Dynamic Fairness: Mobility, Inequality, and the Distribution of Prospects

Baochun Peng[1]

The Hong Kong Polytechnic University

Haidong Yuan

The Chinese University of Hong Kong

Abstract

This paper clarifies the relationship between dynamic fairness, inequality, and mobility. Suppose agents assess the degree of fairness in a society based on the distribution of expected future income, conditional on current income. Within Markovian environments characterised by stochastically monotone transition rules, in which a globally stable steady-state income distribution exists, we show that the degree of dynamic fairness is jointly determined by inequality and mobility. When mobility is held constant, inequality harms fairness; on the contrary, when inequality is held constant, mobility enhances fairness. Moreover, any particular degree of fairness can be obtained by one of many combinations of inequality and mobility, and trade-offs always exists between any two such pairs. Our results hold true with both the standard Lorenz curve and the generalised Lorenz curve and extend to multi-period settings.

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[1] School of Accounting and Finance, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, Telephone: (852) 2766 7068, Email: baochun.peng@polyu.edu.hk . We thank participants at the workshop on Advances in stochastic dominance for welfare analysis in Clermont Ferrand and seminar participants at The University of Hong Kong, The Chinese University of Hong Kong and The Hong Kong Polytechnic University for their helpful comments. The work described in this paper was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. PolyU 599213).

1. Introduction

Fairness is a widely shared concern. For example, consider the following debate.

A: Income inequality has been increasing around the world and the top 1% is getting an ever-increasing portion of the GDP. Things are certainly becoming more and more unfair.

B: The rise of inequality does not necessary pose a concern. What really matters is mobility. As long as there is enough mobility, everyone stands a good chance of becoming the top 1%, so it can be quite fair, after all.

Although the two sides may appear to be talking past each other, they are united in a common concern for fairness. Their disagreement stems from different assessments of the relative importance of inequality and mobility in dynamic fairness.

This paper provides a framework that facilitates the conduct of this debate. In particular, we clarify the relationship between inequality, mobility, and dynamic fairness. Dynamic fairness in this context is defined as the distribution of expected future incomes, conditional on the current distribution of income.

Using a new method that is able to completely separate the effects of inequality and mobility, we show that dynamic fairness is a function of inequality and mobility. When mobility is held constant, inequality harms fairness. When inequality is held constant, mobility enhances fairness. Moreover, any particular degree of fairness can be obtained by one of many combinations of inequality and mobility, and trade-offs always exists between any two such pairs.

Fairness is an assessment of the outcome of the entire society, not merely an assessment of the outcome of the particular individual making the assessment. It concerns the distribution of all likely outcomes. It is related to, but is also different from, the current distribution of income. Current income typically provides information on the likelihood of future outcomes. Nevertheless, how much future information the current information can provide depends on the degree of mobility. The higher the mobility is, the less informative current income becomes for the distribution of future income.

The relationship between asset price and the current yield of the assent can act as an analogy for this. In the famous Lucas (1978) tree model, all trees are Markovian and are stochastically identical. However, this does not imply that all trees have the same price. Instead, the price of each tree is a function of its current yield, as in a Markovian environment, the current yield of a tree summarises all of the currently available information on future yields. The distribution of the prices of the trees hinges on, and is typically different from, the distribution of their current yields. Factors that can affect the asset prices of trees include (i) how dispersed current yields are, (ii) the extent to which current yields can predict future yields, and (iii) the way future yields are discounted and evaluated.

In this light, when located in a dynamic Markovian economy, the current income of any agent acts as a summary of the currently available information concerning her future prospects. The information content also depends on the degree of the income dependence between two consecutive periods.

Completely separating inequality from mobility can be difficult. In the standard approach, the change in mobility is usually measured by the degree of the dependence between the outcomes of two consecutive periods. It follows that changes in inequality inevitably affect mobility and vice versa; thus, it is not possible to completely separate mobility from inequality. In contrast, if mobility is defined in terms of the copula of income across successive periods in a steady state, then it can be shown that inequality can be completely separated from mobility. In particular, it is straightforward to find the Markov process that generates any particular combination of inequality and mobility as its steady-state outcomes.[2] Based on this, we show that dynamic fairness, defined as the distribution of prospects, depends on mobility and inequality. More specifically, it depends on mobility positively and on inequality negatively, as when the invariant distribution is constant, a rise in mobility makes the distribution of prospects more equal, whereas when the degree of mobility is held constant, a more unequal invariant distribution of income shares also leads to a more unequal distribution of prospects. Consequently, trade-offs between equality and mobility always exist for the attainment of any particular level of dynamic fairness.

