeta_{14} & 0 \\ 0 & \zeta_{12} & 0 & \zeta_{13} & 0 & \zeta_{14} \end{bmatrix} \end{aligned}$$

• The placement matrix when the firm places new debt to H types only is given by

$$\mathbf{S}^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\mathbf{S}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\mathbf{S}^{(1)} = [1,0]$$

• The discount rate matrix for creditors is $\mathbf{R}^{(4)} = \mathbf{R}$ with

	$r_H(1) + m$	0	0	0	0	0	0	ך 0
	0	$r_L\left(1 ight) + m$	0	0	0	0	0	0
	0	0	$r_H(2) + m$	0	0	0	0	0
P _	0	0	0	$r_L\left(2\right) + m$	0	0	0	0
n =	0	0	0	0	$r_H\left(3\right) + m$	0	0	0
	0	0	0	0	0	$r_L(3) + m$	0	0
	0	0	0	0	0	0	$r_H\left(4\right) + m$	0
	0	0	0	0	0	0	0	$r_L(4) + m$

• The discount rate matrix for equity holders is

$$\mathbf{RR}^{(4)} = \mathbf{RR} = \begin{bmatrix} r_H(1) + m & 0 & 0 & 0 \\ 0 & r_H(2) + m & 0 & 0 \\ 0 & 0 & r_H(3) + m & 0 \\ 0 & 0 & 0 & r_H(4) + m \end{bmatrix}$$

2.2 Programming notes

Programming should take advantage of build in matrix functionality – all functions should be defined as matrices/vectors where possible. The eigenvalue-eigenvector decomposition is a standard operation in both Mathematica and MATLAB. There are a few issues that warrant further comments: (1) Difficulties arise from the matrix **KK** which has to be defined row-by-row, and thus it cannot be as efficiently implemented. (2) It is important to use a matrix exponential function instead of a simple exponential functions. (3) The boundary conditions (44) and (62) can best be implemented by only defining a vector/matrix $\gamma^{(n)}/\Gamma^{(n)}$ that contains negative entries and a vector/matrix $\gamma\gamma^{(n)}/\Gamma\Gamma^{(n)}$ that only contains entries equal to or less than 1. A simple sort functions should do the job. (4) The eigenvectors that come out of the eigenvector/eigenvalue decompositions (59) and (59) should be cut in half in that the lower half of rows should be discarded.

Next, both debt and equity should be solved for an arbitrary vector δ_B , so that $\mathbf{c}^{(i)}$ and $\mathbf{cc}^{(i)}$ are functions of the vector δ_B . The solutions to $\mathbf{c}^{(i)}$ and $\mathbf{cc}^{(i)}$ will come out of a linear system of equations that can be rapidly solved (but of course, they both are highly nonlinear functions of δ_B). Then, we can numerically optimize equity. We can do this in two ways: (1) we simply impose the smooth pasting condition (16) or (2) we pick a value $\delta \in I_i$ and optimize equity in any one (alive) state $s \leq i$ over the vector δ_B , i.e. $\max_{\delta_B} E_s^{(i)}(\delta; \delta_B)$ of course with an ordering restriction on δ_B so that $\delta_B(1) \leq \delta_B(2) \leq \ldots \leq \delta_B(n)$. Approach (2) can be significantly faster as MATLAB and Mathematica have pretty good optimization routines, and approach (1) can lead to a highly nonlinear system of equations that might be inaccurate to solve.