A theory of robust experiments for choice under uncertainty

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Running Example: Ellsberg Single Urn

- Ellsberg (1961) asks the reader to "imagine an urn *known* to contain 30 red balls and 60 black and yellow balls, the latter in *unknown* proportion." (*Emphasis* added.)

- Ball to be drawn from the urn.

Pblm 1 Choose between

- b_R – bet pays $\begin{cases} \$100 & \text{if ball drawn is red} \\ \$0 & \text{otherwise} \end{cases}$
- b_B – bet pays $\begin{cases} \$100 & \text{if ball drawn is black} \\ \$0 & \text{otherwise} \end{cases}$

Running Example: Ellsberg Single Urn

- Ellsberg (1961) asks the reader to "imagine an urn *known* to contain 30 red balls and 60 black and yellow balls, the latter in *unknown* proportion." (*Emphasis* added.)

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Pblm 2 Choose between

- b_{RY} – bet pays $\begin{cases} \$100 & \text{if ball drawn is red or yellow} \\ \$0 & \text{otherwise} \end{cases}$
- b_{BY} – bet pays $\begin{cases} \$100 & \text{if ball drawn is black or yellow} \\ \$0 & \text{otherwise} \end{cases}$

Violation of "probabilistic sophistication"

- Common response $b_R \succ b_B$ and $b_{BY} \succ b_{RY}$ interpreted by Ellsberg as saying participant is "simply not acting 'as though' they assigned numerical or even qualitative probabilities to the events in question."

- Ellsberg has in mind probability model $(C, \mathcal{A})$, where
  - Consequence matrix

$$C = \begin{array}{c} \\ b_R \\ b_B \\ b_{RY} \\ b_{BY} \end{array} \begin{array}{c} \begin{array}{ccc} s_R & s_B & s_Y \end{array} \\ \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 100 & 0 & 100 \\ 0 & 100 & 100 \end{bmatrix} \end{array},$$

  - Admissible parameters $\mathcal{A}$ is collection of pairs (u, p):
    1. u is (Bernoulli) utility function with $u(0) < u(100)$.
    2. p is prob. with $p(s_R) = \frac{1}{3}$, $p(s_B) = q$ and $p(s_Y) = \frac{2}{3} - q$, for some q, $1/90 \leqslant q \leqslant 59/90$.

An alternative version

- Suppose participant has in mind probability model $(C', \mathcal{A}')$:
- Consequence matrix

$$C' = \begin{array}{c} \\ b_R \\ b_B \\ b_{RY} \\ b_{BY} \end{array} \begin{array}{c} \begin{array}{cccc} s_R & s_B & s_Y & s^* \end{array} \\ \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 100 & 0 & 100 & 0 \\ 0 & 100 & 100 & 100 \end{bmatrix} \end{array},$$

- Admissible parameters $\mathcal{A}'$ is collection of pairs (u, p') for which there is $(u, p) \in \mathcal{A}$ with $p'(s) = p(s)$ for all $s \in \{s_R, s_B, s_Y\}$, thus making $p'(s^*) = 0$.

(Statistically Equivalent) Versions

Consider the alternative model $(C', \mathcal{A}')$.

- Each admissible parameter (u, p') maps each bet b to a lottery $\ell_b^{up'}$ over utilities $(u(0), u(100))$.

  E.g. (u, p') with $p' = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ maps bet b_R to lottery $\left(\frac{2}{3}, \frac{1}{3}\right)$ over the utilities $(u(0), u(100))$.

- Let $\left(\ell_{b_R}^{up'}, \ell_{b_B}^{up'}, \ell_{b_{RG}}^{up'}, \ell_{b_{BG}}^{up'}\right)$ denote the vector of lotteries over bets induced by (u, p').

Versions

$(C, \mathcal{A})$ and $(C', \mathcal{A}')$ are called *versions* if the set of vectors of lotteries over bets induced from the parameters in $\mathcal{A}$ is equivalent to those induced from the parameters in $\mathcal{A}'$.

Experiments in Belief Form

Salient Aspects of an Experiment $(C, \mathcal{A})$.

- Bets $B = \{b_R, b_B, b_{RG}, b_{BG}\}$.
- Possible consequences of a bet $C_b = \{0, 100\}$
- Admissible parameters $\mathcal{A}$ - defines the theory being tested, e.g., expected value, CARA, etc.

Sample Space $S = \{s_R, s_B, s_G\}$:

- **less salient, hypothetical, subjective**.

‘Perturbed’ Beliefs

Recall in *every* version $(C', \mathcal{A}')$ of $(C, \mathcal{A})$:

$b_R \succ b_B$ and $b_{BG} \succ b_{RG}$ cannot be induced by any $(u, p') \in \mathcal{A}'$.