For the set of environments in which a unique and globally stable invariant distribution exists in the steady state, each steady state is characterised by (1) the steady-state distribution of income and (2) the stochastic kernel. The stochastic kernel, as shown by Peng (2014), has a one-to-one relationship with the copula. Thus, the stochastic kernel, or the copula, contains all of the information about mobility, whereas the invariant distribution contains all of the information about inequality. The complete separability of inequality and mobility ensures that it is possible to carry out the comparative statics studies of their effects on dynamic fairness by holding one constant and allowing the other to vary.

In addition, as the mobility analysis is conducted in the *rank space*, whereas the invariant distribution is characterised in the *outcome space*, the results obtained here apply to any outcome variable that 'really matters'. This might include, although it is not confined to, income, wealth, or utility. The extension to multi-period settings is also straightforward.

The literature on mobility is not as developed as the literature on inequality.[3] There are several types of mobility, including movement mobility and exchange mobility. Stochastic processes have often been used in the study of mobility, especially those involving Markov chains with

[2] See Peng (2014).
[3] For a survey of the literature on mobility, see Fields and Ok (1999).

finite state spaces. These analyses often make use of monotone Markov matrices, which capture the situation in which the economic outcomes of all agents are random and a more favourable outcome in one period leads to a stochastically more favourable conditional distribution of the random outcomes in the following period. Important examples include Dardanoni (1993) and D'Agostino and Dardanoni (2009).

Our paper builds on the framework provided by Dardanoni (1993). We differ from Dardanoni in the following respects. (1) Our results apply to general state spaces, not just discrete state spaces. (2) Dardanoni's results are based on placing greater weights on less well-off agents, whereas we apply the same weights to all agents. (3) Dardanoni compares mobility holding the stationary distribution constant. It is not clear whether Markov processes with different stationary distributions can be compared with respect to mobility. This is similar to 'parallel universes', as each is indexed by a particular income distribution. Our approach allows all of the barriers between the parallel universes to be removed.

More generally, our approach differs from the literature on mobility and inequality indexes as an index is a mapping from infinite-dimensional spaces to one-dimensional space, whereas our analysis compares distributions and mobility processes in their original infinite-dimensional spaces.

D'Agostino and Dardanoni (2009) conducted their analysis on rank space, whereas we separate the effects on outcomes and rank spaces.

Benabou and Ok (2001) introduced the study of prospects. They showed that the concavity in prospect functions played an important role in understanding the redistribution policy. In contrast to this paper, Benabou and Ok did not explicitly study fairness or the distribution of expected income conditional on current income.

The remaining parts of the paper are organised as follows. Section 2 lays out the model. Section 3 analyses the relationship between dynamic fairness, inequality, and mobility using the standard Lorenz curve. Section 4 shows that the results can be extended when the generalised Lorenz curve is used. Section 5 extends the analysis to multi-period settings. Section 6 concludes.

2. Model

Consider an economy with a given population, normalised to measure one. The income of agent i at time t is given by $y_{i,t}$, which is distributed on $\Omega_y = [0, \infty)$. The density and cumulative distribution of income are denoted $\phi_t(y)$ and $\Phi_t(y)$, respectively.[4]

Income evolves according to the following law of motion:

$$y_{i,t+1} = F(y_{i,t}, a_{i,t}),$$

where $a_{i,t}$ denotes the endowed ability, which is distributed on $\Omega_a = [0, \infty)$ and is assumed to be independently and identically distributed across agents and across time, with a known cumulative distribution function, (CDF) $\Psi(.)$. Furthermore, the distribution of ability is assumed to be atomless and has a strictly positive density everywhere. Specified in this way, the distribution of $y_{i,t+1}$ follows a Markov process and this paper investigates its steady state.

The function $F(.,.): \Omega_y \times \Omega_a \to \Omega_y$ is understood to be a reduced form, after all of the equilibrium actions of the agents have been endogenised. Changes in the economic environment,

[4] More generally, our results hold true when either Ω_y or Ω_a is an interval in $\mathbb{R}$, or even if $\Omega_y = \mathbb{R}$, provided appropriate changes are made to the way the F function is defined.

such as technological changes or policy changes, can potentially change the shape of the $F(.,.)$ function.

We shall focus on situations for which an invariant distribution exists and is globally stable. Peng (2014) identified a set of conditions for which this is the case. One of the conditions that establishes global stability is that the function $F(.,.)$ is continuous and strictly increasing, which also implies that the Markovian process is stochastically monotone.[5]

When the invariant distribution is globally stable, the system always converges to it irrespective of the initial income distribution. Consequently, although our analyses are confined to the steady state, global stability ensures that the results can be more generally applied.

Mobility

In a Markovian framework, mobility can be naturally seen as the relationship between the state variable over two consecutive periods. We focus on the copula of the system after it has reached the invariant distribution and propose to use this to characterise mobility in an economic environment.

