On the Ellsberg paradox

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Abstract: This paper proposes a model of expected utility maximization which accounts for the Ellsberg paradox and for Machina's extension of it. In the model, decision makers use a commutative ring in which the real numbers are embedded as a subring, and they do so in order to decompose their beliefs into 'ambiguous' or 'unambiguous' parts – with unambiguous beliefs being defined on the reals, and ambiguous beliefs defined otherwise. Decision makers whose beliefs are formed on the ring and who maximize expected utility on it, are then shown to behave in ways that are predicted by the Ellsberg paradox. The major paradoxical cases in Ellsberg's seminal paper are resolved, along with some additional cases owed to Machina and Blavatskyy. Furthermore, it's shown that Ellsberg's reduced form solution to his own paradox is implied by the model.

Keywords: Ellsberg paradox, ambiguity aversion, commutative ring

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0. Introduction

In his seminal contribution to the literature on ambiguity, Ellsberg (1961) proposed a number of cases in which decision makers behave in ways that contravene the postulates of subjective expected utility theory. As is well known, repeated experimental evidence has shown that decision makers often act in the ways that he predicted they would (Machina & Siniscalchi, 2014, §13.4.2).

In this paper, we characterize decision makers' belief formation in a novel way in order to account for Ellsberg's 'paradox'. We do so by allowing decision makers to use a ring of numbers – first introduced by Izhakian & Izhakian (2014) – that embeds and extends the reals.[1] These extended numbers operate in some ways like the real numbers, but they also have properties which capture the idea that decision makers think about ambiguity in ways that can't be entirely captured by the field of reals. In particular, the ring allows decision makers to decompose their beliefs into 'ambiguous' and 'unambiguous' parts, which it's not possible to do when only the real numbers are available. They then incorporate these component values into their decision making. Decision makers who maximize a form of expected utility defined over this ring of numbers – holding some beliefs to be ambiguous and some not – are then able to generate well-defined preference orderings over the choices before them. Moreover, they behave in ways that are consistent with the Ellsberg paradox cases – including extensions of those cases, such as Machina's (2009) 'reflection' and '50:51' examples, and Blavatskyy's (2013) twist of the former. This ability to explain decision makers' behaviour in these further paradoxical cases is in contrast to some other well-known solutions to the paradox which have been proposed over the years, such as Choquet expected utility theory, maxmin expected utility, and prospect theory (see the discussions in Machina, 2009, and Baillon, l'Haridon, & Placido, 2011).

In order to explore this set of propositions, the paper is structured in the following way. The first section introduces the ring and describes the essentials of its algebraic structure and its order relations. The second section describes the way in which decision makers' beliefs are formed on the ring. In this section, we show how decision makers decompose their beliefs into 'ambiguous' and 'unambiguous' parts, and we look at the implications of this decomposition for the relative ordering of, and for the additivity of, beliefs. The third section examines how these beliefs impact on decision makers' optimizing behaviour – in particular, we examine the way in which decision makers maximize expected utility when their beliefs are defined on the ring. In this section we're able to show that the (somewhat neglected) decision rule that Ellsberg postulated in his 1961 paper is implied by our model of expected utility maximization. In section four, the model developed in the previous sections is used to account for a number of paradoxes presented by Ellsberg and Machina. Specifically, we account for

[1] A ring is an algebraic structure composed of a set of elements with two binary operations – addition and multiplication – in which a multiplicative inverse is not necessarily defined for each (non-zero) element. The ring with which economists will be most familiar is the set of 2x2 matrices – where the subset of elements which lack inverses is the set of singular matrices.

the three best known cases that appear in Ellsberg's 1961 article – namely, the 3- and 4-colour problems, and the 2-urn problem – and we also account for Machina's 'reflection example', Blavatskyy's twist of it, and Machina's 50:51 example. In the final section, we discuss the results of the preceding analysis, and conclude.

1. The ring

The ring, E, is composed of elements that have the form: $a + b\acute{e}$, where $a, b \in \mathbb{R}$, and $\acute{e}$ is an operator on elements, as defined below.[2] The first part of the element, $a + b\acute{e}$, is referred to as the real part, and the second as the é-real part. The intuition here is that the real parts of é-real numbers are used to characterize situations that decision makers regard as unambiguous, whilst the é-real parts are used to capture situations that are regarded as ambiguous.

Addition (+) and multiplication (· or juxtaposition) on E are defined as follows:

$$(a + b\acute{e}) + (c + d\acute{e}) = (a + c) + (b + d)\acute{e}$$

$$(a + b\acute{e}) \cdot (c + d\acute{e}) = (ac) + (ad + bc + bd)\acute{e}$$

Multiplication on the ring implies that the é-real numbers 'absorb' the reals – i.e., when a pure real number multiplies a pure é-real number, the product is a pure é-real number (and when one pure é-real number multiplies another, that product is also purely é-real). This captures the intuitively plausible idea that ambiguity propagates at the expense of unambiguity through multiplication.

E can be equally as well represented by putting its elements in matrix form and subjecting them to standard matrix algebra, in which case we have:

$$\mathcal{M}(a + b\acute{e}) = \begin{pmatrix} a & 0 \\ b & a + b \end{pmatrix}$$

We note that the first column of the matrix gives the real and é-real parts of the number, and the fourth quadrant gives what we call the number's 'right angle' (or 'grid') value. The additive and multiplicative identity elements, along with element associated with $\acute{e}$ are, accordingly:

$$\mathcal{M}(0 + 0\acute{e}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{M}(1 + 0\acute{e}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{M}(0 + 1\acute{e}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

[2] In Izhakian & Izhakian (2014), the ring is denoted $\mathbb{PH}$ and the operator as $\wp$. Elements of the ring are referred to as "phantom numbers". We prefer to keep the nomenclature that we've developed independently (but belatedly) since it seems closer to the standard usage in the literature on hypercomplex numbers – in particular, the use of 'e' is adopted and adapted from Yaglom (1968, ch.1, §5).

E is a commutative ring with unity since it satisfies the following conditions. Addition on E associates and commutes, and multiplication also associates and commutes; multiplication distributes over addition to the right and to the left. E has the additive identity element: $0 = 0 + 0é$; and the multiplicative identity element: $1 = 1 + 0é$. The additive inverse of $a + bé$ equals $-a - bé$; and the multiplicative inverse of $a + bé$ equals $\left(\frac{1}{a}\right) - \left(\frac{b}{a^2+ab}\right) é$ when $a \neq 0$ and $a \neq -b$. There is no defined multiplicative inverse of $a + bé$ when $a = 0$ or $a = -b$ (the determinant of the matrix form of $a + bé$ equals zero when $a = 0$ or $a = -b$). However, we note that: $(bé) \cdot \left(\frac{1}{b}\right) é = 1é = é$, which is idempotent.

E embeds $\mathbb{R}$, which is to say, there's an injective ring homomorphism: $\sigma: \mathbb{R} \rightarrow E$ with: $\sigma(a) = a + 0é, \sigma(a + b) = (a + b) + 0é, \sigma(a \cdot b) = a \cdot b + 0é, \sigma(1) = 1 + 0é$ and $\ker(\sigma) = \{0\}$. Notably, also: $\sigma(-a) = -\sigma(a)$, and $\sigma\left(\frac{1}{a}\right) = \frac{1}{\sigma(a)}$. Thus, E preserves the inverse operations of the reals.

