High Frequency Quoting:
Short-Term Volatility in Bids and Offers

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Abstract

High-frequency changes, reversals, and oscillations can lead to volatility in a market's bid and offer quotes. This volatility degrades the informational content of the quotes, exacerbates execution price risk for marketable orders, and impairs the reliability of the quotes as reference marks for the pricing of dark trades. This paper examines variance on time scales as short as fifty milliseconds for the National Best Bid and Offer (NBBO) in the US equity market. On average, in a 2011 sample, NBBO variance at the fifty millisecond time scale is approximately four times larger than can be attributed to long-term fundamental price variance. The historical picture is complex. There is no marked upward trend in short-term quote volatility over 2001-2011. Its character, though, has changed. In the early years (and especially prior to Reg NMS) quote volatility is driven by large spikes in bids and offers. In later years it is more a consequence of high-frequency oscillations comparable to the bid-offer spread in magnitude.

I. Introduction

Recent developments in market technology have called attention to the practice of high-frequency trading. The term is used commonly and broadly in reference to all sorts of fast-paced market activity, not just "trades", but trades have certainly received the most attention. There are good reasons for this, as trades signify the actual transfers of income streams and risk. Quotes also play a significant role in trading process, however. This paper accordingly examines short-term volatility in bids and offers of US equities, a consequence of what might be called high frequency quoting.

By way of illustration, Figure 1 depicts the bid and offer for AEP Industries (a NASDAQ-listed manufacturer of packaging products) on April 29, 2011.[1] In terms of broad price moves, the day is not a particularly volatile one, and the bid and offer quotes are stable for long intervals. The placidity is broken, though, by several intervals where the bid undergoes extremely rapid changes. The average price levels, before, during and after the episodes are not dramatically different. Moreover, the episodes are largely one-sided: the bid volatility is associated with an only moderately elevated volatility in the offer quote. Nor is the volatility associated with increased executions. These considerations suggest that the volatility is unrelated to fundamental public or private information. It appears to be an artifact of the trading process.

It is not, however, an innocuous artifact. Bids and offers in all markets represent price signals, and, to the extent that they are firm and accessible, immediate trading opportunities. From this perspective, the noise added by quote volatility impairs the informational value of the public price. Most agents furthermore experience latency in ascertaining the location of the bid and offer price and in timing of their order delivery. Elevated short-term volatility increases the execution price risk associated with these delays. In US equity markets the bid and offer are particularly important, because they are used as

[1] The bid is the National Best Bid (NBB), the maximum bid across all exchanges. The offer is the National Best Offer (NBO), the minimum offer. They are often jointly referred to as the NBBO. Unless otherwise noted, or where clarity requires a distinction, "bid" and "offer" indicate the NBBO.

benchmarks to assign prices in so-called dark trades, a category that includes roughly thirty percent of all volume.[2]

In the context of the paper's data sample, the AEPI episode does not represent typical behavior. Nor, however, is it a singular event. It therefore serves to motivate the paper's key questions. What is the extent of short-term volatility? How can we distinguish fundamental (informational) and transient (microstructure) volatility? Finally, given the current public policy debate surrounding low-latency activity, how has it changed over time?

These questions are addressed empirically in a broad sample of US equity market data using summary statistics that are essentially short-term variances of bids and offers. Such constructions, though, inevitably raise the question of what horizon constitutes the "short term" (a millisecond? a minute?). The answer obviously depends on the nature of the trader's market participation, as a collocated algorithm at one extreme, for example, or as a remotely situated human trader at the other. The indeterminacy motivates empirical approaches that accommodate flexible time horizons. This analysis uses time scale variance decompositions to measure bid and offer volatility over horizons ranging from under 50 ms to about 27 minutes.

The next section establishes the economic and institutional motivation for the consideration of local bid and offer variances with sliding time scales. Section III discusses the statistical framework. The paper then turns to applications. Section IV presents an analysis of a recent sample of US equity data featuring millisecond time stamps. To extend the analysis to historical samples in which time stamps are to the second, Section V describes estimation in a Bayesian framework where millisecond time stamps are simulated. Section VI applies this approach to a historical sample of US data from 2001 to 2011. Connections to high frequency trading and volatility modeling are discussed in Section VII. A summary concludes the paper in Section VIII.

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[2] Dark mechanisms do not publish visible bids and offers. They establish buyer-seller matches, either customer-to-customer (as in a crossing network) or dealer-to-customer (as in the case of an internalizing broker-dealer). The matches are priced by reference to the NBBO: generally at the NBBO midpoint in a crossing network, or at the NBB or the NBO in a dealer-to-customer trade.

II. Economic effects of quote volatility.

High frequency quote volatility may be provisionally defined as the short-term variance of the bid or offer, the usual variance calculation applied to the bid or offer *level* over a relatively brief window of time. This section is devoted to establishing the economic relevance of such a variance in a trading context. The case is a simple one, based on the function and uses of the bid and offer, the barriers to their instantaneous availability, the role of the time-weighted price mean as a benchmark, and the interpretation of the variance about this mean as a measure of risk.

In current thinking about markets, most timing imperfections are either first-mover advantages arising from market structure or delays attributed to costly monitoring. The former are exemplified by the dealer's option on incoming orders described in Parlour and Seppi (2003), and currently figure in some characterizations of high-frequency traders (Biais, Foucault and Moinas (2012); Jarrow and Protter (2011)). The latter are noted by Parlour and Seppi (2008) and discussed by Duffie (2010) as an important special case of inattention which, albeit rational and optimal, leads to infrequent trading, limited participation, and transient price effects (also, Pagnotta (2009)).

As a group these models feature a wide range of effects bearing on agents' arrivals and their information asymmetries. An agent's market presence may be driven by monitoring decisions, periodic participation, or random arrival intensity. Asymmetries mostly relate to fundamental (cash-flow) information or lagged information from other markets. Agents in these models generally possess, however, timely and extensive market information. Once she "arrives" in a given market, an agent accurately observes the state of that market, generally including the best bid and offer, depth of the book and so on. Moreover, when she contemplates an action that changes the state of the book (such as submitting, revising or canceling an order), she knows that her action will occur before any others'.

In reality, of course, random latencies in receiving information and transmitting intentions combine to frustrate these certainties about the market and the effects of her orders. The perspective of this paper is that for some agents these random latencies generate randomness in the execution prices, and that short-term quote variances can meaningfully measure this risk. Furthermore, although all agents incur random latency, the distributions of these delays vary among participants. An agent's latency distribution can be summarized by time scale, and this in turn motivates time scale decompositions of bid and offer variances.

While random latencies might well affect strategies of all traders, the situation is clearest for someone who intends to submit a marketable order (one that seeks immediate execution) or an order to a dark pool. In either case, ignoring hidden orders, an execution will occur at the bid, the offer or at an average of the two. A trader whose order arrival time is uniformly distributed on a given interval faces price risk over that interval. For a marketable sell order, the variance of the bid over the interval quantifies the price risk, relative to a benchmark equal to the average bid.

The use of an average price in the presence of execution timing uncertainty is a common principle in transaction cost analysis. Perold's implementation shortfall measure is usually operationally defined for a buy order as the execution price (or prices) less some hypothetical benchmark price (and for a sell order as the benchmark less the execution price, Perold (1988)). As a benchmark price, Perold suggests the bid-offer midpoint prevailing at the time of the decision to trade. Many theoretical analyses of optimal trading strategies use this or a similar pre-trade benchmark. Practitioners, however, and many empirical analyses rely on prices averaged over some comparison period. The most common choice is the value-weighted average price (VWAP), although the time-weighted average price (TWAP) is also used. One industry compiler of comparative transaction cost data notes, "In many cases the trade data which is available for analysis does not contain time stamps. …. When time stamps are not available, pension funds and investment managers compare their execution to the volume weighted average price of the stock on the day of the trade" (Elkins-McSherry (2012)). This quote attests to the importance of execution time uncertainty, although a day is certainly too long to capture volatility on the scale of transmission and processing delays. Average prices are also used as objectives by certain execution strategies. A substantial portion of the orders analyzed by Engle, Ferstenberg and Russell (2012) target VWAP, for example.

The situations discussed to this point involve a single trader and single market. In a fragmented market, the number of relevant latencies may be substantially larger. In the US there are presently about 17 "lit" market centers, which publish quotes. A given lit market's quotes are referenced by the other lit markets, dark pools (currently around 30 in number), by executing broker-dealers (approximately 200), and by data consolidators (U.S. Securities and Exchange Commission (2010)). The National Best Bid and Offer (NBBO) is in principle well-defined. The NBBO perceived by any given market center, consolidator or other agent, however, comprises information subject to random transmission delays that differ across markets and receiving agents. These delays introduce noise into the determination. Local time-averaging (smoothing) can help to mitigate the effects of this noise, while the local variance indicates the magnitude of the noise.

If the execution price risk associated with quote volatility is zero-mean and diversifiable across trades, it might appear to be economically trivial. In general, however, agents do not have symmetric exposure to this risk. Market-order traders with faster technology possess a systematic advantage relative to those with slower technology. This can be viewed as an information asymmetry that leads (in the usual fashion) to a transfer of wealth from the slower to the faster participants.

Asymmetric exposure to quote volatility is also likely to place customers at a disadvantage relative to their dealers. The recent SEC concept release notes that virtually all retail orders are routed to OTC market-makers, who execute the orders by matching the prevailing NBBO (U.S. Securities and Exchange Commission (2010)). Stoll and Schenzler (2006) note that these market-makers have flexibility in delaying executions to obtain favorable reference prices. They describe this as a look-back option, and find support for this behavior in a 1999 sample. Dark trading venues also face this sort of problem. A customer sending a sell order to a dark pool or crossing network can submit a *buy* order to a lit market center that will briefly boost quote midpoint, thereby achieving a better price if he receives a dark execution of his sell order. This practice (a form of "spoofing") is forbidden in the Dodd-Frank framework, but difficult to detect and prove in the presence of timing uncertainties.

The SEC's Reg NMS ruling on trade-through protection recognized the problem of "flickering quotes", and mandated a one-second grace period: "… pursuant to Rule 611(b)(8) trading centers would be entitled to trade at any price equal to or better than the least aggressive best bid or best offer, as applicable, displayed by the other trading center during that one-second window." Sub-second intervals were considered, but the benefits were not believed sufficient to justify the costs (U.S. Securities and Exchange Commission (2005)). Clearly, quote volatility within the one-second window weakens the trade-through protection.[3]

[3] The SEC has recently mandated a consolidated audit trail intended to track all events in an order's life cycle (such as receipt, routing, execution and cancellation) (U.S. Securities and Exchange Commission (2012)). In the final rule, the Commission recognized the importance of accurately sequencing the events, and mandated time-stamps at least to the granularity of the millisecond, and, "to the extent that the order

III. Time scale variance decompositions

Consider a price process, p_t, in discrete time. The n-period mean ending at time t is $S(n,t) = n^{-1}\sum_{i=0}^{n-1} p_{t-i}$. The deviations about this mean are $R(n,t,s) = p_s - S(n,t)$ for $t-n+1 < s \le t$. The mean square of this deviation is $MSD(n,t) = n^{-1}\sum_{s=t-n+1}^{t}\left[R(n,t,s)\right]^2$. These are simply computational definitions, summarizing calculations that might be performed on any segment of the price path. To interpret these quantities in a probabilistic framework, we now assume that the first differences of the price process constitute a stationary (but not necessarily uncorrelated) stochastic process. With this assumption, the expectation of $MSD(n,t)$ is finite, time-invariant and equal to the variance of R: $E\left[MSD(n,t)\right] = Var\left[R(n,t,s)\right] \equiv \sigma_n^2$.