We define a two-dimensional copula, $C: [0,1]^2 \to [0,1]$, as a joint CDF of a two-dimensional random vector, with the marginal distributions being uniform on [0,1]. Sklar's well-known theorem states that given any two-dimensional random vectors with marginal distributions $\Lambda_1(.)$ and $\Lambda_2(.)$, its joint distribution $H(.,.)$ can be written as

$$H(x_1, x_2) = C\big(\Lambda_1(x_1), \Lambda_2(x_2)\big).$$

The copula is invariant under strictly increasing transforms (Nelson, 2006; Theorem 2.4.3.).

[5] Hopenhyne and Prescott (1992) used stochastic monotonicity to ensure the existence of a unique invariant distribution.

For any given cumulative distribution $\Phi(.)$, define the rank as $\rho_t \equiv \Phi(y_t)$, such that $y_t = \Phi^{-1}(\rho_t)$. Then, for any given $\{F, \Phi\}$ pair, $y_{t+1} = F(y_t, a_t)$ can be rewritten as $\Phi^{-1}(\rho_{t+1}) = F\left(\Phi^{-1}(\rho_t), a_t\right)$ or $\rho_{t+1} = \Phi\left[F\left(\Phi^{-1}(\rho_t), a_t\right)\right]$. As the distribution of a_t is known as $\Psi(.)$, define $\alpha_t \equiv \Psi(a_t)$. With the given $\{F, \Phi\}$ pair, we can now define $K(.,.)$: $[0,1] \times [0,1] \rightarrow [0,1]$ as

$$K(\rho_t, \alpha_t) \equiv \Phi\left[F\left(\Phi^{-1}(\rho_t), \Psi^{-1}(\alpha_t)\right)\right].$$

As $\Phi(.)$ and $\Psi(.)$ are distributions with strictly positive supports, the function $K(.,.)$ inherits many characteristics of the $F(.,.)$ function. In particular, $K(.,.)$ strictly increases in both arguments. Moreover, as Φ is the invariant distribution under F, it follows that for all $\{\rho_t, \alpha_t\} \in [0,1] \times [0,1]$, the following condition is satisfied:

$$K(\rho_t, \alpha_t) = \int \mathbb{I}[K(x_1, x_2) \leq K(\rho_t, \alpha_t)] dx_1 dx_2,$$

where $\mathbb{I}[.]$ denotes the indexing function. This condition ensures that the population mass that moves into any rank is equal to the mass that moves out.

With this definition of the $K(.,.)$ function, it follows that $\rho_{t+1} = K(\rho_t, \alpha_t)$ and $y_{t+1} = F(y_t, a_t)$ can be written as

$$y_{t+1} = \Phi^{-1}\left(K\left(\Phi(y_t), \Psi(a_t)\right)\right).$$

We refer to the $K(.,.)$ function as the ***positional kernel***.

Peng (2014) showed that any Markov process that has reached its steady state can be decomposed into an invariant distribution and a positional kernel—which contains the same information as the copula on the process across two successive periods. All of the information relating to inequality is contained in the invariant distribution, whereas all of the information relating to mobility is contained in the copula.

The decomposition is also useful when working backwards, as with any combination of the invariant distribution and copula, it is straightforward to derive the Markov process that generates both as its steady-state outcomes. As by definition,

$$F(y_t, a_t) = \Phi^{-1}\left[K\left(\Phi(y_t), \Psi(a_t)\right)\right],$$

the function $F(.,.)$ is easily obtained for any given pair of the invariant distribution Φ and positional kernel $K(.,.)$.

Inequality and Mobility Orderings

We endow well-known partial orders on the space of distributions and copulas. With respect to income distributions, a distribution is said to Lorenz dominate another if its Lorenz curve is strictly above the Lorenz curve of the other distribution. A distribution that Lorenz dominates another is also more equal. With respect to copulas, a copula dominates another if it is greater than the other copula everywhere. Copula dominance corresponds to the notion of immobility.

We consider the copula $c(z,\rho)$ that lies between the independent copula $\Pi(z,\rho) = z\rho$ and the Fréchet-Hoeffding upper bound $M(z,\rho) = min[z,\rho]$. The independent copula corresponds to the case of perfect mobility, whereas the Fréchet-Hoeffding upper bound corresponds to zero mobility.

3. Distribution of Prospects

The distribution of prospects lies at the heart of our concept of dynamic fairness. At a more general level, prospect is defined as the current evaluation of future outcomes, conditional on this period's outcome. The outcome, denoted by y, is very broadly defined, ranging from the income, income share, wealth, wealth share, or level of utility.