So far, we've accounted for the algebraic structure of E. Its order structure is defined by:

$$a + bé = c + dé \ \Leftrightarrow a = c \ and \ b = d$$

$$a + bé < c + dé \Leftrightarrow \ a < c \ and \ b < d$$

Hence, E has the product order. However, we can define the 'right-angle' or 'grid' value of each element of E as follows:

$$\lfloor a + bé \rfloor = a + b \in \mathbb{R}$$

We recall that the right angle value of an é-real number is given by the fourth quadrant entry of the matrix form of that number. The elements of E are completely preordered by their right angle values.

The existence of right angle values gives rise to right angle (or grid) addition since:

$$\lfloor a + bé \rfloor + \lfloor c + dé \rfloor = (a + b) + (c + d) = \lfloor (a + c) + (b + d)é \rfloor = \lfloor (a + bé) + (c + dé) \rfloor$$

The existence of right angle values and right angle addition is useful in characterizing the epistemic evaluations that decision makers make.

2. Beliefs

The epistemic evaluations – or beliefs – of decision makers are structured as follows: given a state space, Ω, and an associated algebra, 2^Ω (Ω, finite), and given that $A, \ B, ... \in 2^\Omega$ denote events, there's a composite mapping: $\mu_é = \ \hat{\mu} \circ \lfloor \mu_é \rfloor$, with $\lfloor \mu_é \rfloor: 2^\Omega \rightarrow \mathbb{R}$ and $\hat{\mu}: \mathbb{R} \rightarrow E$, with $\lfloor \mu_é \rfloor$ satisfying: $0 \leq \lfloor \mu_é(A) \rfloor \leq 1$; $\lfloor \mu_é(A \cup B) \rfloor = \lfloor \mu_é(A) \rfloor + \lfloor \mu_é(B) \rfloor = \lfloor \mu_é(A) + \mu_é(B) \rfloor$ when $A \cap B = \emptyset$; and

$\lfloor \mu_{\acute{e}}(\Omega) \rfloor = 1$; whilst $\hat{\mu}$ ensures that: $\lfloor \mu_{\acute{e}}(A) \rfloor = \lfloor a + b\acute{e} \rfloor$ and $a, b \geq 0$. Thus, $\mu_{\acute{e}}(A)$, can be thought of as the composition of a probability measure defining the right angle values of beliefs, $\lfloor \mu_{\acute{e}}(A) \rfloor$, along with a decomposition of those beliefs into real and é-real parts (or components), $a + b\acute{e}$. Our interpretation is that the real part is the 'unambiguous' component of belief, and the é-real part is the 'ambiguous' component.

That being said, $\mu_{\acute{e}}$ isn't itself a probability measure. To see this, we can consider Ellsberg's 3-colour paradox (which is described in the first table below). In this problem, there's a single urn containing 90 balls of three different colours: red, black, and yellow. There are 30 red balls, and the other 60 balls are either black or yellow in unknown proportions. A reasonable set of beliefs – in accord with the principle of indifference – is as follows (where beliefs with an $\acute{e}$ attached are 'ambiguous', and those *sans*-$\acute{e}$, are 'unambiguous'): $\mu_{\acute{e}}(\emptyset) = 0$, $\mu_{\acute{e}}(r) = \frac{1}{3}$, $\mu_{\acute{e}}(b) = \mu_{\acute{e}}(y) = \frac{1}{3}\acute{e}$, $\mu_{\acute{e}}(r \cup b) = \mu_{\acute{e}}(r \cup y) = \frac{1}{3} + \frac{1}{3}\acute{e}$, $\mu_{\acute{e}}(b \cup y) = \frac{2}{3}$, and, $\mu_{\acute{e}}(\Omega) = 1$. We note that the right angle values of beliefs are non-negative, additive, and sum to unity over Ω. However, we also note, for example, that $\mu_{\acute{e}}(b) + \mu_{\acute{e}}(y) = \frac{2}{3}\acute{e} \neq \frac{2}{3} = \mu_{\acute{e}}(b \cup y)$, and $\mu_{\acute{e}}(b) = \mu_{\acute{e}}(y) = \frac{1}{3}\acute{e} \nleq \frac{2}{3} = \mu_{\acute{e}}(b \cup y)$. Hence, we see that, while the right angle values satisfy the probability laws, the component values do not.

Partly, this has to do with the fact that, since the real and é-real parts of numbers are incommensurable, E is only partially ordered – consequently, the epistemic evaluations of decision makers, defined on E, are also only partially ordered. In adopting an algebra which has this kind of ordering over its elements, we're following the suggestion of Keynes, (1921, ch.III, §14), who argued: "I maintain, then, in what follows, that there are some pairs of probabilities between the members of which *no* comparison of magnitude is possible; that we can say, nevertheless, of some pairs of relations of probability that the one is greater and the other less, although it is not possible to measure the difference between them [along a single dimension]; and that in a very special type of case, to be dealt with later, a meaning can be given to a *numerical* comparison of magnitude [this is the case of canonical, real-valued probabilities]". Thus, because we're effectively following up on Keynes' implicit suggestion that we locate beliefs on a partially ordered ring of numbers, we find that beliefs aren't totally ordered, as are canonical probabilities (by which we mean probabilities defined in the sense of Kolmogorov, 1950, ch.I, §1, and ch.II, §1, Axioms I-VI).

Apart from the issue of incommensurability and the consequent partial ordering of 'probabilities' (in Keynes', not Kolmogorov's, sense of the word), we also find that epistemic evaluations aren't strictly additive. This reflects the fact that some information about the likelihood of events isn't entirely reducible to information about those events' 'constituent parts', and that extra information is only revealed when the parts form unions. For example, in the Ellsberg 3-colour problem, knowing that, say, $\mu_{\acute{e}}(b) = \mu_{\acute{e}}(y) = \frac{1}{3}\acute{e}$, isn't enough for the decision maker to know that it's reasonable for her to believe that $\mu_{\acute{e}}(b \cup y) = \frac{2}{3}$ unambiguously (this is a reasonable belief for her to hold since $\frac{\#(b \cup y)}{\#(r \cup b \cup y)} = \frac{60}{90}$, so that, by direct inference, we have: $\mu_{\acute{e}}(b \cup y) = \frac{2}{3}$). This knowledge is only realized upon consideration of the union itself. And it's this feature of the way that reasonable belief formation operates which is reflected in the non-additivity of decision makers' epistemic evaluations.

Thus, to recapitulate: decision makers' beliefs aren't canonical probabilities. Despite this, those beliefs are sufficiently coherent for them to form the basis of rational decisions, as we'll now see.

3. Decisions

There are three key assumptions that we make about the decision maker's decision rules.

Assumption 1: Beliefs depend on the way acts partition the state space. For any given act, there's a unique partition which generates a bijection between events and the consequences of the act. Decision makers use this partition to determine their component beliefs when calculating the expected utility of the act.

An assumption such as this – whereby the decision maker settles upon a particular partition of the state space when considering how to value an act – is required since beliefs aren't entirely consistent across partitions, as the previous section has shown. The particular partition chosen is a natural one to make in light of the next assumption.

Assumption 2: Partition-specific expected utility maximization on E. The decision maker maximizes a particular form of expected utility whose value lies on the ring:

$$\sum_{A \in \pi^f} \mu_{\acute{e}}(A) \cdot u(x(A))$$

Where: f denotes an act, which maps states to consequences; π^f is the partition of the state space, Ω, generated by f as specified in the previous assumption; A is an event of the given partition; x denotes a consequence, or outcome, with $x \in X$; $x(A)$ is the consequence of an act for a given event, A; $u(x)$ is the utility of a consequence; and, $\mu_{\acute{e}}(A)$ is the belief in the likelihood of an event's occurring. The (expected) value of an act is denoted by v.