Given her technology and a contemplated order, an individual trader might well focus on the particular horizon associated with her order arrival uncertainty period. In the present analysis, though, it more useful to consider a broad range of averaging periods that spans horizons of general interest. Any increasing sequence of n might be chosen (such as, 1 sec, 5 sec, 20 sec). For various mathematical reasons, though, it is convenient to let n be dyadic (increasing by powers of two): $n_j = n_0 2^j$ for $j = 0,1,\ldots$, where $n_0 \ge 1$ defines the starting resolution for the analysis. The corresponding sequence of R variances $\sigma_{n_j}^2$ is denoted σ_j^2 for $j = 0,1,\ldots$ It will also be useful to define $\nu_j^2 = \sigma_j^2 - \sigma_{j-1}^2$ as the incremental variance change associated with moving from averaging interval n_{j-1} to n_j .

The random-walk special case is useful both as an illustration of the calculations and also as benchmark in interpreting the empirical results. Suppose that the price evolves in continuous time with variance per unit time σ_u^2 (not necessarily a Gaussian diffusion) and that the prices are initially averaged over successive intervals of length M_0 units of time. It is shown in the appendix that $\sigma_j^2 = 2^{j-1} M_0 \sigma_u^2 / 3$ and $\nu_j^2 = 2^{j-2} M_0 \sigma_u^2 / 3$.

The characterization of a time series according to time scale was historically based on Fourier (frequency domain) analysis, in which a series is decomposed as a sum of sine and cosine basis functions. Modern approaches use broader classes of basis functions, called wavelets. The essential distinction is

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handling and execution systems of any SRO or broker-dealer utilize time stamps in increments finer than the minimum required by the NMS plan time stamps, such SRO or member must use time stamps in such finer increments when reporting data to the central repository."

that while trigonometric functions cycle repeatedly over the full sample, wavelets are localized, and are therefore better suited to picking up phenomena like the AEPI movements that are very concentrated in time.

Percival and Walden (2000) is a comprehensive textbook discussion of wavelets that emphasizes connections to conventional time series analysis. The present notation mostly follows their conventions. *S* and *R* denote smoothed and rough components (or simply "smooths" and "roughs"). The dyadic convention facilitates the use of standard computationally efficient methods (wavelet transforms) to compute the σ_j^2 and ν_j^2 estimates. The quantity ν_j^2 is formally considered a wavelet variance. It is important to emphasize, though, that it can be defined (as above) and even computed (albeit suboptimally) without wavelet transforms. Denoting it as the wavelet variance simply places it in an extensive and well-developed literature.[4]

The rough variance σ_j^2 reflects variation persistent over all time scales shorter than the averaging period $n_j = n_0 2^j$. The wavelet variance ν_j^2, though, defined as an incremental quantity, and reflects variation only at time scale $\tau_j = n_0 2^{j-1}$. The market-order trader exposed to timing risk at a particular time scale would generally also be exposed to risk at all shorter time scales. The usefulness of the wavelet

[4] Wavelet transformations, also known as time scale or multi-resolution decompositions are widely used across many fields. Gençay, Selçuk and Whitcher (2002) discuss economic and financial applications in the broader context of filtering. Nason (2008) discusses time series and other applications of wavelets in statistics. Ramsey (1999) and Ramsey (2002) provides other useful economic and financial perspectives. Walker (2008) is clear and concise, but oriented more toward engineering applications.

Studies that apply wavelet transforms to the economic analysis of stock prices loosely fall into two groups. The first set explores time scale aspects of stock comovements. A stock's beta is a summary statistic that reflects short-term linkages (like index membership or trading-clientele effects) and long-term linkages (like earnings or national prosperity). Wavelet analyses can characterize the strength and direction of these horizon-related effects (for example, Gençay, Selçuk and Whitcher (2005); In and Kim (2006)). Most of these studies use wavelet transforms of stock prices at daily or longer horizons. A second group of studies uses wavelet methods to characterize volatility persistence (Dacorogna, Gencay, Muller, Olsen and Pictet (2001); Elder and Jin (2007); Gençay, Selçuk, Gradojevic and Whitcher (2010); Gençay, Selçuk and Whitcher (2002); Høg and Lunde (2003); Teyssière and Abry (2007)). These studies generally involve absolute or squared returns at minute or longer horizons. Wavelet methods have also proven useful for jump detection and jump volatility modeling Fan and Wang (2007). Beyond studies where the focus is primarily economic or econometric lie many more analyses where wavelet transforms are employed for ad hoc stock price forecasting (Atsalakis and Valavanis (2009); Hsieh, Hsiao and Yeh (2011), for example). An early draft of Hasbrouck and Saar (2011) used wavelet analyses of message count data to locate periods of intense message traffic on NASDAQ's Inet system.

variance, which reflects variation over a single time scale, may therefore not be readily apparent. Sometimes there may be economic or technological reasons why an effect can't occur on a shorter time scale. Intermarket feedback effects, for example, can't operate over scales shorter than the intermarket transmission time. Alternatively, if the short-term components are regarded as mostly noise, then a variance based solely on longer-term components may be viewed as a more reliable indication of fundamental volatility.

Fan and Gencay (2010) apply this principle to unit root tests based on time scale decompositions. Gencay and Signori (2012) explore the use of variance ratios at different time scales to test for serial correlation. In the present application, multi-scale variance ratios can be used to assess the excess high frequency volatility relative to what would be implied by a random-walk calibrated to a low frequency. Consider the variance ratio

$$V_{j,J} = 2^{J-j} \nu_j^2 \big/ \nu_J^2$$

Here, J denotes the largest index (longest time scale) considered in the study, j $\left(0 \le j \le J\right)$ generally denotes a shorter time scale, and 2^{J-j} is a scaling factor. If the process follows a random walk with uncorrelated increments, $V_{j,J} = 1$. To the extent that $V_{j,J}$ exceeds unity, there is excess short-term volatility. This variance ratio is defined in terms of the wavelet variances. A similar normalization can be defined for the rough variances as

$$VR_{j,J} = 2^{J-j-1} \sigma_j^2 \big/ \nu_J^2$$

Note that while the variance in the numerator is a rough variance, the denominator is a wavelet variance. This term reflects variation only at the longest time scale, and is in principle stripped of all short-term components.

There is a long tradition of variance ratios in empirical market microstructure (Amihud and Mendelson (1987); Barnea (1974); Hasbrouck and Schwartz (1988)).[5] Microstructure effects are

[5] Return variance ratios are also used more broadly in economics and finance to characterize deviations from random-walk behavior over longer horizons (Charles and Darné (2009); Faust (1992); Lo and MacKinlay (1989)).

generally thought to induce transitory mispricing, which generally inflates short-term variances relative to long-term variances. Ratios constructed from wavelet variances give a more precise and nuanced characterization.

The wavelet covariance between two processes is defined analogously to the wavelet variance. Of particular importance is the covariance between the bid and offer, denoted $\nu^2_{bid,offer,j}$. The wavelet correlation, denoted $\rho_j = \nu^2_{bid,offer,j} \Big/ \sqrt{\nu^2_{bid,j}\nu^2_{offer,j}}$, is used to assess the extent to which the bid and offer co-move at different time scales.

Percival and Walden characterize the asymptotic distributions of wavelet variance estimates. By most standards, the number of observations in the present application is more than sufficient to rely on asymptotic results. (With a 50 ms observation interval, a six-hour trading day contains 432,000 observations.) The data exhibit, however, bursts of activity, long periods with no changes ("too many zeroes," as some have noted), and other features that suggest convergence to the asymptotic results might be very slow. Accordingly, the results reported here are based on cross-firm means and standard errors of these means.

IV. A cross-sectional analysis

From a trading perspective, stocks differ most significantly in their general level of activity (volume measured by number of trades, shares or values). The first analysis aims to measure the general level of high frequency quote volatility and to relate the measures to trading activity in the cross-section for a recent sample of firms.

IV.A. Data and sample construction.

The analyses are performed for a subsample of US firms using trading data from April, 2011 (the first month of my institution's subscription.) The subsample is constructed from all firms present on the CRSP and TAQ databases from January through April of 2011 with share codes of 10 or 11, with closing prices between two and one thousand dollars, and with a primary listing on the New York, Amex or

NASDAQ exchanges.[6] I compute the daily average dollar volume based on trading in January through March, and randomly select 15 firms from each decile. For brevity, reported results are grouped into quintiles.

The U.S. equity market is highly fragmented, but all exchanges post their quotes to the Consolidated Quote System (CQS).[7] The CQ and NBBO files from the NYSE's daily TAQ dataset used here are definitive transcripts of the consolidated activity, time-stamped to the millisecond.[8] A record in the consolidated quote (CQ) file contains the latest bid and offer originating at a particular exchange. If the bid and offer establish the NBBO this fact is noted on the record. If the CQ record causes the NBBO to change for some other reason, a message is posted to another file (the NBBO file). Thus, the NBBO can be obtained by merging the CQ and NBBO files. It can also be constructed (with a somewhat more involved computation) directly from the CQ file. Spot checks verified that these two approaches were consistent.

Studies using TAQ data have traditionally used error filters to throw out quotes that appear spurious. Recent daily TAQ data, though appear to be much cleaner than older samples. In particular, the NBBO construction provided by the NYSE clearly defines what market participants would have perceived. Some quotes present in the CQ file are not incorporated into the NBBO because they are not firm, indicative or otherwise deemed "not NBBO-eligible". Beyond these exclusions, however, I impose no additional filters for the estimates discussed in this section. Error filters are used, however, in the subsequent historical analysis, and will be discussed in greater detail at that point.

Table 1 reports summary statistics. Post-Reg NMS US exchanges have become more similar in structures and trading mechanisms. With respect to listing characteristics, though, differences persist. The

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[6] The American Stock Exchange merged with NYSE Euronext in 2008, and was renamed NYSE Amex LLC. In May, 2012, the name was changed to NYSE MKT LLC. It will be identified in this paper as "Amex".

[7] At the same time that an exchange sends a quote update to the consolidated system, it can also transmit the update on its own subscriber line. For subscribers this can reduce the delay associated with consolidation and retransmission (which is on the order of about five milliseconds). Thus, while the CQS is a widely-used single-source of market data, it is not the fastest. Moreover, bids and offers with sizes under 100 shares are not reported.

[8] The "daily" reference in the Daily TAQ dataset refers to the release frequency. Each morning the NYSE posts files that cover the previous day's trading. The Monthly TAQ dataset, more commonly used by academics is released with a monthly frequency and contains time stamps in seconds.

NYSE "classic" has the largest proportion of high-volume stocks, NYSE Amex has the smallest, and NASDAQ falls in the middle.

Market event counts (trades, quotes, and so forth) display some interesting patterns. There are large numbers of quote records, since one is generated when any market center changes its best bid, best offer, or size at the bid or offer. If the action establishes the bid and offer as the NBBO this fact is noted on the quote record. But if the action causes some other change in the aggregate prices or sizes at the NBBO, an NBBO record is generated. Since many quote records don't induce such a change, there are substantially fewer NBBO records. Finally, many actions might change one of sizes or one side of the quote. Thus, the numbers of NBB and NBO changes are smaller yet.