When the planning horizon is given by $T > 0$, the prospects are given by $\sum_{t=1}^{T} \beta^t E_0(y_t|y_0)$, where the operator E_0 indicates that the expectation is conditional on the information set at time 0. The value of T could be infinity.

Dynamic fairness is concerned with the distribution of prospects in an entire society, conditional on the distribution of the past outcomes. In a Markovian steady state, this information is captured by the mapping of the rank of the outcome in the previous period to the weighted sums of the future outcomes. Moreover, this mapping consists of two components, the mapping of the past ranks to the future ranks and the mapping of the future ranks to the future outcomes. The details of these two components correspond to mobility and inequality, respectively. Taken together, the mapping from the past ranks to the expected outcomes contains exactly the information needed for the Lorenz curve of the prospects.

Single Period with Standard Lorenz Curve

Our analysis begins with the single period prospect using the standard Lorenz curve. In other words, $T = 1$ and y is defined as the income share. It turns out that the results obtained here can be extended to more general cases.

Let $\Phi: \mathbb{R}^+ \rightarrow [0,1]$ be the distribution function of the income share of the invariant distribution. Thus, $\int_0^1 \Phi^{-1}(\rho) d\rho = 1$ and the Lorenz curve of the invariant distribution is given by

$$L(z) = \int_0^z \Phi^{-1}(\rho) d\rho, z \in [0,1].$$

The prospects of an agent with the current position z are given by

$$E(y_{t+1} | \rho_t = z) = \int_0^1 \Phi^{-1}\big(K(z, \alpha)\big) d\alpha.$$

Due to stochastic monotonicity, the expected income share rises with the current income share. The Lorenz curve of the prospects can therefore be written as

$$L_y(z) = \int_0^z \int_0^1 \Phi^{-1}\big(K(\rho, \alpha)\big) d\alpha d\rho,\ z \in [0,1].$$

Using the change in variables by defining $\rho_{t+1} = K(\rho, \alpha)$, it is straightforward to see that the cumulative density of ρ_{t+1} on $(\rho, \alpha) \in [0, z] \times [0,1]$ is exactly given by the copula $c(z, \rho)$. Therefore, the Lorenz curve of the prospects can be rewritten as

$$L_y(z) = \int_0^1 \Phi^{-1}(\rho) [\partial c(z, \rho) / \partial \rho] d\rho,\ z \in [0,1].$$

The function Φ completely characterises inequality, the copula $c(z, \rho)$ completely characterises mobility, and these two can be completely separated. On this basis, the equation above implies that the distribution of prospects is a function of equality and mobility, or, putting it loosely, fairness = M (inequality, mobility), with M(.) being a functional equation. This makes it possible to examine the effects of the changes in inequality and mobility on the distribution of the prospects separately.

The class of copulas $c(z,\rho)$ that are consistent with the assumption of stochastic monotonicity lies between the independent copula $\prod(z,\rho) = z\rho$ and the Fréchet-Hoeffding upper bound $M(z,\rho) = min[z,\rho]$. With these notations, we have the following results.

Proposition 1 *(Mobility reduces the inequality of the prospects for any given distribution of income.) Consider two economies with identical steady state distributions of income shares. Suppose the copulas are represented by $c(z,\rho)$ and $\hat{c}(z,\rho)$. The corresponding Lorenz curves for the distribution of the prospects are denoted $L_y(z)$ and $\hat{L}_y(z)$, respectively. If $c(z,\rho) \geq \hat{c}(z,\rho)$ for all $(z,\rho) \in \mathbb{I}^2$, then $L_y(z) \leq \hat{L}_y(z)$ for all $z \in [0,1]$.*

Proof *Using integration by parts, the Lorenz curve for the distribution of prospects can be written as $L_y(z) = \int_0^1 [z - c(z,\rho)]\Phi^{-1}(d\rho)$. As $\Phi^{-1}(d\rho)$ is non-negative, it clearly follows that $c(z,\rho) \geq \hat{c}(z,\rho)$ implies $L_y(z) \leq \hat{L}_y(z)$.*

Recall that $c(z,\rho) = \prod(z,\rho)$ corresponds to the case of perfect mobility, whereas $c(z,\rho) = M(z,\rho)$ corresponds to the case of zero mobility. The following results show that with perfect mobility, the Lorenz curve of the distribution of prospects reaches its upper bound, whereas with zero mobility, the Lorenz curve of the distribution of prospects becomes identical to the Lorenz curve of the steady state distribution.