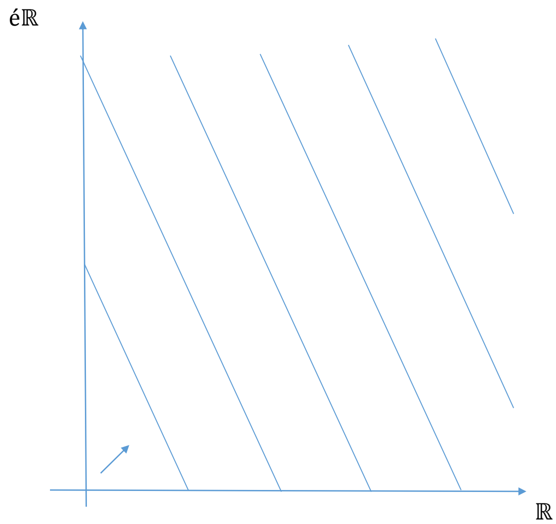
For our limited purposes here, we assume that utilities are defined on $\mathbb{R}$, whilst beliefs are given on E (we recall that $\mathbb{R}$ is a subring of E). The utility function has the usual properties (and utilities are state independent and are normalized to take positive values). As $u(.)$ and $\mu_{\acute{e}}(.)$ are well-defined, and the operations of addition and multiplication on E are also well-defined, so too is the value function, v.

Given the ring on which the decision maker operates, it's necessary for her to have an appropriately formed attitude to ambiguity if she is to characterize her optimizing decision rule completely.

Assumption 3: Consistent attitude to ambiguity. To understand this condition, we suppose that there's a real number, $\alpha > 0$, which reflects the ambiguity attitude of the decision maker. Next, we consider the line with slope $-\alpha$ that passes through some point, $v' = a' + b'é \in E$. This line forms an equivalence class for the decision maker in the following sense: the decision maker is indecisive over (or is indifferent between) any two acts whose expected utility values both map into the set: $\left\{a + bé \mid \frac{b-b'}{a-a'} = -\alpha\right\}$; moreover, any act that maps its value into the open upper half-plane of E formed by $-\alpha(a - a') = b - b'$ is definitely chosen over any act that maps its value into that set, and any act that maps its value into the open lower half-plane created by $-\alpha(a - a') = b - b'$ is never chosen in preference to an act that has its value in that set. Since our choice of v' was arbitrary, we see that the decision maker's (constant co-efficient) attitude to ambiguity defines an equivalence relation on E.

We note that, if the decision maker is ambiguity averse, then $\alpha > 1$, and if the decision maker is ambiguity seeking, then $\alpha < 1$. The intuition here is that, if $\alpha > 1$, the decision maker needs a greater increase in the é-real value of an act to compensate for a given loss in the real value of that act if she is to remain indifferent between the two acts. If $\alpha < 1$ the decision maker needs less than proportionate compensation for taking on greater ambiguity. If ambiguity has no impact on the decision maker, we have: $\alpha = 1$, in which case, maximizing expected utility on E reduces to maximizing expected utility on $\mathbb{R}$ (as the decision maker is then simply maximizing the right angle value of expected utility, and right angle values lie entirely in the reals).

The combination of assumptions one to three implies that the decision maker has a completely preordered ranking of acts. The following Cartesian graph shows a map of equivalence classes defined on E, with $\alpha > 1$. (The assumption underlying this representation is that E is isomorphic to $\mathbb{R}^2$, a proposition which is shown to be true in the appendix.)



The above three assumptions also allow us to define the linear transformation: $\phi: E \to \mathbb{R}$ with $a + bé \mapsto \alpha a + b$, so that the complete decision rule for the decision maker becomes:

$$\max_{f \in F} \ \phi\left(\sum_{A \in \pi^f} \mu_é(A) \cdot u\big(x(A)\big)\right)$$

Where F is the set of available acts.

This real-valued maximand usefully decomposes to:

$$\sum_{A \in \pi^f} \lfloor \mu_é(A) \rfloor \cdot u\big(x(A)\big) + (\alpha - 1) \sum_{A \in \pi^f} a(A) \cdot u\big(x(A)\big)$$

Or, more simply:

$$\lfloor U \rfloor + (\alpha - 1)\mathfrak{A}$$

Where $a(A)$ is the value of the real part of the belief that the decision maker has in the likelihood of event A's occurring, $\lfloor U \rfloor$ is the right angle value of expected utility, and $\mathfrak{A}$ is the value of 'unambiguous' expected utility.

Putting the maximand in this form allows us to see more clearly the structure of the decision maker's thinking. The first summation term (i.e., the term before the plus sign) is the right angle value of expected utility, and the second summation term tells us how much of the right angle value of expected utility is 'unambiguous' in nature. Thus, this latter term is a measure of the degree of 'reliability' of the expected utility of the act. The extent to which this measure matters to the decision maker is given by the value, $(\alpha - 1)$, which multiplies it. If the decision maker is indifferent to ambiguity – i.e., is neither ambiguity avid or averse – then, $\alpha = 1$, and the entire term following the plus sign vanishes. The decision maker then simply maximizes (the right angle value of) expected utility, as earlier remarked. Alternatively, if it's the case that all probabilities are unambiguous throughout the decision problem, so that we have, for all events: $a(A) = \lfloor \mu_{\acute{e}}(A) \rfloor$, then the decision maker's problem also reduces to one of choosing the act which maximizes (the right angle value of) expected utility. These two cases – where the decision maker is ambiguity neutral, or where she holds no ambiguities about her beliefs – are behaviourally equivalent, as might be expected.

Another way of thinking about the decision maker's decision rule which the above formulation puts one in mind of is the following: the maximand is a weighted sum of expected utility, $\lfloor U \rfloor$, and reliable expected utility, $\mathfrak{A}$. When the decision maker is indifferent to ambiguity, she simply maximizes expected utility. As her ambiguity aversion increases, i.e., as $\alpha \to \infty$, she increases the weight placed on attaining at least a minimum of reliable expected utility, and, in the limit, she solely maximizes the latter form of utility. If one then associates 'reliable expected utility' with 'minimum expected utility', the model suggests that ambiguity averse decision makers span the range from expected utility maximization to minimum-expected utility maximization as α ascends from 1 to ∞. Interestingly, this decision rule replicates the one postulated by Ellsberg in his original article (1961, p.664), where 'his' ρ and 'our' α are related by: $\rho = \frac{1}{\alpha}$. The interpretation of the parameter, ρ or α, however, is rather different in the two cases. Ellsberg supposes that it reflects the confidence that the decision maker has in her 'estimated distribution', whereas, for us, that confidence is represented by the quotient: $\sum_{A \in \pi f} a(A)$, which tells us how much of the total expectation is unambiguous in nature. We call this value the 'reliability measure' of the decision maker's expectation. In our analysis, the parameter, α, rather reflects the decision maker's attitude towards ambiguity. Moreover, in our case, the parameter isn't lower bounded by 1 but by 0, reflecting the possibility of the decision maker having an attitude of ambiguity avidity. In that case, the decision maker becomes increasingly ambiguity avid as $\alpha \to 0$, and, in the limit, she maximizes only purely ambiguous value, and discounts real/unambiguous value entirely.