Volatility and spreads tend to be elevated at the start and end of trading sessions (9:30 to 16:00). To remove the effect of these deterministic effects, I confine the variance estimates to the 9:45 to 15:45 subperiod. The estimates are computed using the maximal overlap Haar transform.[9] I assume no overlap across days, and discard boundary values affected by wrap-around. Estimates are computed separately for the bid and offer, and then averaged for convenience in presentation. Reported means are generally computed across-firms, and the standard errors of these means are constructed in the usual fashion, assuming independence across observations. Due to volatility commonalities, this is likely to bias the standard errors downwards. Market commonalities of all sorts weaken at shorter horizons, however, and this is likely to be especially true of the extremely brief intervals considered here.

To facilitate economic interpretation, the time scale variances are reported in several alternative ways. I report wavelet and rough variances in three ways: mils ($0.001) per share, basis points relative to average price, and as a short/long-term variance ratio. The mils per share scaling is useful because many trading fees (such as commissions and clearing fees) are assessed on a per share basis. Access fees, the charges levied by exchanges on taker (aggressor) sides of executions are also assessed per share. US SEC Regulation NMS caps access fees at 3 mils ($0.003) per share, and in practice most exchanges are close to this level. Practitioners regard access fees as significant to the determination of order routing decisions,

[9] The computations were performed in Matlab using the WMTSA package (Cornish (2006)). These routines conform closely to Percival and Walden. Although Matlab has its own wavelet toolbox, the data structures and other conventions differ significantly from those of Percival and Walden. I also found the Mathematica wavelet functions to be consistent with Percival and Walden.

and this magnitude therefore serves an approximate threshold of economic importance. Basis point scaling is meaningful because most analyses involving investment returns or comparison across firms assume that share normalizations are arbitrary. Variance ratios provide a summary measure of short-term variance inflation relative to what would be expected from a random-walk calibrated to long-term variance.

IV.B. Results

Table 2 summarizes the averages for all time scales of wavelet and rough variances under all three normalizations. As an illustrative calculation, a trader facing arrival time uncertainty of 50 milliseconds is exposed to a price risk standard deviation of $\sqrt{0.32} \approx 0.57$ mils per share (from column (1)), or $\sqrt{0.11} \approx 0.33\text{bp}$ (from column (2)). The entry in column (3), 3.99, implies that the price risk is roughly four times what would be consistent with a random-walk calibrated to longest time scale in the analysis (27.3 minutes). At 200 ms, the risk crosses the one mil threshold (1.17, column (2)). At 800 ms, it is on the order of one basis point. The variance ratios (columns (3) and (6)) increase monotonically in moving to shorter time scales.

Column (7) of Table 2 reports the wavelet correlations between bids and offers. If the bid and offer always moved in lock step, this correlation would be unity at every time scale. At longer time scales this correlation is indeed quite high, but at shorter time scales it is only moderately positive.

Table 3 reports results for a subset of the measures and time scales, but provides more detail across dollar volume subsamples, and also includes standard errors. Panels A and B report estimates of rough variances in mils per share and basis points, respectively. Stocks in the two lowest dollar volume quintiles have sharply higher short-term volatility. In comparing the two normalizations, it is apparent that variance in mils per share (Panel A) at the shorter scales is more stable across dollar volume quintiles than variance in basis points (Panel B). The latter decline by a factor of about twenty in moving from the lowest to highest quintile. This decline appears, therefore, to be explained mostly by the increase in share prices across the quintiles. Put another way, it appears that quote volatility is best characterized as a "mils per share" phenomenom, perhaps due to the tick size effects or the use of per-share cost schedules in assessing trading fees.

Table 3 Panel C reports selected variance ratios across dollar volume quintiles. Figure 2 graphs the fill set. For the highest volume quintile, the excess variance seems to be about 30% at the shortest time scales. For the lowest volume quintile, however, the excess is, at ten or above, substantially higher. The wavelet bid-offer correlations are reported in Table 3 Panel D, and graphed in Figure 3. These also exhibit marked variation across dollar volume. For the highest quintile, they are close to unity at a time scale of 25.6 seconds; for the lowest, the correlation at 27.4 minutes is a modest 0.51. This suggests a pronounced de-coupling of the bid and offer.

Hansen and Lunde note that to the extent that volatility is fundamental, we would expect bid and offer variation to be perfectly correlated, that is, that a public information revelation would shift both prices by the same amount (Hansen and Lunde (2006)). Against this presumption, the short-term correlation estimates are striking. At time scales of 200 ms or lower, the correlation is below 0.7 for all activity quintiles. For the shortest time scales and lower activity quintiles, the correlation is only slightly positive. This suggests that substantial high-frequency quote volatility is of a distinctly transient nature.

V. Truncated time stamps.

The analysis in the preceding section relies on a recent one-month sample of daily TAQ data. For addressing policy issues related to low-latency activity, it would be useful to conduct a historical analysis, spanning the period over which low-latency technology was deployed. Extending the analysis backwards, however, is not straightforward. Millisecond time-stamps are only available in the daily TAQ data from 2006 onwards. Monthly TAQ data (the standard source used in academic research) is available back to 1993 (and the precursor ISSM data go back to the mid-1980s). These data are substantially less expensive than the daily TAQ, and they have a simpler logical structure.

The time stamps on the Monthly TAQ and ISSM datasets are reported only to the second. At first glance this might seem to render these data useless for characterizing sub-second variation. This is unduly pessimistic. It is the purpose of this section to propose, implement and validate an approach for estimating sub-second characteristics of the bid and offer series using the second-stamped data. This is possible because the data generation and reporting process is richer than it initially seems.

Specifically, the usual sampling situation in discrete time series analysis involves either aggregation over periodic intervals (such as quarterly GDP) or point-in-time periodic sampling (such as the end-of-day S&P index). In both cases there is one observation per interval, and in neither case do the data support resolution of components shorter than one interval. In the present situation, however, quote updates occur in continuous time and are disseminated continuously. The one second time-stamps arise as a truncation (or equivalently, a rounding) of the continuous event times. The Monthly TAQ data include all quote records, and it is not uncommon for a second to contain ten or even a hundred quote records.

Assume that quote updates arrive as a Poisson process of constant intensity. If the interval $(0, t)$ contains n updates, then the update times have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$ (Ross (1996), Theorem 2.3.1). Within a one-second interval containing n updates, therefore, we can simulate continuous arrival times by drawing n realizations from the standard uniform distribution, sorting, and assigning them to quotes (in order) as the fractional portions of the arrival times. These simulated time-stamps are essentially random draws from true distribution. This result does not require knowledge of the underlying Poisson arrival intensity.

We make the additional assumption that the quote update times are independent of the updated bid and offer prices. (That is, the "marks" associated with the arrival times are independent of the times.) Then all estimates based on the simulated time stamp series constitute draws from their corresponding posterior distributions. This procedure can be formalized in a Bayesian Markov-Chain Monte Carlo (MCMC) framework. To refine the estimates, we would normally make repeated simulations ("sweeps") over the sample, but due to computational considerations and programming complexity, I make only one draw for each CQ record.

It is readily granted that few of the assumptions underlying this model are completely satisfied in practice. For a time-homogeneous Poisson process, inter-event durations are independent. In fact, inter-event times in market data frequently exhibit pronounced serial dependence, and this feature is a staple of the autoregressive conditional duration and stochastic duration literature (Engle and Russell (1998); Hautsch (2004)). In NASDAQ Inet data, Hasbrouck and Saar (2011) show that event times exhibit intra-second deterministic patterns. Suboordinated stochastic process models of security prices suggest that transactions (not wall-clock time) are effectively the "clock" of the process (Shephard (2005)).

We can assess the reliability of the randomization approach, however, by a simple test. The time-stamps of the data analyzed in the last section are stripped of their millisecond remainders. New millisecond remainders are simulated, the random-time-stamped data are analyzed, and we examine the correlations between the two sets (original and randomized) of estimates. Let $\nu_{j,i,d}^{2}$ denote the bid variance estimate for firm *i* on day *d* at level *j* based on the original time stamps, and let $\nu_{j,i,d}^{2}$ denote the estimate based on the simulated time stamps. (Results for offer variances are similar.) Table 4, Panel A reports estimates across firms and days of $Corr\left(\nu_{j,i,d}^{2}, \nu_{j,i,d}^{2}\right)$.The agreement between original and randomized estimates is very high for all time scales and in all subsamples. Even at the time scale of less than fifty ms, the mean correlation is 0.952. At time scales above one second, the agreement is nearly perfect.

Given the questionable validity of some of the assumptions, and the fact that only one draw is made for each second's activity, this agreement might seem surprising. It becomes more reasonable, however, when one considers the extent of averaging underlying the construction of both original and randomized estimates. There is explicit averaging in that each wavelet variance estimate formed over a sample of roughly 120 hours. As long as the order is maintained, a small shift in a data point has little impact over the overall estimate.[10]

Agreement between original and randomized bid-offer covariances is slightly weaker. The correlation of under-50 ms components is 0.775 (in the full sample), this climbs to 0.979 at a time scale of 200 ms. The reason for the relatively poorer performance of the randomized covariance estimates is simply that the wavelet covariance between two series is sensitive to alignment. For a given CQ record, the bid and offer quotes are paired, but in a typical record sequence the NBB and NBO are not changed in the same record. When a bid change is shifted even by a small amount relative to the offer, the inferred pattern of co-movement is distorted.

Across dollar volume quintiles, the correlations generally improve for all time scales. This is true for both wavelet variances and covariances, but is more evident in the latter. This is a likely consequence of the greater incidence, in the higher quintiles, of multiple quote records within the same second. Specifically, for a set of *n* draws from the uniform distribution, the distribution of any order statistic tightens as *n* increases. (For example, the distribution of the first order statistic in a sample of five

[10] Also, inherent in the wavelet transformation is an (undesirable) averaging across time scales known as leakage, wherein the variance at one time scale affects to a small degree the estimate at neighboring time scale (Percival and Walden, p. 303).

hundred in a given second is tighter than the distribution of the first order statistic in a sample of one.) Essentially, an event time can be located more precisely within the second if the second contains more events. This observation will have bearing on the analysis of historical samples with varying numbers of events.

In working with Monthly TAQ data, Holden and Jacobsen (2012, HJ) suggest assigning sub-second time stamps by evenly-spaced interpolation. If there is one quote record in the second, it is assigned a millisecond remainder of 0.500 seconds; if two records, 0.333 and 0.667 seconds, and so on. HJ show that interpolation yields good estimates of effective spreads. It is not, however, equivalent to the present approach. Consider a sample in which each one-second interval contains one quote record. Even spacing places each quote at its half-second point. As a result, the separation between each quote is one second. For example, a sequence of second time stamps such as 10:00:01, 10:00:02, 10:00:03 … maps to 10:00:01.500, 10:00:02.500, 10:00:03.500, and so on. The interpolated time stamps are still separated by one second, and therefore the sample has no information regarding sub-second components. In contrast, a randomized procedure would sweep the space of all possibilities, including 10:00:01.999, 10:00:02.000, …, which provides for attribution of one-millisecond components. Of course, as the number of events in a given one-second interval increases, the two approaches converge: the distribution of the *kth* order statistic in a sample of *n* uniform observations collapses around its expectation, $k/(n+1)$ as *n* increases.[11]

[11] For one class of time-weighted statistics in this setting, interpolated time stamps lead to unbiased estimates. Consider a unit interval where the initial price, p_0, is known, and there are *n* subsequent price updates $p_i, i=1,\ldots,n$ at occurring at times $0<t_1<\cdots<t_n<1$. The time-weighted average of any price function $f(p)$ is $Avg^{TW}=\sum_{i=0}^{n} f(p_i)(t_{i+1}-t_i)$, where $t_0\equiv 0$ and $t_{n+1}\equiv 1$. Assuming a time-homogeneous Poisson arrival process, the t_i are distributed (as above) as uniform order statistics. This implies $Et_i=i/(n+1)$, the linear interpolated values. If the marks (the p_i) are distributed independently of the t_i, $E\left[Avg^{TW}\right]=(n+1)^{-1}\sum_{i=0}^{n} f(p_i)$. This result applies to time-weighted means of prices and spreads (assuming simultaneous updates of bids and offers). It also applies to wavelet transforms and other linear convolutions. It does not apply to variances (or wavelet variances), however, which are nonlinear functions of arrival times.