Proposition 2 *(The Lorenz curve of the distribution of prospects lies between the Lorenz curve of the steady state distribution and the 45 degree line.) When $c(z,\rho) = \prod(z,\rho)$, $L_y(z) = z$. When $(z,\rho) = M(z,\rho)$, $L_y(z) = L(z)$.*

Proof *When $(z,\rho) = \prod(z,\rho)$, $L_y(z) = \int_0^1 \Phi^{-1}(\rho) z d\rho = z \int_0^1 \Phi^{-1}(\rho) d\rho = z$.* When $(z,\rho) = M(z,\rho)$, $L_y(z) = \int_0^z \Phi^{-1}(\rho) d\,\rho = L(z)$.

Now, holding mobility fixed and allowing inequality to change, the following results show that the distribution of prospects changes in the same direction as inequality.

Proposition 3 (*Inequality in the distribution of prospects rises with the inequality of the income distribution.) Consider two economies with identical copulas $c(z,\rho)$ for which the steady state distributions of income shares are denoted Φ and $\widehat{\Phi}$, with $\Phi\ {}_{cx}\!\preccurlyeq \widehat{\Phi}$. If we denote the corresponding Lorenz curve for the distribution of prospects as $L_y(z)$ and $\hat{L}_y(z)$, respectively, then $L_y(z) \geq \hat{L}_y(z)$ for all $z \in [0,1]$.*

Proof As $\Phi\ {}_{cx}\!\preccurlyeq \widehat{\Phi}$, *it follows that* $\int_0^z \left[\Phi^{-1}(\rho) - \widehat{\Phi}^{-1}(\rho)\right] d\rho \geq 0$ *and* $\int_0^1 \left[\Phi^{-1}(\rho) - \widehat{\Phi}^{-1}(\rho)\right] d\rho = 0$. *As* $\Phi^{-1}(\rho)$ *and* $\widehat{\Phi}^{-1}(\rho)$ *are non-decreasing functions, they can only intersect a finite number of times, at most. The stochastic monotonicity of the Markov process implies that* $c(z,\rho)$ *is concave in* ρ *for all values of z (Nelson, 2006; Corollary 5.2.11); therefore,* $L_y(z) - \hat{L}_y(z) = \int_0^z \left[\Phi^{-1}(\rho) - \widehat{\Phi}^{-1}(\rho)\right][\partial c(z,\rho)/\partial\rho] d\rho \geq 0$ *and thus* $L_y(z) \geq \hat{L}_y(z)$ *for all* $z \in [0,1]$.

Proposition 4 (*Any pair of distributions of prospects and any mobility process—as long as some mobility exists—uniquely determines an invariant distribution that is consistent with both.) For any distribution of prospects Φ_p and any copula $c(z,\rho)$ that satisfies $c(z,\rho) > M(z,\rho)$ for some (z,ρ), there exists a unique income distribution Φ, which, together with the mobility process characterised by $c(z,\rho)$, generates Φ_p as the steady state distribution of prospects.*

Proof: *Define the set* $\Omega(\Phi_p) = \left\{\Phi^{-1} \middle| \int_0^z \Phi^{-1}(\rho) d\rho \leq \int_0^z {\Phi_p}^{-1}(\rho) d\rho, \text{ for all } z \in [0,1]\right\}$. *When endowed with the total variation norm,* $\Omega(\Phi_p)$ *is a closed and bounded subset in a Banach space. Now define the mapping as*

$$T\left(\Phi^{-1}(z) \middle| \Phi_p, c(z,\rho)\right) = \int_0^z \Phi_p^{-1}(\rho) d\rho + \int_0^1 \Phi^{-1}(\rho)[\mathbb{I}(\rho \leq z) - \partial c(z,d\rho)/\partial\rho]\, d\rho.$$

It can be verified that $T: \Omega(\Phi_p) \to \Omega(\Phi_p)$. *Now,* $0 \le |\mathbb{I}(\rho \le z) - \partial c(z,\rho)/\partial\rho| < 1$ *(Nelson, 2006; Theorem 2.2.7); therefore, for any* $\Phi, \widehat{\Phi} \in \Omega(\Phi_p)$, *we have* $\|\Phi - \widehat{\Phi}\| = \int_0^1 |\Phi^{-1}(\rho) - \widehat{\Phi}^{-1}(\rho)| d\rho > \int_0^1 \int_0^1 \left| \left(\Phi^{-1}(\rho) - \widehat{\Phi}^{-1}(\rho)\right) [\mathbb{I}(\rho \le z) - \partial c(z,\rho)/\partial\rho] \right| d\rho dz = \|T\Phi - T\widehat{\Phi}\|$. *It follows that T is a contraction mapping and, based on the Banach fixed-point theorem, there exists a unique* $\Phi \in \Omega(\Phi_p)$, *such that* $T\Phi = \Phi$. *It is straightforward to verify that this* Φ *and* $c(z,\rho)$ *together generate* Φ_p *as the steady state distribution of prospects.*

Remark This proof is based on the contraction mapping theorem. In practice, the actual distribution Φ can be therefore found recursively. Take any Φ^{-1} that is Lorenz dominated by Φ_p. According to Proposition 4, $\lim_{n\to\infty} T^n \left(\Phi^{-1}(z) \middle| \Phi_p, c(z,\rho)\right)$ will converge to Φ.