More informally, the above formulation tells us that the decision maker obeys the intuitive maxim: 'maximize expected utility plus an adjustment factor for ambiguity', which is a plausible model of agency, and is one which is present in a wide variety of models that express the ambiguity attitudes of decision makers.

With these preliminaries in hand, we're now ready to account for the decision problems we set out to explain.

4. Paradoxes

In this section, we successively show that decision makers who behave in the manner described in the previous section behave in the ways captured by the paradox posed by Ellsberg and extended by Machina. The first three cases are from Ellsberg's 1961 article; the fourth and fifth cases are Machina's reflection example and Blavatskyy's variation of it (Machina, 2009, p.390; Blavatskyy, 2013), whilst the final case is Machina's 50:51 example (Machina, 2009, p.385).

In each case, we proceed in six stages: first, a verbal description of the decision situation is given; secondly, a table describing the decision problem is provided; thirdly, the beliefs of the decision maker are specified for the relevant events; fourthly, the value of each act is given; fifthly, the decision rule is used to determine the optimal (pairwise) choices; and, finally, a Cartesian graph is used to represent the paradox and its resolution.

Case 1: the Ellsberg 3-colour problem:

In this decision problem, there's a single urn containing 90 balls. 30 of those balls are red, and the remaining 60 are black or yellow in unknown proportions. The payoffs are as given in the table. Acts f and g are to be compared, and f' and g' are to be compared. Decision makers who maximize expected utility utilizing (real-valued) probabilities obey the choice relation: f is chosen over g if and only if f' is chosen over g'. Evidence has shown that this implication often fails to hold in practice.

	30 balls	60 balls	
	red	black	yellow
f	\$100	\$0	\$0
g	\$0	\$100	\$0
f'	\$100	\$0	\$100
g'	\$0	\$100	\$100

The beliefs of the decision maker which are relevant to her decision making are as follows: $\mu_{\acute{e}}(\emptyset) = 0$, $\mu_{\acute{e}}(red) = \frac{1}{3}$, $\mu_{\acute{e}}(black) = \mu_{\acute{e}}(yellow) = \left(\frac{1}{3}\right)\acute{e}$, $\mu_{\acute{e}}(red \cup black) = \mu_{\acute{e}}(red \cup yellow)$ $= \frac{1}{3} + \left(\frac{1}{3}\right)\acute{e}$, $\mu_{\acute{e}}(black \cup yellow) = \frac{2}{3}$.

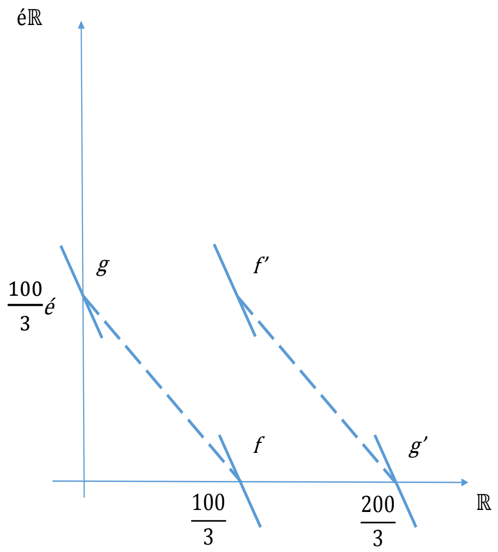
We may, without prejudice, identify utilities with dollar payoffs, in which case, we have:

$f: \frac{100}{3}$ $f': \frac{100}{3} + \frac{100}{3}é$

$g: \frac{100}{3}é$ $g': \frac{200}{3}$

Thus, if f is chosen over g, we have $\alpha > 1$, so g' is chosen over f', which is the behaviour to be explained.

The following Cartesian graph represents the situation. In the graph, each dashed line is the convex hull of a pairwise choice confronting the decision maker; and the parallel, negatively-sloped solid lines are equivalence classes (loosely speaking, 'indifference curves') defined by the decision maker's attitude to ambiguity. (Note that all the negatively-sloped solid lines in the diagram are actually parallel, and any appearance to the contrary is an instance of the Zöllner illusion at work.)



Case 2: the Ellsberg 2-urn problem:

In this decision problem, there are two urns, each containing 100 balls. In urn I, the balls are coloured red or black and are distributed in unknown proportions. In urn II, 50 of the balls are red, and the remaining 50 are black. The payoffs are as given in the table. Acts f and g are to be compared, and f' and g' are to be compared. Decision makers who maximize expected utility utilizing (real-valued) probabilities obey the choice relation: f is chosen over g if and only if g' is chosen over f'. Evidence has shown that this implication often fails to hold in practice.

	Urn I		Urn II	
	100 balls		50 balls	50 balls
	red	black	red	black
f	\$100	\$0	-	-
g	-	-	\$100	\$0
f'	\$0	\$100	-	-
g'	-	-	\$0	\$100

The beliefs of the decision maker which are relevant to her decision making are as follows:

$\mu_{é}(\emptyset) = 0, \mu_{é}(urn\ I{:}\ red) = \mu_{é}(urn\ I{:}\ black) = \frac{1}{2}é, \mu_{é}(urn\ II{:}\ red) = \mu_{é}(urn\ II{:}\ black) = \frac{1}{2}$

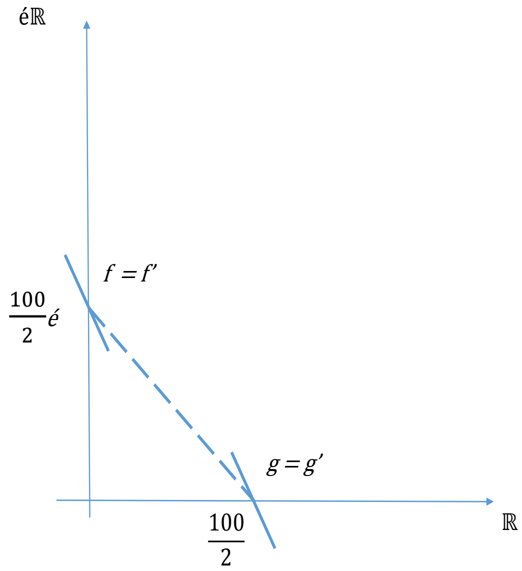
We may, without prejudice, identify utilities with dollar payoffs, in which case, we have:

$f{:}\ \frac{100}{2}é$ $\quad$ $f'{:}\ \frac{100}{2}é$

$g{:}\ \frac{100}{2}$ $\quad$ $g'{:}\ \frac{100}{2}$

Thus, if g is chosen over f, we have $\alpha > 1$, so g' is chosen over f', which is the behaviour to be explained.

The following Cartesian graph represents the situation. In the graph, the dashed line is the convex hull of the pairwise choices confronting the decision maker and the parallel, negatively-sloped solid lines are equivalence classes defined by the decision maker's attitude to ambiguity.



Case 3: the Ellsberg 4-colour problem:

In this decision problem, there's a single urn, containing 200 balls. 100 of the balls are coloured red or black and are distributed in unknown proportions. 50 of the balls are green, and 50 are yellow. The payoffs are as given in the table. Acts f and g are to be compared, and f' and g' are to be compared. Decision makers who maximize expected utility utilizing (real-valued) probabilities obey the choice relation: f is chosen over g if and only if f' is chosen over g'. Evidence has shown that this implication often fails to hold in practice.