VI. Historical evidence

This section describes the construction and analysis of variance estimates for a sample of US stocks from 2001 to 2011. In each year, I construct variance estimates for a single representative month (April) for a subsample of firms.

The period covers significant changes in market structure and technology. Decimalization had been mandated, but was not completely implemented by April, 2001. Reg NMS was proposed, adopted, and implemented.[12] Dark trading grew over the period. Market information and access systems were improved, and latency emerged as a key concern of participants. The period also includes many events related to the financial crisis, which are relatively exogenous to equity market structure.

The regulatory and technological shifts over the period caused changes in the fundamental nature of bid and offer quotations. Markets in 2001 were still dominated by what would later be called "slow" procedures. Quotes were often set manually. Opportunities for automated execution against these quotes were few (cf. the NYSE's odd-lot system, and NASDAQ's Small Order Execution System). Trade-through protection was limited and weakly enforced. Quotes for 100 shares or less were not protected. With the advent of Reg NMS, the bids and offers became much more accessible (for automated execution). These considerations are important in interpreting the results that follow.

VI.A. Data

The data for this phase of the analysis are drawn from CRSP and *Monthly* TAQ datasets. The sample selection procedure in each year is essentially identical to that described for the 2011 cross-sectional sample. In each year, from all firms present on CRSP and TAQ in April, with share codes in (10 and 11), and with primary listings on the NYSE, Amex and NASDAQ exchanges, I draw fifteen firms from each dollar trading volume decile.[13] Quote data are drawn from TAQ.

Table 5 reports summary statistics. The oft-remarked increase in the intensity of trading activity is clearly visible in the trends for median number of trade and quote records. From 2001 to 2011, the

[12] Reg NMS was proposed in February, 2004) and adopted in June 2005 with an effective date of August 2005. It was implemented in stages, mostly over 2006.
[13] As of April, 2001, NASDAQ had not fully implemented decimalization. For this year, I do not sample from stocks that traded in sixteenths.

average annual compound growth rate is about 25% percent for trades, and about 36% for quotes. As described in the last section, all of a firm's quote records in a given second are assigned random, but order preserving, millisecond remainders. The NBBO is constructed from these quote records. This yields a NBBO series with (simulated) millisecond time stamps. The 2011 numbers differ slightly from those reported in Table 1. These differences are a consequence of different error filters.

Prior to the construction of the NBBO the bid and offer are filtered for extreme values. The following quotes (bids or offers) are eliminated: those with zero size and/or zero price; those quotes priced at 20% or lower of the smallest closing price reported on CRSP in the month; those priced at 500% or higher of highest closing price. Quotes that crossed the market are only eliminated if the crossing is a dollar or more, or more than 10 percent of the midpoint price. Other filters use the previously prevailing bid and offer midpoint as a benchmark. For stocks priced at ten dollars or less, the bid and offer has to be within forty percent of the benchmark; for stocks between ten and one hundred dollars, the cutoff is twenty percent; for stocks between one hundred and 250, ten percent; above 250, five percent.[14] These filters do not eliminate all suspicious bids and offers, a point to which the discussion will subsequently return.

VI.B. Results

In analyzing 2001-2011, it is best to begin with the wavelet variance ratios. By construction they are normalized with respect to long-term variance, and over this period there are large swings in market-wide long-term volatility (evident from a cursory examination of the VIX). These would be expected to affect the short term variances as well. Table 6 Panel A reports the mean normalized wavelet variances for shorter time scales in the analysis. As in the 2011 sample, there is substantial variance inflation relative to the random-walk in all years. Perhaps surprisingly, though, the excess variance is high in all years, including the early years of the decade. The pattern does not suggest an increasing trend.

[14] The error filters are applied uniformly for the Monthly TAQ data in all years 2001-2011. For 2011 this causes a small apparent discrepancy in the counts for NBB and NBO changes, between Tables 1 and 5. The inputs to Table 5 are filtered, and hence have slightly fewer NBB and NBO changes relative to the unfiltered inputs to Table 1.

Given the recent media attention devoted to low-latency activity and the undeniable growth in quote volume, the absence of a strong trend in quote volatility seems surprising. There are several possible explanations. In the first place, "flickering quotes" drew comment well before the start of the sample, in an era when quotes were dominated by human market makers (Harris (1999); U.S. Commodities Futures Trading Commission Technology Advisor Committee (2001)). Also an artifact of this era is the specialist practice of "gapping" the quotes to indicate larger quantities at worse prices (Jennings and Thirumalai (2007)). In short, the quotes may have in reality been less unwavering than popular memory holds. The apparent discrepancy between quote volatility and quote volume can be explained by appealing to the increase in market fragmentation and consequent growth in matching quotes.

Bid-offer plots for firm-days in each year that correspond to extreme realizations of the variances exhibit an interesting pattern. In later years, these outlier plots tend to resemble the initial AEPI example, with rapid oscillations of relatively low amplitude. In the earlier years, they are more likely to feature small number of prominent spikes associated with a sharply lower bid or elevated offer that persists for a minute or less.

As an example, Figure 4 (Panel A) depicts the NBBO for PRK (Park National Corporation, Amex-listed) on April 6, 2001. At around 10:00 there is a downward spike in the NBB. Shortly after noon there is a sharp drop in the NBB of roughly three dollars and a sharp rise in the NBO of about one dollar. To better document this behavior, Table 7 details the CQ records in the vicinity of the noon episode. There are multiple exchanges active in the market, but Amex (*A*) is the apparent price leader. At 12:02:22, *A* establishes the NBB at 86.74. At 12:03:11, *A* bids 83.63, exposing the previous *T* (NASDAQ) bid of 86.68 as the new NBB. At 12:03:16, *T* backs off, leaving *A*'s 83.63 as best. Within half a minute, however, the NBB is back at 86.50. The lower bid is not marketed by any special mode flag. It is not a penny ("stub") bid. The size of the bid at two (hundred shares) is typical for the market on that day. A similar sequence of events sends the NBO up a dollar for about one second.

These quotes are not so far off the mark as to be clearly erroneous. We must nevertheless question whether they were "real"? Did they reliably indicate the consensus market values at those instances? Were they accessible for execution? Were they truly the best in the market? There were no trades between 11:38 and 12:13, but if a market order had been entered, would it in fact have been executed at the NBBO?[15] These are meaningful questions because they bear directly on market quality. Ultimately, though, the record is unlikely to provide clear answers. The US equity market in 2001 reflected a blend of human and automated mechanisms, practices and conventions that defies detailed description even at a distance of only twelve years.

Discerning whether or not quote volatility increased over the period, therefore, requires that we sharpen the question. The quote volatility in the initial AEPI example is of high frequency, but low amplitude. This is visually distinct from the spikes of high frequency and high amplitude found in PRK. The latter is sometimes called “pop” noise, in reference to its sound in audio signals (Walker (2008)). As in the de-noising of audio signals, the goal is to remove the pops from the signals of lower amplitude. The wavelet literature has developed many denoising approaches (see Percival and Walden, Gençay et al, and Walker). When the stochastic properties of the noise and signal processes are known, optimal methods can often be established. In the present case, though, I adopt a simpler method.

Wavelet transforms facilitate the direct computation of smooth and rough components of the observed process. This process, known as multiresolution analysis, isolates components at different time scales. As an example, Panel B of Figure 4 plots the rough component of the PRK bid at a time scale of 51.2 seconds. It is zero mean by construction, and the spikes are cleanly resolved. On the principle that high-frequency quoting (as in the AEPI example) should not be substantially larger than the bid-offer spread in magnitude, I set acceptance bands at $\pm Min\left(1.5\times\left(average\,spread\right),\$0.25\right)$. The minimum of \$0.25 is set to accommodate stocks with very tight spreads. For PRK, the bands are approximately ±\$0.33, and they are indicated in the figure by horizontal black lines. Values lying outside of the band are set to the band limits. This clips the high-amplitude peaks, while leaving the low-amplitude components, some of which are highly oscillatory, untouched. The signal (bid or offer) is reconstituted using the clipped rough, and analysis proceeds on this denoised signal. I recompute all estimates for all firms using the denoised bids and offers.

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[15] The Amex (like the NYSE) had specialists in 2001. Specialists generally had affirmative price continuity obligations that would have discouraged (though not expressly forbidden) trades occurring at prices substantially different from those prevailing immediately before and immediately after. A broker-dealer, however, would not have been subject to this restriction.

Table 6 Panel B reports the wavelet variance ratios for the denoised quotes. The results are striking. In the early years, the variance ratios computed from the denoised quotes are much lower than those computed from the raw data. In later years, however, the reduction associated with the denoising is small. For the 200 ms variance ratio, for example, the 2001 drop is from 5.28 (for the raw quotes) to 1.56 (for the denoised quotes), but the 2011 value only drops from 3.74 to 3.57.

These results are consistent with the view that over the decade, the nature of quote volatility, if not the overall level, has changed. In the early years, the volatility was of relatively high amplitude but non-oscillatory. It is removed by the pop-denoising procedure. The procedure does not attenuate the low-amplitude highly oscillatory components, however, which drive quote volatility in the later years. The difference between the raw and denoised ratios generally declines throughout the decade, but the largest drops occur around the Reg NMS period.

VII. Discussion

From an economic perspective, high frequency quote volatility is connected most closely to other high frequency and low latency phenomena in modern markets. From a statistical perspective, it is connected to volatility modeling.

VII.A. High-frequency quoting and high-frequency trading

Most definitions of algorithmic and high-frequency trading encompass many aspects of market behavior (not just executions), and would be presumed to cover quoting as well.[16] Executions and quotations are nevertheless very different events. It is therefore useful to consider their relation in the high-frequency context.

Quote volatility is not necessarily associated with high-frequency executions. One can envision regimes where relatively stable quotes are hit intensively when fundamental valuations change, and

[16] A CFTC draft definition reads: "High frequency trading is a form of automated trading that employs: (a) algorithms for decision making, order initiation, generation, routing, or execution, for each individual transaction without human direction; (b) low-latency technology that is designed to minimize response times, including proximity and co-location services; (c) high speed connections to markets for order entry; and (d) high message rates (orders, quotes or cancellations)" (U.S. Commodities Futures Trading Commission (2011)).

periods (such as Figure 1) where frenetic quoting occurs in the absence of executions. Nevertheless, the same technology that makes high-frequency executions possible also facilitates the rapid submission, cancellation and repricing of the nonmarketable orders that define the bid and offer. One might expect this commonality of technology to link the two activities in practice.

Executions are generally emphasized over quotes when identifying agents as high-frequency traders. For example, Kirilenko, Kyle, Samadi and Tuzun (2010) select on high volume and low inventory. The low inventory criterion excludes institutional investors who might use algorithmic techniques to accumulate or liquidate a large position. The NASDAQ HFT dataset uses similar criteria (Brogaard (2010); Brogaard, Hendershott and Riordan (2012)). Once high-frequency traders are identified, their executions and the attributes of these executions lead to direct measures of HF activity in panel samples.