- Consider, however, an admissible parameter $(u, p') \in \mathcal{A}'$ with $p' = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ and the inadmissible belief $\hat{p} = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right)$
- Since the inadmissible parameter $(u, (1-\varepsilon)p' + \varepsilon\hat{p})$ induces the preferences

$$b_{BG} \succ b_{RG} \succ b_R \succ b_B \text{ for all } \varepsilon \in (0,1) ,$$

  we say this inadmissible preference ordering is ε-admissible.
- As we can take any belief $\hat{p}$ over the sample space S', ε-admissibility means we can consider small perturbations of any admissible belief in any direction in $\Delta(S')$.

Robust Experiments

- A preference ordering over bets $\succsim$ is ε-admissible in $(C', \mathcal{A}')$ if for some $(u', p') \in \mathcal{A}'$, some $\hat{p} \in \Delta(S')$ and some $\varepsilon \in (0,1)$, the parameter $(u', (1-\delta)p' + \delta\hat{p})$ induces $\succsim$ for all $\delta \in (0, \varepsilon)$.

- A preference ordering over bets is *weakly-admissible* if it is ε-admissible for *some* version of the experiment.

- An inadmissible preference ordering is *robustly inadmissible* if it is not weakly-admissible.

- An experiment is *robust* if every inadmissible preference ordering is robustly inadmissible.

Characterizing Robustness

For any pair of distinct $\succsim$ and $\succsim'$ over a set of bets B,
we say that $\succsim$ is ***finer*** than $\succsim'$ if for any pair of bets $b, \hat{b} \in B$,

$b \succ' \hat{b}$ implies $b \succ \hat{b}$ (i.e., strict preferences are preserved).

Similarly, we say $\succsim$ is ***coarser*** than $\succsim'$ if $\succsim'$ is *finer* than $\succsim$.

Theorem 1

An inadmissible preference ordering is robustly inadmissible if and only if no coarser preference ordering is admissible.

Corollary 1.2

An experiment in belief form is robust if and only if every preference ordering that is finer than an admissible preference ordering is also admissible.

“Robustifying” Ellsberg

Consider slightly perturbed consequence matrx:

$$\hat{C} = \begin{array}{c} \\ b_R \\ b_B \\ b_{RY} \\ b_{BY} \end{array} \begin{array}{c} \begin{array}{ccc} \hat{s}_R & s_B & s_Y \end{array} \\ \begin{bmatrix} 99 & 0 & 0 \\ 0 & 100 & 0 \\ 100 & 0 & 100 \\ 0 & 100 & 100 \end{bmatrix} \end{array},$$

Notice for any admissible preference ordering $\succsim$, we have:

$$b_R \succsim b_B \Rightarrow b_{RY} \succ b_{BY}$$

Thus for the inadmissible preference pattern

$$b_R \,\hat{\succ}\, b_B \text{ and } b_{BY} \,\hat{\succ}\, b_{RY}$$

no coarsening is admissible.

What about other Experiments?

Consider Allais Common Consequence.

Problem 1: Choose between

$$b_1 - \{\ \$1m \quad \textit{for sure}$$

$$b_2 - \begin{cases} \$5m & \textit{with prob. } 0.10 \\ \$1m & \textit{with prob. } 0.89 \\ \$0 & \textit{with prob. } 0.01 \end{cases}$$

Problem 2: Choose between

$$b_3 - \begin{cases} \$1m & \textit{with prob. } 0.11 \\ \$0 & \textit{with prob. } 0.89 \end{cases}$$

$$b_4 - \begin{cases} \$5m & \textit{with prob. } 0.10 \\ \$1m & \textit{with prob. } 0.90 \end{cases}$$

Common response $b_1 \succ b_2$ *and* $b_4 \succ b_3$.

- But, how do we handle Allais Common Consequence?
  - There is not a clear cut sample space.
- Yes, it's all in your head!
  - That's our point: sample space is hypothetical.

Experiment $(C, \mathcal{A})$ capturing Allais.

- Take sample state $S = \{s_1, s_2, s_3\}$ with corresponding consequence matrix and unique admissible belief

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 0 & 1 \\ 1 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{100} \\ \frac{89}{100} \end{bmatrix}.$$

  Thus, $\mathcal{A}$ is the set of pairs (u, p) where u is a utility function satisfying $u(0) < u(1) < u(5)$

- Notice coarser preference $b_1 \sim b_2 \succ b_4 \sim b_3$ is admissible.

  e.g. $u(5) = 1\,, u(1) = \frac{10}{11}\,, u(0) = 0\,.$

- Thus by Corollary 1.2, Allais CC is not robust!
  - Similar problem to Ellsberg (too many indifferences).

Experiment $(C, \mathcal{A})$ capturing Allais.

- Take sample state $S = \{s_1, s_2, s_3\}$ with corresponding consequence matrix and unique admissible belief

$$C = \begin{bmatrix} 1 & 1 & 1 \\ \color{red}{\mathbf{6}} & 0 & 1 \\ 1 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix}, \quad p = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{100} \\ \frac{89}{100} \end{bmatrix}.$$

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Conclusions:

1. We have formulated a rigorous definition of a robust experiment and robust inadmissibility of a preference ordering.

2. **Key result:**
   If inadmissible preferences can be made admissible by coarsening, then the inadmissibility is not robust.

3. We have shown how to robustify some experiments.