Remark: The distribution of prospects depends on inequality and mobility. The inequality in the distribution of prospects increases with the inequality of the invariant distribution and decreases with mobility. To keep the distribution of prospects unchanged, a rise in mobility needs to be compensated by a rise in the inequality of the invariant distribution. Proposition 4 makes it possible to determine this trade-off.

Remark (The equality-mobility indifference curve): Proposition 4 implies that it is possible to construct 'equality-mobility indifference curves'. For any given distribution of prospects and any given chain of mobility (completely ordered by copula dominance), Proposition 4 implies there is a corresponding chain of invariant distributions, such that for any copula, there is a corresponding invariant distribution, and all such mobility-equality combinations give the same distribution of prospects. Any mobility-equality combination 'to the north-east' of this indifference curve worsens fairness, whereas any mobility-equality combination to the 'south-west' of the indifference curve improves fairness. Moreover, a rise in inequality and mobility may lead to a more or less equal distribution of prospects, depending on their relative strengths—i.e., the side of the indifference curve on which it eventually lies.

4. Single Period with a Generalised Lorenz Curve

A possible concern for the results obtained so far is that Lorenz dominance is not able to pick up information related to the absolute levels of income. It is known that, as a criterion, Lorenz dominance is neither necessary nor sufficient for ranking social welfare. In contrast, social welfare can be ranked by the use of the generalised Lorenz curve.[6] It turns out that our results continue to apply when generalised Lorenz curves are used.

The key reason this can be done is because of our inequality mobility decomposition. Inequality is represented by the mapping from the rank space to the outcome space at any given period, whereas mobility is represented by the mapping between rank spaces across two periods. Both the standard and generalised Lorenz curves are mappings from the rank space to some outcome space. Thus, when the distribution of prospects is examined, two steps are always involved. The first is the self-mapping of the rank space onto itself across two periods, the nature of which depends on mobility alone. The second is the mapping from the rank space onto the desired outcome space, which depends on the invariant distribution alone. When the outcome space is income shares, the distribution of prospects is captured by the standard Lorenz curve. When the outcome space is income itself, the distribution of prospects is captured by the generalised Lorenz curve.

For the purpose of this section, define $\Phi: \mathbb{R}^+ \to [0,1]$ as the distribution function of the income of the invariant distribution. Thus, $\int_0^1 \Phi^{-1}(\rho) d\rho = \mu$ is the mean of the income distribution and the generalised Lorenz curve of the invariant distribution is given by $G(z) = \int_0^z \Phi^{-1}(\rho) d\rho$, $z \in \mathbb{R}^+$. By defining prospect as the expected income, rather than the expected income share, conditional on the income of the previous period, it is easy to see that the generalised Lorenz curve of prospects can be rewritten as

[6] Generalised Lorenz curve dominance is equivalent to second order stochastic dominance (Shorrocks, 1983; Thistle, 1989).

$$G_y(z) = \int_0^1 \Phi^{-1}(\rho)[\partial c(z,\rho)/\partial\rho]d\rho,\ z \in \mathbb{R}^+.$$

It is straightforward to verify that, by replacing $L(z)$ with $G(z)$ and replacing $L(z)$ with $G_y(z)$, the results obtained in Propositions 1-3 continue to hold. We summarise these results as follows.

Proposition 5 *(Mobility reduces the inequality of prospects for any given distribution of income.) Consider two economies with identical steady-state distributions of incomes. Suppose the copulas are represented by $c(z,\rho)$ and $\hat{c}(z,\rho)$, respectively. The corresponding generalised Lorenz curves for the distribution of prospects are denoted $G_y(z)$ and $\hat{G}_y(z)$, respectively. If $c(z,\rho) \geq \hat{c}(z,\rho)$ for all $(z,\rho) \in \mathbb{I}^2$, then $G_y(z) \leq \hat{G}_y(z)$ for all $z \in \mathbb{R}^+$.*

Proposition 6 *(The generalised Lorenz curve of the distribution of prospects lies between the generalised Lorenz curve of the steady state distribution and the line corresponding to a perfectly equal income distribution.) When $(z,\rho) = \prod(z,\rho)$, $L_y(z) = z\mu$. When $c(z,\rho) = M(z,\rho)$, $L_y(z) = L(z)$.*

Proposition 7 (*Inequality in the distribution of prospects rises with the inequality of the income distribution.) Consider two economies with identical copulas $c(z,\rho)$, in which the steady state distributions of income are denoted Φ and $\hat{\Phi}$, with $\Phi \ {}_{cx}\!\preccurlyeq \hat{\Phi}$. If we denote the corresponding generalised Lorenz curve for the distribution of prospects as $G_y(z)$ and $\hat{G}_y(z)$, respectively, then $G_y(z) \geq \hat{G}_y(z)$ for all $z \in \mathbb{R}^+$.*

5. Multiple-Period Planning Horizon

In this section we show that our results extend to the case in which the agents have a planning horizon of $T > 1$ periods, for which the value of T can potentially be infinite. The subjective discount factor is assumed to be given by $\beta \in (0,1)$.