	100 balls		50 balls	50 balls
	red	black	green	yellow
f	\$100	\$100	\$0	\$0
g	\$100	\$0	\$100	\$0
f'	\$0	\$100	\$0	\$100
g'	\$0	\$0	\$100	\$100

The beliefs of the decision maker which are relevant to her decision making are as follows:

$\mu_é(\emptyset) = 0$, $\mu_é(red) = \mu_é(black) = \frac{1}{4}é$, $\mu_é(green) = \mu_é(yellow) = \frac{1}{4}$, $\mu_é(red \cup black) = \mu_é(green \cup yellow) = \frac{1}{2}$, $\mu_é(red \cup green) = \mu_é(black \cup yellow) = \frac{1}{4} + \frac{1}{4}é$.

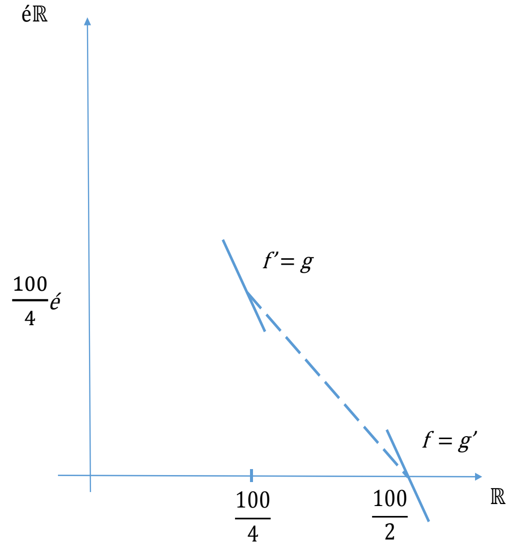
We may, without prejudice, identify utilities with dollar payoffs, in which case, we have:

$f{:}\,\frac{100}{2}$ $\qquad\qquad$ $f'{:}\,\frac{100}{4}+\frac{100}{4}é$

$g{:}\,\frac{100}{4}+\frac{100}{4}é$ $\qquad\qquad$ $g'{:}\,\frac{100}{2}$

Thus, if f is chosen over g, we have $\alpha > 1$, so g' is chosen over f', which is the behaviour to be explained.

The following Cartesian graph represents the situation. In the graph, the dashed line is the convex hull of the pairwise choices confronting the decision maker; and the parallel, negatively-sloped solid lines are equivalence classes defined by the decision maker's attitude to ambiguity.



Case 4: the Machina reflection example:

In this decision problem, there's a single urn, containing 100 balls. 50 of the balls are labelled E_1 or E_2 and are distributed in unknown proportions, whilst the remaining 50 balls are labelled E_3 and E_4, and are also distributed in unknown proportions. The payoffs are as given in the table. Acts f and g are to be compared, and f' and g' are to be compared. Decision makers who maximize expected utility utilizing (real-valued) probabilities obey the choice relation: f is chosen over g if and only if f' is chosen over g'. Evidence has shown that this implication can fail to hold in practice (l'Haridon & Placido, 2010).

	50 balls		50 balls	
	E_1	E_2	E_3	E_4
f	\$4,000	\$8,000	\$4,000	\$0
g	\$4,000	\$4,000	\$8,000	\$0
f'	\$0	\$8,000	\$4,000	\$4,000
g'	\$0	\$4,000	\$8,000	\$4,000

The beliefs of the decision maker which are relevant to her decision making are as follows: $\mu_{é}(\emptyset) = 0$, $\mu_{é}(E_1) = \mu_{é}(E_2) = \mu_{é}(E_3) = \mu_{é}(E_4) = \frac{1}{4}é$, $\mu_{é}(E_1 \cup E_2) = \mu_{é}(E_3 \cup E_4) = \frac{1}{2}$, $\mu_{é}(E_1 \cup E_3) = \mu_{é}(E_2 \cup E_4) = \frac{1}{2}é$.

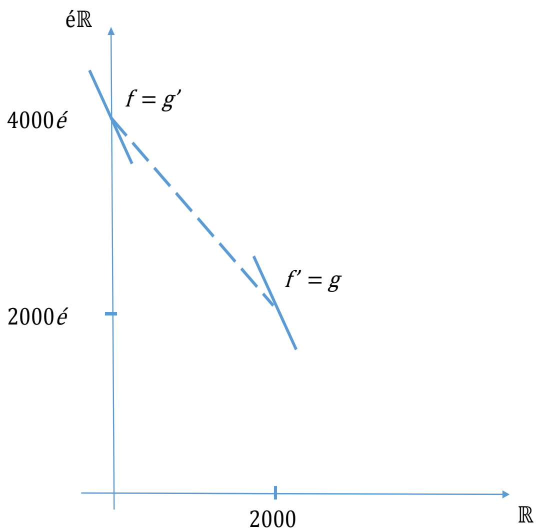
We may, without prejudice, identify utilities with dollar payoffs, in which case, we have:

$f: 2{,}000é + 2{,}000é$ $\quad$ $f': 2{,}000 + 2{,}000é$

$g: 2{,}000 + 2{,}000é$ $\quad$ $g': 2{,}000é + 2{,}000é$

Thus, if g is chosen over f, we have $\alpha > 1$, so f' is chosen over g', which is the behaviour to be explained.

The following Cartesian graph represents the situation.



Case 5: the Blavatskyy twist of the Machina reflection example

In this decision problem, there's a fair coin, and a bag containing an unknown number of black and white marbles. There are four possible states: $\{heads \,\&\, black, heads \,\&\, white,\ tails \,\&\, black,$ $tails \,\&\, white\}$. In the table, the top left quadrant describes act, f_1, which is to be compared to act, g_1, in the top right of the table; while acts, f_2 and g_2, in the bottom half of the table, are to be compared. It's assumed that \$4,000 < \$x. Hypothetical reasoning suggests that, if f_1 is chosen over g_1, then f_2 is chosen over g_2; however, many decision models can't generate this choice profile (Blavatskyy, 2013).

f_1	black	white	g_1	black	white
heads	\$4,000	\$4,000	heads	\$4,000	\$0
tails	\$0	\$0	tails	\$4,000	\$0
f_2	black	white	g_2	black	white
heads	\$4,000	\$4,000	heads	\$4,000	\$x
tails	\$x	\$0	tails	\$4,000	\$0

The beliefs of the decision maker which are relevant to her decision making are as follows: $\mu_{é}(hb) = \mu_{é}(hw) = \mu_{é}(tb) = \mu_{é}(tw) = (1/4)é$, $\mu_{é}("heads") = \mu_{é}("tails") = 1/2$, and $\mu_{é}("black") = \mu_{é}("white") = (1/2)é$.