In some situations, however, identifications based on additional, non-trade information are possible. Menkveld (2012) identifies one Chi-X participant on the basis of size and prominence. The Automated Trading Program on the German XETRA system allows and provides incentives for designating an order as algorithmic (Hendershott and Riordan (2012)). Other studies analyze indirect measures of low-latency activity. Hendershott, Jones and Menkveld (2011) use NYSE message traffic. Hasbrouck and Saar (2011) suggest strategic runs (order chains) of cancel and replace messages linked at intervals of 100 ms or lower.

Most of these studies find a positive association between low-latency activity and market quality. Low-latency activity, for example, tends to be negatively correlated with as posted and effective spreads, which are inverse measures of market quality. Most also find a zero or negative association between low-latency activity and volatility, although the constructed volatility measures usually span intervals that are long relative to those of the present paper. With respect to algorithmic or high-frequency activity: Hendershott and Riordan (2012) find an insignificantly negative association with the absolute value of the prior 15-minute return; Hasbrouck and Saar (2011) find a negative association with the high-low difference of the quote midpoint over a 15-minute interval; Brogaard (2012) finds a negative relation with absolute price changes over intervals as short as ten seconds.

The time-scaled variance estimates used here clearly aim at a richer characterization of volatility than the high/low or absolute return proxies used in the studies above. The present study does not, on the other hand, attempt to correlate the variance measures with intraday proxies for high-frequency trading. One would further suspect, of course, that the ultimate strategic purpose of high-frequency quoting is to facilitate a trade or to affect the price of a trade. The mechanics of this are certainly deserving of further research.

The discussion in Section II associates short-term quote volatility with price uncertainty for those who submit marketable orders, use dark mechanisms that price by reference, or face monitoring difficulties. From this perspective, quote volatility is an inverse measure of market quality. This paper's finding that volatility of the low-amplitude rapidly oscillating sort has increased over the last decade therefore suggests a decline in this aspect of market quality has accompanied the increased use of technology.

VII.B. High-frequency quoting and volatility modeling

Security prices at all horizons are a mix of integrated and stationary components. The former are usually identified with persistent fundamental information innovations; the latter, with transient microstructure effects. The former are important to long-term hedging and investment; the latter, to trading and market-making. The dichotomy is sometimes reflected in different statistical tools and models.

Between the two approaches, the greatest common concerns arise in the analysis of realized volatility (Andersen, Bollerslev, Diebold and Ebens (2001); Andersen, Bollerslev, Diebold and Labys (2003a); Andersen, Bollerslev, Diebold and Labys (2003b)). RVs are calculated from short-term price changes. They are useful as estimates of fundamental integrated volatility (IV), and typically serve as inputs to longer-term forecasting models. RVs constructed directly from trade, bid and offer prices are typically noisy, however, due to the presence of microstructure components. Local averaging moderates these effects. (See Hansen and Lunde (2006) and the accompanying comments. Other approaches are discussed in Ait-Sahalia, Mykland and Zhang (2011); Zhang (2006); Zhang, Mykland and Aït-Sahalia (2005).)

The present study draws on several themes in the RV literature. The volatility ratio plots in Figure 2 serve a purpose similar to the volatility signature plots introduced by Fang (1996) and used in Andersen, Bollerslev, Diebold and Ebens (2002) and Hansen and Lunde (2006). Hansen and Lunde also articulate the connection between bid-offer comovement and fundamental volatility: since the bid and offer have economic fundamentals in common, divergent movements must be short-term, transient, and unconnected to fundamentals.

One strand in the RV literature emphasizes analysis of multiple time-scales. Zhang, Mykland and Aït-Sahalia (2005) posit a framework consisting of a Brownian motion with time-varying parameters, $dX_t = \mu_t dt + \sigma_t dz,$ and a discretely-sampled noisy observation process, $Y_{t_i} = X_{t_i} + \varepsilon_{t_i}$. The Y_{t_i} are viewed as transaction prices, and ε_{t_i} constitute i.i.d. microstructure noise. The objective is estimation of the integrated volatility $\int \sigma_t^2 dt$ over a sample. Zhang et al propose a two-scale variance estimator in which a long-scale estimate is corrected for bias with an adjustment based on properties of the noise estimated at a short scale. While the present analysis also features multiple time scales, there are major differences in the perspective. In the present situation, execution price risk is caused by volatility in the observed process (the quote, not the underlying latent value, X_t); the quote process is right-continuous (and continuously observable); the noise is not necessarily i.i.d. (cf. the AEPI episodes in Figure 1); and, the noise is possibly correlated with the X_t increments.

The paper also departs from the RV literature in other respects. The millisecond time scales employed in this paper are several orders of magnitude shorter than those typically encountered. Most RV studies also focus on relatively liquid assets (index securities, Dow-Jones stocks, etc.). The low-activity securities included in the present paper's samples are important because, due to their larger spreads and fewer participants, they are likely to exhibit relatively strong, persistent and distinctive microstructure-related components.

VIII. Conclusion and outstanding questions

High-frequency volatility in the bid and offer quotes induces risk for agents who experience delay in communicating with the market. The risk may be quantified as the price variance over the interval of delay, relative to the average price over the interval. This volatility degrades the informational value of the quotes. Furthermore, because the bid and offer are often used as reference prices for dealer trades against customers, the volatility increases the value of a dealer's look-back option and exacerbates monitoring problems for customers, exchanges, and regulators.

This study is a preliminary analysis of short-term quote volatility in the US equity market. Estimates of sub-second high frequency variance for the National Best Bid and Offer (NBBO) are well in excess of what would be expected relative to random-walk volatility estimated over longer intervals. The excess volatility is more pronounced for stocks that have lower average activity. Furthermore, the correlations between bids and offers at these time scales are positive, but low. That the bid and offer are not moving together also suggests that the volatility is not fundamental.

The paper proposes a simulation approach to measuring millisecond-level volatility in US equity data (like the Monthly TAQ) that possess all quote records, but are time-stamped only to the second. In data time-stamped to the millisecond I compare two sets of estimates: one set based on the original time-stamps; the other based on simulated time stamps. I find high correlations between the two estimates, establishing the reliability of the simulation procedure.

With these results, the paper turns to a longer US historical sample, 2001-2011, with one-second time-stamps. Despite the current public scrutiny of high-frequency trading, the rapid growth in the number of quote records, and the presumption that low-latency technology is a new and recent phenomenon, the excess short-term quote volatility found in the 2011 data also appears in earlier years.

The nature of the volatility has changed markedly over the decade, however. Prior to Reg NMS, the volatility appears attributable to spikes associated with bids and offers that are neither clearly erroneous nor reliably valid. Post Reg NMS, the volatility is more attributable to oscillatory low-amplitude changes: rapid movements not substantially larger than the spread. Of course, this does not imply that Reg NMS caused growth in oscillatory quoting. The growth has occurred concomitantly with technological developments, and perhaps as an inevitable result of them.

Appendix: Deviations about averages of random walks

Consider a price series that evolves as $p_t = p_{t-1} + u_t$ where u_t is a white-noise process with unit variance. Without loss of generality, we initialize $p_0 = 0$ and consider the mean-squared deviations about the mean over the first n observations.

$$MSD(n) = \frac{1}{n}\sum_{i=1}^{n} p_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} p_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{i} u_i\right)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{i} u_i\right)\right)^2$$

Taking expectations (noting that $Eu_iu_j = 1$ for $i = j,$ and zero otherwise) and simplifying the sums gives

$$\sigma_n^2 \equiv E\left(MSD\left(n\right)\right) = \frac{n+1}{2} - \frac{\left(n+1\right)\left(2n+1\right)}{6n} = \frac{n^2-1}{6n}$$

For the sequence of averaging periods $n_j = n_0 2^j$ for $j = 0,1,2,\ldots$, the corresponding sequence of variances is

$$\sigma_j^2 = \frac{4^j n_0^2 - 1}{3n_0 2^{j+1}}$$

In moving from $j-1$ to j the incremental change in variance (also known as the wavelet variance) is

$$\nu_j^2 = \sigma_j^2 - \sigma_{j-1}^2 = \frac{4^j n_0^2 + 2}{3n_0 2^{j+2}}$$

We now reinterpret these results in a slightly expanded framework. Suppose that the original time subscript t indexes periods of Δ time units ("milliseconds") and that the variance per unit time of the u_t process is σ_u^2. Let M denote the averaging period measured in units of time, and correspondingly, $M_j = M_0 2^j$ for $j = 0,1,\ldots$ Then the rough and wavelet variances become

$$\sigma_j^2 = \frac{(4^j M_0^2 - \Delta^2)\sigma_u^2}{3M_0 2^{j+1}} \text{ and } \nu_j^2 = \frac{(4^j M_0^2 + 2\Delta^2)\sigma_u^2}{3M_0 2^{j+2}}.$$

In the continuous time limit, as $\Delta \to 0$, that $\sigma_j^2 = 2^{j-1} M_0 \sigma_u^2 / 3$ and $\nu_j^2 = 2^{j-2} M_0 \sigma_u^2 / 3$. These results suffice to define and characterize the variances considered in the paper.

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Table 1. Sample Summary Statistics

Source: CRSP and Daily TAQ data, April 2011. The sample is 100 firms randomly selected from CRSP with stratification based on average dollar trading volume in the first quarter of 2011, grouped in quintiles by dollar trading volume over the first quarter of 2011. NBB is the National Best Bid; NBO the National Best Offer. Except for the counts (first four rows), all table entries are cross-firm medians.

		Dollar trading volume quintile				
	Full sample	1 (low)	2	3	4	5 (high)
No. of firms	150	30	30	30	30	30
NYSE	47	0	5	7	16	19
Amex	6	2	2	0	1	1
NASDAQ	97	28	23	23	13	10
Avg. daily CT records (trades)	1,331	31	431	1,126	3,478	16,987
Avg. daily CQ records (quotes)	23,928	967	7,706	24,026	53,080	181,457
Avg. daily NBBO records	7,138	328	3,029	7,543	16,026	46,050
Avg. daily NBB changes	1,245	120	511	1,351	2,415	4,124
Avg. daily NBO changes	1,164	103	460	1,361	2,421	4,214
Avg. price (bid-offer midpoint)	$15.62	$4.87	$5.46	$17.86	$27.76	$51.60
Market capitalization of equity, $ Million	$683	$41	$202	$747	$1,502	$8,739

Table 2. Time scale variance estimates, 2011

Summary estimates of time scale variances and related measures for the 2011 sample (150 US firms, April, 2011). The wavelet variances are estimates of the price variance at the indicated time scale. They are presented as originally estimated, in [mils ($0.001) per share]^2, in [basis points relative to price, 0.01%]^2, and as a ratio normalized by the longest time scale (27.3 minutes) and scaled to be unity assuming a random-walk with uncorrelated increments. The rough variances are cumulative over the indicated and lower time scales, but are otherwise similarly interpreted. The bid-offer correlation is the wavelet correlation (correlation between detail components) at the indicated time scale. All entries are cross-firm means.