Again, define $\Phi: [0,1] \to \mathbb{R}^+$ as the distribution function of the income shares of the invariant distribution, with $\int_0^1 \Phi^{-1}(\rho) d\rho = 1$. We wish to examine the distribution of prospects, which is now defined as $\sum_{i=1}^{T} \beta^i E(y_{t+i} | \rho_t = z)$, where y_{t+i} denotes the income at time $t + i$.

Based on Darsow et al. (1992) and Nelson (2006), we define the Markov process in terms of the copula and let c^s denote the copula of y_t and y_{t+s}, where $c^s = c^{s-n} * c^n$ for all $0 < n < s$, and where the product $c^{s-n} * c^n$ is defined as $c^{s-n} * c^n(u, v) = \int_0^1 D_2 c^{s-n}(u, t) D_1 c^n(t, v) dt$, with D_i denoting the partial derivative with respect to the ith argument. Then, the Lorenz curve of prospects with T periods and a discount factor of β can be written as

$$L_y(z; T, \beta) = \sum_{i=1}^{T} \beta^i \int_0^1 \Phi^{-1}(\rho) \left[\partial c^i(z, \rho) / \partial \rho \right] d\rho, \; z \in [0,1].$$

With this, we now show that our results extend to multiple period cases.

If we define $x(y; T, \beta, \mu) = \mu \sum_{i=1}^{T} \beta^i y$, the following results are immediately obtained.

Lemma 1 *For any given distribution of an income y with the distribution function given by Φ, the distribution function of the income share of $x(y; T, \beta, \mu)$ is also given by Φ.*

Proof *It is easily verified, as* $x(y;T,\beta,\mu)=\mu\sum_{i=1}^{T}\beta^{i}y=\mu\left[\frac{\beta-\beta^{T}}{1-\beta}\right]y$*. As* $\mu\left[\frac{\beta-\beta^{T}}{1-\beta}\right]$ *is a constant, the result immediately follows.*

Proposition 8 *If* $\hat{c}(u,v)>c(u,v)$ *for all* $(u,v)\in[0,1]^2$*, then* $\hat{c}^{s}(u,v)>c^{s}(u,v)$ *for all* $s>1$.

Proof *The proof consists of two steps. In Step 1, we show that if the copulas* $c^{s-1}(u,v)$ *and* $c(u,v)$ *are both concave functions of u and v, then their product* $c^{s}=c^{s-1}*c$ *is also a concave function of u and a concave function of v. Using integration by parts, we obtain* $c^{s}=\int_0^1 D_2c^{s-1}(u,t)D_1c(t,v)dt=\int_0^1 c(t,v)[-D_{22}c^{s-1}(u,t)]dt$*. As stochastic monotonicity implies that* $c(u,v)$ *is a concave function of u and a convvcave function of v (Nelson, 2006; Corollary 5.2.11), then for any* $\eta>1$*, it follows that* $c(t,\eta v)\leq\eta c(t,v)$*. Thus,* $c^{s}(t,\eta v)=\int_0^1 c(t,\eta v)[-D_{22}c^{s-1}(u,t)]dt\leq=\eta\int_0^1 c(t,v)[-D_{22}c^{s-1}(u,t)]dt=\eta c^{s}(t,v)$*, so* c^{s} *is indeed concave. In Step 2, we show that if* $\hat{c}^{s-1}(u,v)>c^{s-1}(u,v)$ *for some s,* $\hat{c}(u,v)>c(u,v)$*, and all of these functions are concave, then* $\hat{c}^{s}=\hat{c}^{s-1}*\hat{c}>c^{s-1}*c=c^{s}$*. To see this, write* $\hat{c}^{s}-c^{s}=\hat{c}^{s-1}*\hat{c}-c^{s-1}*c=(\hat{c}^{s-1}-c^{s-1})*\hat{c}+c^{s-1}*(\hat{c}-c)$*. Now,* $c^{s-1}*(\hat{c}-c)=\int_0^1 D_2c^{s-1}(u,t)[D_1\hat{c}(t,v)-D_1c(t,v)]dt$*. Concavity ensures that* $D_2c^{s-1}(u,t)$ *is non-increasing and* $\hat{c}(u,v)>c(u,v)$ *ensures that* $\int_0^1 D_2c^{s-1}(u,t)[D_1\hat{c}(t,v)-D_1c(t,v)]dt>0$*. Similarly,* $(\hat{c}^{s-1}-c^{s-1})*\hat{c}>0$ *can also be established by symmetry. Therefore,* $\hat{c}^{s}>c^{s}$*. By combining steps 1 and 2 and using induction, we obtain* $\hat{c}^{s}(u,v)>c^{s}(u,v)$ *for all* $s>1$*.*