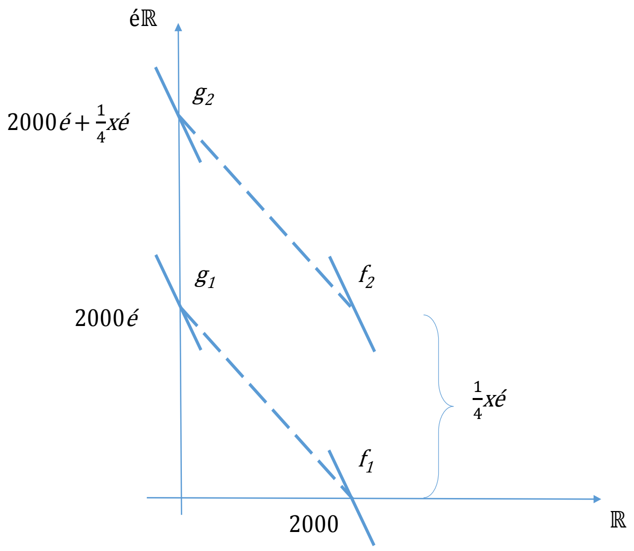
We may, without prejudice, identify utilities with dollar payoffs, in which case, we have:

$f_1\colon \frac{1}{2} \cdot 4{,}000 = 2{,}000$ $\qquad$ $f_2\colon 2{,}000 + \frac{1}{4} é \cdot x$

$g_1\colon \frac{1}{2} é \cdot 4{,}000 = 2{,}000é$ $\qquad$ $g_2\colon 2{,}000é + \frac{1}{4} é \cdot x$

Thus, if f_1 is chosen over g_1, we have $\alpha > 1$, so f_2 is chosen over g_2, which is the behaviour to be explained.

The following Cartesian graph represents the situation.



Case 6: the Machina 50:51 example:

In this decision problem, there's a single urn, containing 101 balls. 50 of the balls are labelled E_1 or E_2 and are distributed in unknown proportions, whilst the remaining 51 balls are labelled E_3 and E_4, and are also distributed in unknown proportions. The payoffs are as given in the table. Acts f and g are to be compared, and f' and g' are to be compared. Decision makers who maximize expected utility utilizing (real-valued) probabilities obey the choice relation: f is chosen over g if and only if f' is chosen over g'. Hypothetical reasoning suggests that this implication might be expected to fail to hold in practice; specifically, it's reasonable to suppose that an ambiguity averse decision maker will prefer f to g, but will generally prefer g' to f' (Machina, 2009).

	50 balls		51 balls	
	E_1	E_2	E_3	E_4
f	\$8,000	\$8,000	\$4,000	\$4,000
g	\$8,000	\$4,000	\$8,000	\$4,000
f'	\$12,000	\$8,000	\$4,000	\$0
g'	\$12,000	\$4,000	\$8,000	\$0

The beliefs of the decision maker which are relevant to her decision making are as follows: $\mu_{\acute{e}}(\emptyset) = 0$, $\mu_{\acute{e}}(\mathrm{E}_1) = \mu_{\acute{e}}(\mathrm{E}_2) = \frac{50}{202}\acute{e}$, $\mu_{\acute{e}}(\mathrm{E}_3) = \mu_{\acute{e}}(\mathrm{E}_4) = \frac{51}{202}\acute{e}$, $\mu_{\acute{e}}(\mathrm{E}_1 \cup \mathrm{E}_2) = \frac{100}{202}$, $\mu_{\acute{e}}(\mathrm{E}_3 \cup \mathrm{E}_4) = \frac{102}{202}$, $\mu_{\acute{e}}(\mathrm{E}_1 \cup \mathrm{E}_3) = \mu_{\acute{e}}(\mathrm{E}_2 \cup \mathrm{E}_4) = \frac{101}{202}\acute{e}$.

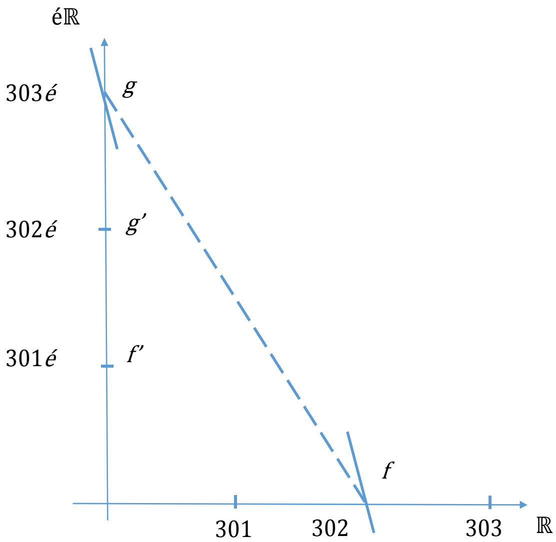
We may, without prejudice, identify utilities with dollar payoffs, in which case, we have:

$f\colon \frac{302}{202} \cdot 4{,}000$ $\quad$ $f'\colon \frac{151}{202} \cdot 4{,}000\acute{e} + \frac{150}{202} \cdot 4{,}000\acute{e}$

$g\colon \frac{303}{202} \cdot 4{,}000\acute{e}$ $\quad$ $g'\colon \frac{152}{202} \cdot 4{,}000\acute{e} + \frac{150}{202} \cdot 4{,}000\acute{e}$

Thus, f is chosen over g if $\alpha > \frac{303}{302}$, and g' is always chosen over f', which is the behaviour to be explained.

The following Cartesian graph represents the situation (note the displaced origin).



5. Conclusions

In this paper we set out to show that a relatively simple model of expected utility maximization on the ring, E, can account for three well known instances of the Ellsberg paradox and three instances of what has come to be called the Machina paradox. Along the way, we've shown that Ellsberg's own mooted decision rule is implied by the model, which might therefore be taken to provide a warrant of sorts for that rule.

Perhaps unsurprisingly, given that the model implies Ellsberg's own solution to his paradox, the model solves the cases he poses in a straightforward and logically intuitive manner. Moreover, the model is unfazed by Machina's extension of the Ellsberg paradox.

The fundamental contribution of the paper lies in showing that the ring, E, can be used to decompose decision makers' epistemic evaluations. According to the model given above, decision makers form probabilistic expectations – as they do in many other models – but they then go on to deconstruct those expectations to reflect the reliability of the information to which they have access. They do this by decomposing their overall expectations – i.e., the right angle values of the probabilities they hold – into two parts: a real part, which reflects the unambiguous element of their beliefs, and an é-real part, which reflects the ambiguous element of their beliefs. When it comes to decision making, these disparate elements of belief are reconciled by the decision makers' well-defined attitudes to ambiguity, and decision makers are able to form well-defined preference orderings over the options available to them. The model is, therefore, one of 'rational' behaviour in that sense. There's more that can be said about that behaviour, and about the way in which decision makers formulate their thinking in terms of the ring, E, but we haven't the space to expand along those lines here.[3]

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[3] However, we do provide an axiomatic account of decision makers' behaviour in the appendix.

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Appendix:

There are a number of ways of showing that E is isomorphic to $\mathbb{R}^2$ but the briefest goes as follows.

Statement: $E \cong \mathbb{R}^2$.

Argument: We begin by observing that E can be characterized as a factor ring of the ring of polynomials over the reals, $\mathbb{R}[x]$. To see this, consider the homomorphism, $\xi: \mathbb{R}[x] \to E$, which embeds coefficients and sends the indeterminate, $x \rightsquigarrow \acute{e}$. Now, $\xi(x - x^2) = \acute{e} - \acute{e}^{\,2} = 0$ by the idempotency of $\acute{e}$. Furthermore, $(x - x^2)$ is reducible to: $(x) \cdot (1 - x)$. Evidently, also: $(x) + (1 - x) = \mathbb{R}[x]$, so (x) and $(1 - x)$ are comaximal. Hence, $\ker(\xi) = (x - x^2)$, and, by way of the Chinese remainder theorem, we have: $E \cong \mathbb{R}[x]/(x - x^2) \cong \mathbb{R}[x]/(x) \times \mathbb{R}[x]/(1 - x) \cong \mathbb{R} \times \mathbb{R}$, which is as needed to be shown.