		Rough variances, σ_j^2			Wavelet variances, ν_j^2			
		(1)	(2)	(3)	(4)	(5)	(6)	(7)
Level, j	Time scale	(mils per share)^2	(basis points)^2	Ratio, $VR_{j,J}$	(mils per share)^2	(basis points)^2	Ratio, $V_{j,J}$	Bid-Offer Correlation
0	< 50 ms	0.17	0.06	4.22	0.17	0.06	4.22	
1	50 ms	0.32	0.11	3.99	0.15	0.05	3.76	0.32
2	100 ms	0.61	0.22	3.79	0.29	0.10	3.58	0.36
3	200 ms	1.17	0.41	3.53	0.55	0.19	3.27	0.41
4	400 ms	2.19	0.75	3.21	1.03	0.34	2.88	0.44
5	800 ms	4.15	1.38	2.90	1.96	0.63	2.59	0.47
6	1,600 ms	7.93	2.56	2.64	3.78	1.18	2.38	0.51
7	3.2 sec	15.27	4.73	2.40	7.35	2.17	2.16	0.55
8	6.4 sec	29.59	8.57	2.12	14.31	3.85	1.84	0.60
9	12.8 sec	57.62	15.49	1.88	28.03	6.91	1.65	0.64
10	25.6 sec	112.38	28.03	1.70	54.76	12.54	1.51	0.69
11	51.2 sec	219.31	51.17	1.54	106.92	23.14	1.39	0.74
12	102.4 sec	428.81	94.11	1.42	209.50	42.94	1.29	0.79
13	3.4 min	842.72	174.70	1.32	413.91	80.60	1.21	0.83
14	6.8 min	1,668.69	328.05	1.23	825.97	153.35	1.15	0.86
15	13.7 min	3,287.68	618.26	1.16	1,618.99	290.21	1.08	0.88
16 (=J)	27.3 min	6,379.91	1,159.03	1.08	3,092.22	540.77	1.00	0.90

Table 3. Time scale variance estimates across dollar trading volume quintiles, 2011

Estimates of time scale variances and related measures for the 2011 sample (150 US firms, April, 2011) for quintiles constructed on dollar trading volume. The total (or rough) variances are cumulative over the indicated and lower time scales, and are presented in Panel A in units of [basis points relative to price, 0.01%]^2. For brevity, only a subset of the levels are reported. Panel B contains estimates of total (rough) variance ratios, scaled to be unity assuming a random-walk with uncorrelated increments. The bid-offer correlation is the wavelet correlation (correlation between detail components) at the indicated time scale (Panel C). All entries are cross-firm means with standard errors in parentheses.

Panel A. Rough variance, σ_j^2, mils per share

			Dollar trading volume quintiles				
Level, j	Time scale	Full sample	1 (low)	2	3	4	5 (high)
0	< 50 ms	0.17	0.13	0.10	0.14	0.22	0.25
		(0.01)	(0.07)	(0.02)	(0.01)	(0.01)	(0.01)
1	50 ms	0.15	0.12	0.09	0.12	0.20	0.23
		(0.01)	(0.07)	(0.01)	(0.01)	(0.01)	(0.01)
3	200 ms	0.55	0.38	0.29	0.43	0.75	0.89
		(0.04)	(0.22)	(0.03)	(0.02)	(0.04)	(0.05)
5	800 ms	1.96	1.03	1.00	1.53	2.79	3.35
		(0.09)	(0.40)	(0.09)	(0.07)	(0.14)	(0.20)
7	3.2 sec	7.35	3.33	3.45	5.41	10.73	13.42
		(0.28)	(0.93)	(0.30)	(0.23)	(0.57)	(0.84)
10	25.6 sec	54.76	15.25	20.99	34.87	81.37	117.44
		(2.09)	(2.41)	(1.84)	(1.50)	(4.50)	(8.22)
14	6.8 min	825.97	173.25	242.94	445.93	1,273.22	1,930.34
		(42.96)	(28.14)	(21.87)	(21.43)	104.07)	173.36)
16	27.3 min	3,092.22	533.78	798.24	1,665.48	5,425.58	6,786.46
		(188.75)	(87.40)	(73.69)	(82.94)	(634.46)	(639.18)

Panel B. Rough variance, σ_j^2, basis points

			Dollar trading volume quintiles				
Level, j	Time scale	Full sample	1 (low)	2	3	4	5 (high)
0	< 50 ms	0.06	0.20	0.06	0.03	0.02	0.01
		(0.01)	(0.07)	(<0.01)	(<0.01)	(<0.01)	(<0.01)
1	50 ms	0.11	0.38	0.12	0.05	0.03	0.02
		(0.03)	(0.14)	(0.01)	(<0.01)	(<0.01)	(<0.01)
3	200 ms	0.41	1.33	0.44	0.19	0.11	0.06
		(0.09)	(0.47)	(0.02)	(0.01)	(<0.01)	(<0.01)
5	800 ms	1.38	4.26	1.58	0.70	0.40	0.23
		(0.21)	(1.15)	(0.07)	(0.03)	(0.01)	(0.01)
7	3.2 sec	4.73	13.90	5.65	2.59	1.51	0.87
		(0.49)	(2.60)	(0.26)	(0.12)	(0.04)	(0.03)
10	25.6 sec	28.03	71.52	36.85	17.37	11.39	7.27
		(1.46)	(7.44)	(1.74)	(0.78)	(0.34)	(0.25)
14	6.8 min	328.05	694.55	463.45	225.57	173.41	119.32
		(10.59)	(44.14)	(27.35)	(11.01)	(6.47)	(4.60)
16	27.3 min	1,159.03	2,234.94	1,718.14	815.77	685.54	446.54
		(41.18)	143.62)	138.93)	(42.03)	(27.48)	(18.25)

Panel C. Rough variance ratio,

			Dollar trading volume quintiles				
Level, j	Time scale	Full sample	1 (low)	2	3	4	5 (high)
0	< 50 ms	4.22	12.72	3.45	2.62	1.76	1.37
		(1.28)	(6.96)	(0.18)	(0.07)	(0.04)	(0.02)
1	50 ms	3.99	12.01	3.23	2.44	1.69	1.35
		(1.25)	(6.81)	(0.16)	(0.06)	(0.04)	(0.02)
3	200 ms	3.53	10.40	2.83	2.20	1.57	1.30
		(1.06)	(5.77)	(0.11)	(0.05)	(0.03)	(0.02)
5	800 ms	2.90	7.82	2.50	2.02	1.43	1.21
		(0.66)	(3.56)	(0.08)	(0.04)	(0.03)	(0.02)
7	3.2 sec	2.40	5.87	2.17	1.82	1.32	1.15
		(0.38)	(2.08)	(0.06)	(0.04)	(0.02)	(0.02)
10	25.6 sec	1.70	3.06	1.70	1.49	1.19	1.17
		(0.12)	(0.64)	(0.04)	(0.03)	(0.02)	(0.02)
14	6.8 min	1.23	1.58	1.24	1.17	1.04	1.16
		(0.02)	(0.12)	(0.03)	(0.03)	(0.02)	(0.02)
16	27.3 min	1.08	1.19	1.08	1.06	1.01	1.06
		(0.01)	(0.06)	(0.03)	(0.02)	(0.02)	(0.02)

Panel D. Wavelet bid-offer correlations.

			Dollar trading volume quintiles				
Level, j	Time scale	Full sample	1 (low)	2	3	4	5 (high)
1	50 ms	0.32	0.05	0.23	0.31	0.41	0.56
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
3	200 ms	0.41	0.11	0.33	0.42	0.49	0.65
		(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.02)
5	800 ms	0.48	0.15	0.40	0.51	0.56	0.72
		(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)
7	3.2 sec	0.55	0.19	0.47	0.59	0.66	0.82
		(0.01)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
10	25.6 sec	0.70	0.27	0.61	0.75	0.85	0.95
		(0.01)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)
14	6.8 min	0.86	0.44	0.88	0.97	0.99	1.00
		(0.02)	(0.03)	(0.04)	(0.03)	(0.03)	(0.03)
16	27.3 min	0.90	0.51	0.96	0.99	1.00	1.00
		(0.02)	(0.04)	(0.05)	(0.03)	(0.04)	(0.03)

Table 4. Correlations between statistics based on actual vs. simulated millisecond time stamps.

The sample is 150 randomly chosen U.S. stocks, April, 2011. For each stock, I compute two estimates of the wavelet quote variance for bids. One estimate uses the actual millisecond time stamps; for the other estimate, the millisecond portion of the time stamp is simulated. Entries in Panel A are the correlations (across firms) between these two estimates. Panel B reports correlations between bid-offer wavelet covariance estimates constructed from actual and simulated time stamps.

Panel A. Correlations for wavelet variances, ν_j^2

			Dollar trading volume quintiles				
Level, j	Time scale	Full sample	1 (low)	2	3	4	5 (high)
0	< 50 ms	0.952	0.948	0.960	0.958	0.916	0.979
1	50 ms	0.953	0.944	0.952	0.952	0.937	0.982
3	200 ms	0.975	0.965	0.969	0.975	0.977	0.988
5	800 ms	0.994	0.991	0.989	0.995	0.996	0.998
7	3.2 sec	0.999	0.999	0.999	1.000	1.000	1.000
10	25.6 sec	1.000	1.000	1.000	1.000	1.000	1.000
14	6.8 min	1.000	1.000	1.000	1.000	1.000	1.000
16	27.3 min	1.000	1.000	1.000	1.000	1.000	1.000

Panel B. Correlations for wavelet covariances

			Dollar trading volume quintiles				
Level, j	Time scale	Full sample	1 (low)	2	3	4	5 (high)
0	< 50 ms	0.775	0.333	0.768	0.896	0.919	0.943
1	50 ms	0.900	0.662	0.926	0.965	0.972	0.978
3	200 ms	0.979	0.921	0.986	0.995	0.995	0.998
5	800 ms	0.999	0.998	0.999	1.000	1.000	1.000
7	3.2 sec	1.000	1.000	1.000	1.000	1.000	1.000
10	25.6 sec	1.000	1.000	1.000	1.000	1.000	1.000
14	6.8 min	1.000	1.000	1.000	1.000	1.000	1.000
16	27.3 min	1.000	1.000	1.000	1.000	1.000	1.000

Table 5. Summary statistics, historical sample, 2001-2011

From the CRSP file, for each year, 2001-2011 and all stocks present in January through April of that year with share codes equal to 10 or 11, I draw 150 firms in a random sample stratified by dollar trading volume in January through March. NBB is the National Best Bid; NBO, the National Best Offer; CT, Consolidated Trade; CQ, Consolidated Quote. Trade and quote counts are from the Monthly TAQ database (one-second time stamps). Except for the number of firms, table entries are cross-firm medians.

	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
No. firms	146	146	150	150	150	150	150	150	150	150	150
NYSE	108	46	51	44	48	44	55	53	56	54	47
Amex	22	4	11	12	8	15	14	7	5	14	6
NASDAQ	16	96	88	94	94	91	81	90	89	82	97
Avg. daily CT records (trades)	142	122	187	393	425	605	970	1,209	1,790	1,141	1,331
Avg. daily CQ records (quotes)	1,078	534	1,299	3,850	5,828	7,307	12,521	16,328	39,378	23,249	23,928
Avg. daily NBB changes	103	127	203	509	596	761	772	1,144	1,618	1,466	1,210
Avg. daily NBO changes	103	129	213	537	729	751	789	1,119	1,731	1,457	1,126
Avg. price (bid-offer midpoint)	$18.85	$17.94	$14.83	$16.53	$16.10	$21.14	$15.81	$14.01	$10.72	$16.32	$15.62
Market capitalization of equity, $ Million	$745	$302	$189	$345	$325	$411	$480	$405	$316	$478	$683

Table 6. Wavelet variance ratios for bids and offers, 2001-2011

In each year, 150 US firms are randomly selected from CRSP (stratified sampling by average daily dollar trading volume). Quote records for Aprile are taken from the NYSE Monthly TAQ database, and randomly assigned order-preserving millisecond time stamps. Wavelet variances are estimated for each side (bid and offer) and each firm. The wavelet variance ratio $V_{j,J=16}$ reflects the variance (over the indicated time scale) relative to variance of a random-walk calibrated to the variance at the longest interval in the study, 27.3 minutes (at J=16). Under the random-walk hypothesis, the ratio should be unity at all time scales. Estimates in Panel A are constructed from bids and offers that were filtered for errors, but not otherwise adjusted. Estimates in Panel B are constructed from denoised bids and offers (with short-term peaks clipped).