Proposition 8 allows us to extend our previous results to the multi-period setting.

Proposition 9 *(Mobility reduces the inequality of prospects for any given distribution of income shares.) Consider two economies with identical steady state distributions of incomes. Suppose* *that the one-period copulas are represented by* $c(z,\rho)$ *and* $\hat{c}(z,\rho)$*, respectively. Then, the corresponding Lorenz curves for the distribution of prospects are denoted* $L_y(z;T,\beta)$ *and* $\hat{L}_y(z;T,\beta)$*, respectively. If* $c(z,\rho) \geq \hat{c}(z,\rho)$ *for all* $(z,\rho) \in \mathbb{I}^2$*, then* $L_y(z;T,\beta) \leq \hat{L}_y(z;T,\beta)$ *for all* $z \in [0,1]$.

Proof: *Observe that the T-period prospect function* $L_y(z;T,\beta)$ *is similar to a weighted sum of a single-period prospect, after adjusting the appropriate copulas. The stochastic monotonicity of the Markov process implies that* $c(z,\rho)$ *is concave in* ρ *for all values of* z *(Nelson, 2006; Corollary 5.2.11). Based on Lemma 1 and Propositions 1 and 8, the result immediately follows.*

Proposition 10 *(The T-period Lorenz curve of the distribution of prospects lies between the Lorenz curve of the steady state distribution and the line corresponding to a perfectly equal income distribution.) When* $(z,\rho) = \prod(z,\rho)$*,* $L_y(z;T,\beta) = z\mu$*. When* $c(z,\rho) = M(z,\rho)$*,* $L_y(z;T,\beta) = L(z)$.

Proof: *It is straightforward to verify that* $\prod * \prod = \prod$ *and* $\mathrm{M} * \mathrm{M} = \mathrm{M}$*. Based on Proposition 2 and the similarity between the T-period prospect function* $L_y(z;T,\beta)$ *and a weighted sum of a single-period prospect, after adjusting the appropriate copulas the result immediately follows.*

Proposition 11 (*For any given mobility process that is concave, the inequality in the distribution of prospects rises with the inequality of the income distribution.) Consider two economies with identical one-period copulas* $c(z,\rho)$*, for which the steady state distributions of income are denoted* Φ *and* $\widehat{\Phi}$*, with* $\Phi \preccurlyeq_{cx} \widehat{\Phi}$*. We denote the corresponding Lorenz curves for the distribution of prospects as* $L_y(z)$ *and* $\hat{L}_y(z)$*, respectively. If* $c(z,\rho)$ *is concave in* ρ *for all values of* $z \in \mathbb{R}^+$*, then for their two corresponding T-period Lorenz curves of prospects,* $L_y(z;T,\beta) \leq \hat{L}_y(z;T,\beta)$ *for all* $z \in [0,1]$.

Proof: *As the T-period prospect function $L_y(z;T,\beta)$ is similar to a weighted sum of a single-period prospect after adjusting the appropriate copulas, by using Proposition 3, the result immediately follows.*

Remark Using Propositions 5-11, it is straightforward to show that our results also extend to multiple periods with generalised Loren curves. The extension from the standard Lorenz curve to the generalised Lorenz curve hinges the independence of the mapping from the rank space to the outcome space to the mobility structure. The extension from one-period to multiple-period prospects hinges on Proposition 8, which allows single-period copula dominance to be turned into multiple-period copula dominance under the additional assumption of concavity.

6. Conclusion

This paper provides a general framework for analysing the relationship between inequality, mobility, and dynamic fairness. In Markovian economic environments, prospects, defined as the present discounted value of likely future incomes as a function of current income, depend on current income and the nature of mobility. By defining dynamic fairness as the distribution of prospects ranked by the criterion of Lorenz dominance or generalised Lorenz dominance, it is shown that for any given level of inequality, dynamic fairness rises with mobility; in contrast, for any given level of mobility, dynamic fairness falls with inequality. Moreover, for any given level of dynamic fairness, it is possible to identify the exact nature of the inequality-mobility trade-off.

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