Comment: This result allows us to construct a taxonomy of the two-dimensional hypercomplex numbers in the following way. Take the ring of polynomials defined over the field of reals, and consider the following factor rings and their associated isomorphisms: $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$; $E \cong \mathbb{R}[x]/(x^2 - x)$; and $\mathcal{D} \cong \mathbb{R}[x]/(x^2)$, where $\mathcal{D}$ is the ring of dual numbers. The first of these isomorphisms is well known, the second has just been demonstrated, and the last is known if not, perhaps, familiar. Since Yaglom (1968, ch.1, §3) has shown that all two-dimensional hypercomplex number systems are isomorphic to either the complex numbers, the dual numbers, or the split complex numbers, we're minded to think that E might be isomorphic to the last of these (which we denote by H). The following statement and argument show that this thought is justified.

Statement: $E \cong H$.

Argument: To see that this is so, consider the mapping, τ, which sends numbers from the ring, E, to the ring of split complex numbers, H, along with its inverse mapping, τ^{-1}:

$\tau: E \rightarrow H$ with $(a,\ b) \mapsto (a + \frac{1}{2}b,\ \frac{1}{2}b)$

$\tau^{-1}: H \rightarrow E$ with $(a,\ b) \mapsto (a - b,\ 2b)$

Once we recall that the split complex numbers are defined by the following operations:

$(a + bj) + (c + dj) = (a + c) + (b + d)j$

$(a + bj) \cdot (c + dj) = (ac + bd) + (ad + bc)j$

(where j is the split complex operator, with $j^2 = +1$), then it's a straightforward verification to see that τ and τ^{-1} are both homomorphisms. Hence, the é-reals and the split complex numbers are isomorphic, as asserted.

Comment: That E and H are isomorphic is a matter of intrinsic interest, but it's also an interesting fact to know for the following reason: it shows us that the ring we've used to characterize an 'ambiguous' or an 'approximate' calculus of numbers isn't of recent provenance. We've already referred to the contribution of Izhakian and Izhakian (2014), but the use of E, or an isomorphism of it, to derive such a calculus goes back at least to Warmus (1956). It's a matter of some curiosity as to why earlier use hasn't been made of this known algebra to deal with issues of ambiguity in decision theory (à la the Ellsberg and Machina paradoxes) or with issues of approximation in the same theory (à la the Allais paradox).

In the space that remains, we want to provide a behavioural account to accompany the cognitive-functional description of decision makers' motivations and actions that we've used hitherto. In what follows, we make use of two preliminary principles which define the way in which decision makers induce evidence; and, subsequently, we use five premises to characterize decision makers' behaviour. We subsequently show that these behavioural assumptions imply the behaviour discussed in the text.

The first of the two principles is the *principle of indifference*. In discussing this principle, we here limit ourselves to an examination of it as it pertains to the current circumstances – namely, to situations that in which balls are drawn from urns. Further implications are left for another occasion.

We begin by defining the concept of an 'admixture range'. An admixture range is a range of n events: $A, B, C, \ldots \in 2^{\Omega}$, such that all the events belong to a partition of Ω, and the range satisfies the following four conditions: i) the cardinality of $(A \cup B \cup C \cup \ldots)$ is known (i.e., $\#(A \cup B \cup C \cup \ldots)$ is known); ii) the cardinality of any proper subset of $(A \cup B \cup C \cup \ldots)$ is unknown; iii) every finer partition of $A, B, C, \ldots$ satisfies conditions (i)-(ii); and, iv) there is no finer partition satisfying conditions (i)-(iii). This last condition – that the partition be maximally fine – implies that the events that constitute an admixture range are states, otherwise we could always find another, finer, partition satisfying conditions (i)-(iii). With that in mind, decision makers compute their beliefs in the likelihood of states' occurring in the following way. From the cardinality of an admixture range, is deducted the sum: $\min\#(A)+\min\#(B)+\min\#(C)+\cdots$; and so we determine the 'remainder' of $(A,B,C,\dots)\triangleq Rem\#(A,B,C,\dots)=\#(A\cup B\cup C\cup\dots)-\{\min\#(A)+\min\#(B)+\min\#(C)+\cdots\}$. The principle of indifference then suggests, in the current context, that we divide $Rem\#(A,B,C,\dots)$ by n so as to distribute this mass of remaining possibilities in an equitable manner across the range of relevant states. The belief or likelihood that state A will occur is then given by the formula:

$$\lfloor\mu_{é}(A)\rfloor=\frac{\min\#(A)}{\#\Omega}+\frac{Rem\#(A,B,C,\dots)}{n\#\Omega}$$

$$a(A)=\frac{\min\#(A)}{\#\Omega}$$

$$b(A)=\frac{Rem\#(A,B,C,\dots)}{n\#\Omega}$$

$$\mu_{é}(A)=a(A)+b(A)é$$

For the events that constitute the admixture range, we have, of course:

$$\mu_{é}(A\cup B\cup C\cup\dots)=\frac{\#(A\cup B\cup C\cup\dots)}{\#\Omega}$$

In fact, whenever $\#(A\cup B\cup C\cup\dots)$ is known, the belief that $A\cup B\cup C\cup$ ... will occur is given as per the last formula, and this is so even if $A,B,C,$ ... doesn't constitute an admixture range.

For events larger than states, say, event I, where $\#(I)$ *isn't* known, the decision maker proceeds in a number of steps. She begins by considering the set of partitions of I, $\Pi(I)$, with $\pi(I)\in\Pi(I)$. She then computes the epistemic evaluation of each such partition as follows. Let $A\in 2^{\Omega}$ also satisfy: $A\in\pi(I)$; the epistemic evaluation of $\pi(I)$ is, then: $\mu_{é}\big(\pi(I)\big)=\sum_{A\in\pi(I)}\mu_{é}(A)$, where $\mu_{é}(A)$ is determined by the methods given above. The epistemic evaluations of every partition of I necessarily return the same right angle value of belief, but they may return different values of the real part of belief (i.e., their a-values may differ). The reason for this is that an event that's a block of a particular partition might contribute the 'missing' element of a sequence whose cardinality is known, and which is partially contained within another event-block of that partition; but, by dividing up the sequence among the two blocks, the partition fails to count the cardinality of that sequence. Another, different, partition, say, $\pi'(I)\in\Pi(I)$, might treat the union of the two events as a single block, and so *will* count the cardinality of the sequence. The epistemic value of this latter partition will, *ceteris paribus*, have a higher real (a) value than the epistemic evaluation of the first partition even though their right angle values are the same. To see an example of this, consider the Blavatskyy twist and suppose that the decision maker wants to determine the value of $\mu_{é}(tw^{c})$. She observes that: $\mu_{é}(tb)+\mu_{é}(hb)+\mu_{é}(hw)=\frac{3}{4}é=\mu_{é}(\text{"}black\text{"})+\mu_{é}(hw)$; but she also observes that: $\mu_{é}(tb)+\mu_{é}(\text{"}heads\text{"})=\frac{1}{2}+$

$\frac{1}{4}\acute{e}$. Evidently, the right angle values of these epistemic evaluations are the same, but their composite values are different. The reason for this difference is that the last epistemic evaluation explicitly counts the cardinality of "heads" in a way not done by either of the other two candidate measures of $\mu_{\acute{e}}(tw^c)$. On the grounds that she wants to work with the most reliable, or 'unambiguous', beliefs that the totality of evidence makes it possible for her to hold, the decision maker then settles upon $\mu_{\acute{e}}(tw^c) = \frac{1}{2} + \frac{1}{4}\acute{e}$ as her epistemic evaluation of the likelihood that some state other than "tails & white" will occur. More generally, we can say that the decision maker chooses the value which maximizes the 'reliability ratio', $\frac{a}{\lfloor a+b\acute{e} \rfloor}$, from the various candidate values available to her; and she settles upon that value as the one which best expresses her estimation of the likelihood that event *I* will happen.