Panel A. Wavelet variance ratios, $V_{j,J=16}$, computed from raw bids and offers

Level, j	Time scale	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
1	50 ms	5.22	7.16	6.03	10.28	6.69	8.57	6.96	6.06	4.52	7.08	4.70
2	100 ms	5.44	6.58	5.28	9.69	6.51	8.07	6.27	5.38	4.12	6.26	4.32
3	200 ms	5.28	6.28	5.13	9.03	6.22	7.34	5.33	4.64	3.68	5.40	3.74
4	400 ms	4.59	5.23	5.00	8.16	5.75	6.30	4.25	3.84	3.21	4.53	3.07
5	800 ms	3.12	4.04	3.93	5.57	5.03	5.10	3.41	3.11	2.76	3.71	2.56
6	1,600 ms	2.11	2.55	3.25	4.11	4.14	4.05	2.89	2.59	2.43	3.04	2.23
7	3.2 sec	1.98	2.24	2.93	3.38	3.48	3.37	2.56	2.29	2.17	2.53	2.01
8	6.4 sec	1.94	2.11	2.62	2.91	2.93	2.92	2.35	2.08	1.95	2.16	1.82

Panel B. Incremental variance ratios, $V_{j,J=16}$, computed from denoised bids and offers

Level, j	Time scale	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
1	50 ms	1.60	2.37	3.15	7.02	6.09	8.24	6.56	5.83	4.20	6.79	4.46
2	100 ms	1.57	2.32	3.09	6.82	5.89	7.76	5.89	5.17	3.83	6.00	4.07
3	200 ms	1.56	2.27	3.03	6.48	5.61	7.04	4.99	4.45	3.41	5.18	3.57
4	400 ms	1.55	2.23	2.94	5.90	5.16	6.02	3.96	3.68	2.97	4.36	3.00
5	800 ms	1.57	2.19	2.83	5.00	4.47	4.82	3.13	2.98	2.56	3.58	2.52
6	1,600 ms	1.64	2.20	2.71	3.99	3.60	3.79	2.63	2.51	2.27	2.94	2.20
7	3.2 sec	1.81	2.30	2.62	3.44	3.02	3.16	2.33	2.23	2.04	2.46	2.00
8	6.4 sec	2.11	2.51	2.59	3.20	2.65	2.75	2.15	2.04	1.86	2.11	1.82

Table 7. Consolidated quote record for PRK, April 6, 2001.

The table contains the consecutive records from the monthly TAQ consolidated quote file for the Park National Corporation. The first five columns are directly from the CQ file. The national best bid and offer (NBBO), and the exchange(s) at the NBBO are inferred. The NBBO columns contain entries only when there is a change. The size is units of 100 shares. ("4x2" means that 400 shares are bid for and 200 shares are offered.) The exchange codes are "A" (the American Stock Exchange, the primary listing exchange [presently named "NYSE MKT LLC"]); "M," Midwest; "C," Cincinnati; "T," NASDAQ.

Time	Bid	Offer	Size	Ex	Mode	NBB		NBO	
12:01:33	86.73	86.90	4x2	A	12	86.73	A	86.90	A
12:01:34	86.63	87.00	1x1	M	12				
12:01:35	86.35	87.28	1x1	C	12				
12:01:35	86.67	86.96	1x1	T	12				
12:01:35	86.67	86.96	1x1	T	12				
12:02:22	86.74	86.90	3x2	A	12	86.74	A		
12:02:23	86.64	87.00	1x1	M	12				
12:02:25	86.68	86.96	1x1	T	12				
12:02:25	86.68	86.96	1x1	T	12				
12:03:11	83.63	86.90	1x2	A	12	86.68	T		
12:03:13	83.53	87.00	1x1	M	12				
12:03:15	83.25	87.28	1x1	C	12				
12:03:15	83.60	86.90	2x2	A	12				
12:03:16	83.57	86.96	1x1	T	12	83.60	A		
12:03:16	83.57	86.96	1x1	T	12				
12:03:16	83.50	87.00	1x1	M	12				
12:03:21	83.54	86.96	1x1	T	12				
12:03:21	83.54	86.96	1x1	T	12				
12:03:27	83.81	86.90	1x2	A	12	83.81	A		
12:03:29	83.71	87.00	1x1	M	12				
12:03:30	83.43	87.28	1x1	C	12				
12:03:30	83.81	86.90	1x2	A	12				
12:03:32	83.75	86.96	1x1	T	12				
12:03:37	83.75	86.96	1x1	T	12				
12:03:38	86.50	86.90	2x2	A	12	86.50	A		
12:03:39	86.40	87.00	1x1	M	12				
12:03:40	86.50	86.90	2x2	A	12				
12:03:40	86.12	87.28	1x1	C	12				
12:03:45	86.44	86.96	1x1	T	12				
12:03:45	86.44	86.96	1x1	T	12				
12:03:46	86.50	88.00	2x9	A	12			86.96	T
12:03:48	86.40	88.10	1x1	M	12				
12:03:49	86.12	88.38	1x1	C	12				
12:03:51	86.44	88.06	1x1	T	12			88.00	A
12:03:51	86.44	88.06	1x1	T	12				
12:03:52	86.50	86.90	2x2	A	12			86.90	A
12:03:54	86.40	87.00	1x1	M	12				
12:03:55	86.12	87.28	1x1	C	12				
12:03:58	83.00	86.90	2x2	A	12	86.44	T		
12:03:58	86.44	86.96	1x1	T	12				
12:03:58	86.44	86.96	1x1	T	12				
12:04:00	82.90	87.00	1x1	M	12				
12:04:01	82.62	87.28	1x1	C	12				
12:04:01	82.94	86.96	1x1	T	12	83.00	A		
12:04:01	82.94	86.96	1x1	T	12				
12:04:06	86.50	86.90	2x2	A	12	86.50	A		

Table 8. Wavelet bid and offer variances, 2001-2011

In each year, 150 US firms are randomly selected from CRSP (stratified sampling by average daily dollar trading volume). Quote records for Aprile are taken from the NYSE Monthly TAQ database, and randomly assigned order-preserving millisecond time stamps. Wavelet variances are estimated for each side (bid and offer) and each firm. The wavelet variance ratio $V_{j,J=16}$ reflects the variance (over the indicated time scale) relative to variance of a random-walk calibrated to the variance at the longest interval in the study, 27.3 minutes (at J=16). Under the random-walk hypothesis, the ratio should be unity at all time scales. All estimates are constructed from denoised bids and offers (with short-term peaks clipped).

Panel A. Rough variances, mils per share

Level, j	time scale	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
1	50 ms	0.05	0.06	0.07	0.19	0.23	0.08	0.08	0.19	0.25	0.12	0.07
		(<0.01)	(<0.01)	(<0.01)	(0.02)	(0.03)	(<0.01)	(<0.01)	(0.01)	(0.03)	(0.01)	(0.01)
3	200 ms	0.34	0.43	0.45	1.28	1.52	0.49	0.45	1.05	1.50	0.70	0.40
		(0.01)	(0.01)	(0.02)	(0.10)	(0.17)	(0.03)	(0.02)	(0.07)	(0.19)	(0.04)	(0.08)
5	800 ms	1.50	1.83	1.85	4.78	5.56	1.75	1.42	3.42	5.16	2.28	1.42
		(0.05)	(0.06)	(0.07)	(0.34)	(0.53)	(0.08)	(0.07)	(0.20)	(0.42)	(0.14)	(0.22)
7	3.2 sec	6.82	7.35	6.77	13.94	16.03	5.33	4.19	10.38	16.45	6.60	4.74
		(0.52)	(0.30)	(0.22)	(0.79)	(1.25)	(0.19)	(0.16)	(0.48)	(0.96)	(0.32)	(0.49)
10	25.6 sec	80.46	61.96	47.03	75.25	84.18	30.37	25.18	61.34	109.42	34.66	30.76
		(16.17)	(6.98)	(2.57)	(4.08)	(5.75)	(0.85)	(0.98)	(2.92)	(13.16)	(1.19)	(3.22)
14	6.8 min	735.03	682.84	489.43	730.86	862.96	322.69	302.16	607.71	1,638.73	375.31	333.50
		(30.23)	(25.72)	(12.26)	(39.26)	(73.94)	(7.40)	(23.79)	(19.02)	492.40)	(12.01)	(12.05)
16	27.3 min	2,511.15	2,300.46	1,554.80	2,240.06	2,872.45	1,047.78	1,046.55	1,987.67	4,623.58	1,264.94	1,164.98
		(80.99)	(74.89)	(39.18)	123.51)	335.55)	(24.35)	101.74)	(62.06)	849.95)	(46.57)	(41.65)

Panel B. Rough variance ratios

Level, j	time scale	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
1	50 ms	1.60	2.37	3.15	7.02	6.09	8.24	6.56	5.83	4.20	6.79	4.46
		(0.02)	(0.06)	(0.11)	(0.56)	(0.39)	(0.75)	(0.40)	(0.46)	(0.31)	(0.38)	(1.42)
3	200 ms	1.57	2.30	3.06	6.65	5.76	7.42	5.47	4.86	3.65	5.65	3.84
		(0.02)	(0.06)	(0.10)	(0.51)	(0.36)	(0.64)	(0.30)	(0.37)	(0.26)	(0.32)	(1.16)
5	800 ms	1.56	2.23	2.91	5.61	4.94	5.72	3.87	3.58	2.91	4.25	2.94
		(0.03)	(0.07)	(0.09)	(0.37)	(0.27)	(0.43)	(0.17)	(0.23)	(0.16)	(0.23)	(0.69)
7	3.2 sec	1.71	2.25	2.71	4.11	3.64	3.94	2.78	2.63	2.31	3.02	2.28
		(0.09)	(0.15)	(0.10)	(0.23)	(0.15)	(0.22)	(0.09)	(0.10)	(0.09)	(0.12)	(0.35)
10	25.6 sec	2.36	2.70	2.60	3.16	2.42	2.53	2.03	1.89	1.74	1.93	1.70
		(0.37)	(0.60)	(0.29)	(0.54)	(0.07)	(0.09)	(0.05)	(0.05)	(0.04)	(0.05)	(0.12)
14	6.8 min	1.37	1.41	1.50	1.58	1.49	1.52	1.37	1.29	1.32	1.30	1.23
		(0.04)	(0.06)	(0.03)	(0.06)	(0.02)	(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)
16	27.3 min	1.12	1.13	1.16	1.18	1.16	1.17	1.12	1.09	1.11	1.10	1.08
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)

Figure 1. The bid and offer AEPI, April 29, 2011

National Best Bid and Offer (NBBO). Source: NYSE Daily TAQ dataset.

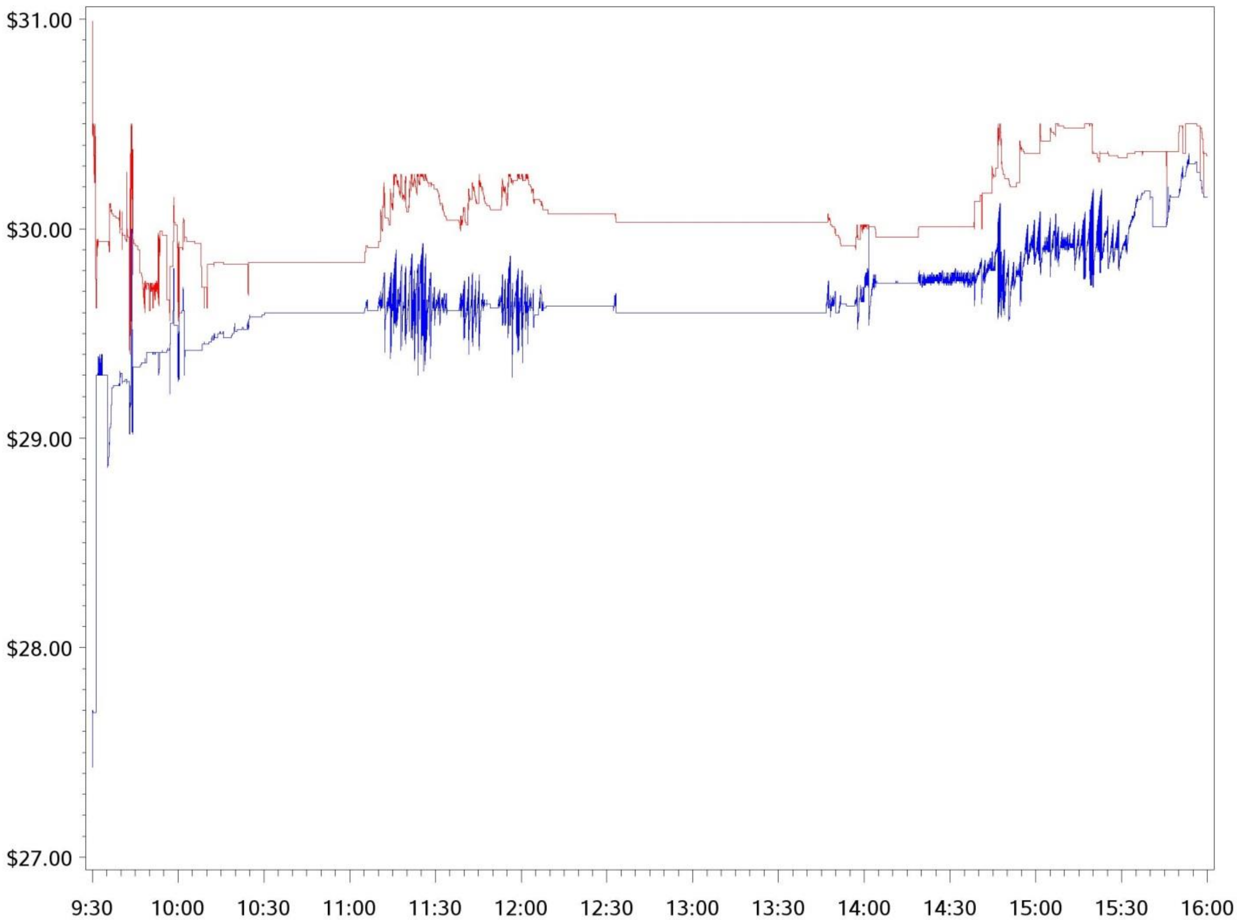


Figure 2. Variance ratios, $V_{j,J=12}$ for the National Best Bid and Offer

The sample is 150 randomly chosen U.S. stocks, over April, 2011. The wavelet variance ratio is $V_{j,J} = 2^{J-j}\nu_j^2/\nu_J^2$ where ν_j^2 is the estimated wavelet variance at level *j* (and time scale $\tau_j = 2^{j-1}\times 50\,ms$) and *J* is the largest level (longest time scale) considered. The estimates of ν_j^2 are averaged over the National Best Bid and National Best Offer. For a random-walk, $V_{j,J} = 1$. The stocks are grouped in quintiles by dollar trading volume, and the figure plots variance ratio averages within each quintile.

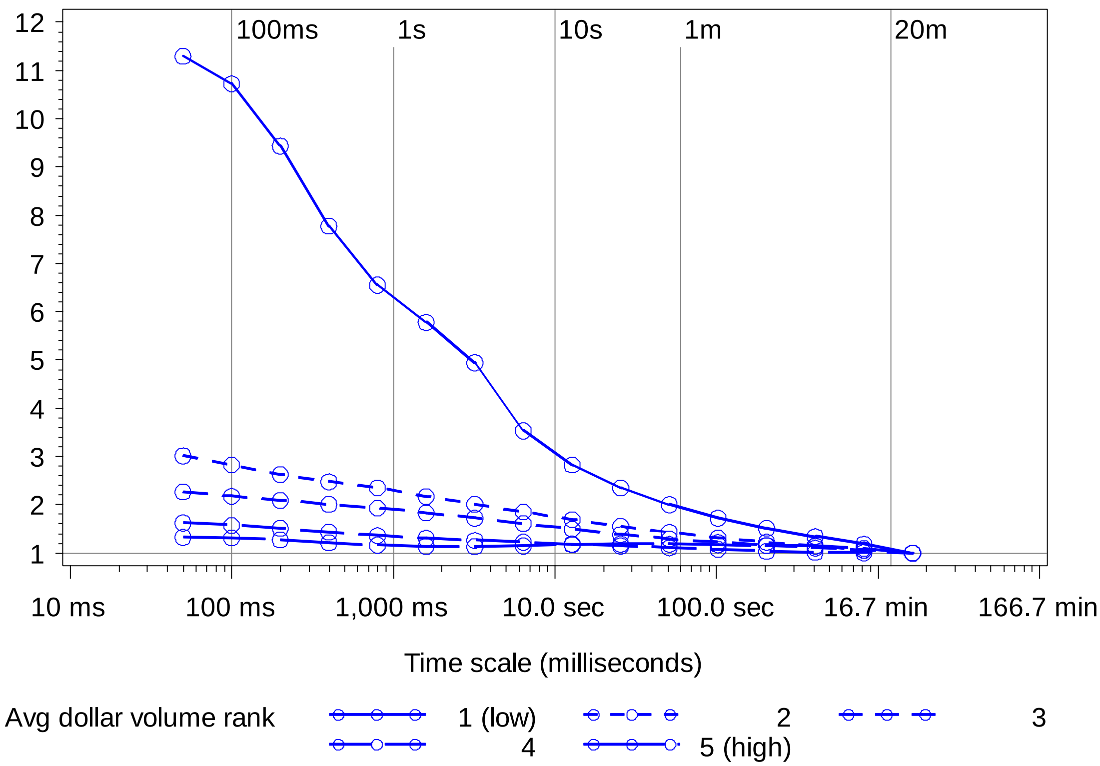


Figure 3. Wavelet correlations between the National Best Bid and National Best Offer

The sample is 150 randomly chosen U.S. stocks, over April, 2011. The wavelet correlation at level j (and time scale $\tau_j = 2^{j-1} \times 50\,ms$) is defined as $\rho_j = \nu^2_{NBB,NBO,j} \Big/ \sqrt{\nu^2_{NBB,j}\nu^2_{NBO,j}}$. The stocks are grouped in quintiles by dollar trading volume, and the figure plots the correlations averaged within each quintile.

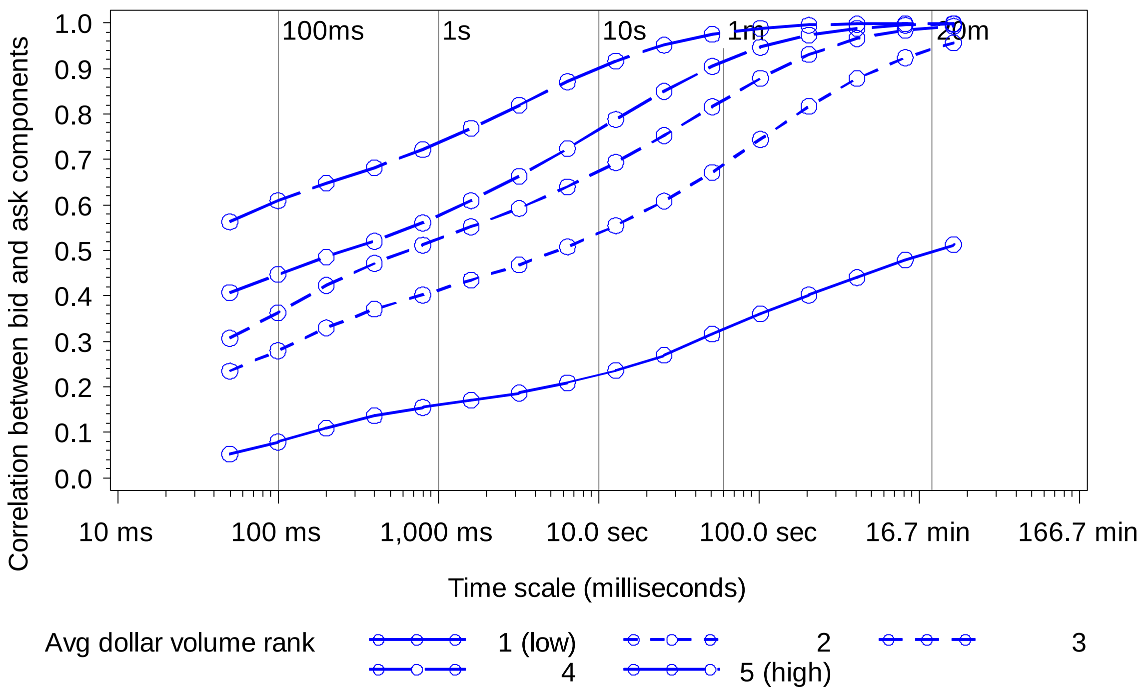
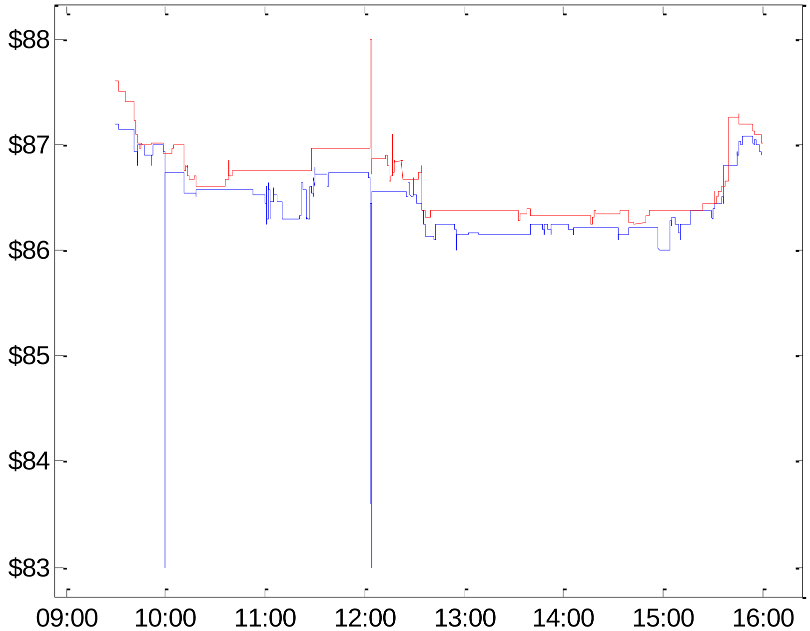


Figure 4. Bid and offer for PRK, April 6, 2001.

Panel A. National Best Bid and Offer (constructed from NYSE monthly TAQ data).



Panel B. Rough component of National Best Bid (timescale of 51.2 seconds)