With this calculus in hand, the decision maker is able to determine her beliefs about the likelihood of events. Of course, all reasonable decision makers who can count and take ratios and who accept the principle of indifference, arrive at the same set of beliefs. Moreover, it's worth noting, *en passant*, that the principle of indifference is essential to the derivation of beliefs in the current context since it determines the quantum and the distribution of the pure é-real components of decision makers' epistemic evaluations. The necessity of the principle in the case where beliefs are formed on the ring of é-real numbers is of some philosophical interest as it's sometimes asserted that the principle of indifference is of heuristic rather than logical value. In our case, it is, rather, the latter.

The second of our principles is the *partition principle*, which deals with partitions of the state space. In discussing this principle, we here limit ourselves to an examination of it as it pertains to the current circumstances – namely, to situations in which the payoffs are dollar values. As indicated in the text, decision makers are assumed to use the 'natural' partition associated with an option when determining their epistemic evaluations. The natural partition is the one in which states are agglomerated so as to put events and outcomes in a one-to-one correspondence.

With these two principles in hand, decision makers hold that each event is associated with a unique epistemic evaluation (or belief), and that there is a unique partition that is appropriate for calculating the value of any given option. This, itself, implies that decision makers may transform their thinking about the options they face in the following way. As a matter of primitive construction, we may suppose – as we've done in the text – that decision makers think in terms of acts, which map states to consequences. But, if each act is associated with the uniquely given 'natural partition' for the purposes of evaluating that act, and if each event of that partition is associated with a uniquely determined belief, then the decision maker may equally well think of her options as 'lotteries', which are defined in the following way:

$$L = (\mu_{\acute{e}}^{L}: X \to \{a + b\acute{e} \mid \lfloor a + b\acute{e} \rfloor \leq 1; 0 \leq a, b; \sum_{x\in \mathrm{X}} \lfloor \mu_{\acute{e}}^{L}(x) \rfloor = 1\})$$

This is to say, each lottery specifies an epistemic evaluation for each of the consequences from the set of available consequences, X (we suppose: $\#X < \infty$). The set of lotteries is denoted by $\mathcal{L}$.

With that in mind, we can then define the five premises which characterize decision makers' behaviour. The first premise is straightforward.

Ordering: there's a (complete) preference ordering, $\gtrsim$, over $\mathcal{L}$.

In order to define the next two premises, we need the following definition.

Definition: if $\lambda = a_\lambda + b_\lambda é$, with $0 \leq a_\lambda, b_\lambda$ and $a_\lambda + b_\lambda \leq 1$, then $\lambda^c = a_{\lambda^c} + b_{\lambda^c} é$, with $0 \leq a_{\lambda^c}, b_{\lambda^c}$; $a_{\lambda^c} + b_{\lambda^c} = 1 - (a_\lambda + b_\lambda)$.

We can think of λ^c as the 'complement of λ' defined with respect to the boundary set: $\{a + bé \mid 0 \leq a, b; \lfloor a + bé \rfloor = 1\}$. λ^c is the concept that we need in E which is analogous to – indeed, generalizes – the concept of $1 - \lambda$ which is defined with respect to λ when: $\lambda \in [0,1]$. Evidently, in general, λ^c isn't unique.

With the given definition in hand, we're now in a position to state the following two premises, which adapt the standard Archimedean and independence assumptions to our present circumstances.

Archimedean: there are: $\theta, \theta^c \in E$, with $0 \leq a_\theta, a_{\theta^c}, b_\theta, b_{\theta^c}$ and $\lfloor \theta \rfloor, \lfloor \theta^c \rfloor \leq 1$ such that for any lotteries, $L, L', L'' \in \mathcal{L}$ with: $L'' \gtrsim L \gtrsim L'$, we have: $\theta L'' + \theta^c L' \sim L$; and, for degenerate or constant lotteries: $L_x \equiv x$, with: $x'' \gtrsim x \gtrsim x'$, whenever we have: $\theta x'' + \theta^c x' \sim x$ then $\theta, \theta^c \in \mathbb{R} \subset E$.

Independence: for all lotteries, $L, L', L'' \in \mathcal{L}$ and for any $\lambda \in E$ such that: $0 \leq a_\lambda, b_\lambda$ and $a_\lambda + b_\lambda < 1$, we have: $\lambda L + \lambda^c L'' \gtrsim \lambda L + \lambda^c L' \Leftrightarrow L'' \gtrsim L'$; and we note that this takes as a special case: for all lotteries, $L, L', L'' \in \mathcal{L}$ and for any $\lambda, (1 - \lambda) \in [0,1)$, we have: $\lambda L + (1 - \lambda) L'' \gtrsim \lambda L + (1 - \lambda) L' \Leftrightarrow L'' \gtrsim L'$.

These assumptions are sufficient to determine a real-valued utility function over prizes or outcomes, and an expected utility function defined over E, as given in the text. The next assumption continues the line of argument as presented there, and defines the decision maker's attitude to ambiguity. In what follows, x^* is the best available prize and x_* the worst, with $x^* \succ x_*$.

Attitude to ambiguity: whenever we have: $L \sim v.x^* + v^c.x_*$, where $v = a^v + b^v é$, then there's a unique value: $r = a^v + \alpha^{-1} b^v$, with $0 < \alpha$, such that: $L \sim r.x^* + (1 - r)x_*$, and the mapping that accomplishes this – i.e., which sends $(v, v^c) \mapsto (r, 1 - r)$ – is denoted by φ.

The following assumption follows naturally from the previous one – it states that a decision maker prefers a lottery which places higher odds on getting the better of two outcomes as opposed to a lottery which places lower odds on getting the better outcome.

Monotonicity: whenever: $L' \sim r'.x^* + (1-r')x_*$ and $L'' \sim r''.x^* + (1-r'')x_*$, then $L'' \gtrsim L' \Leftrightarrow r'' \geq r'$.

In fact, for values of $\alpha \geq 1$, monotonicity is implied by the earlier premises, ordering through to independence; however, since the decision maker may be ambiguity avid (i.e., since it's possible that $0 < \alpha < 1$), we add this premise to those given earlier.

Given these premises, we can state the following.

Statement: The five stated premises imply that there exists a real-valued utility function, $u: X \to \mathbb{R}$, and, moreover, an ambiguity adjusted utility function exists such that, for any two lotteries: $L'' \gtrsim L' \Leftrightarrow \varphi(\sum_{x \in \mathrm{X}} \mu''_{\acute{e}}(x) \cdot u(x)) \geq \varphi(\sum_{x \in X} \mu'_{\acute{e}}(x) \cdot u(x))$.

Argument: We begin by showing that the first three premises can be used to determine a real-valued utility function that accomplishes: $u_y \geq u_x \Leftrightarrow y \gtrsim x$, where $u_y, u_x \in \mathbb{R}$ are the values that solve: $y \sim \left[u_y x^* + \left(1-u_y\right)x_*\right]$ and $x \sim [u_x x^* + (1-u_x)x_*]$, respectively.

To see this, we note that the first two premises (i.e., 'ordering' and 'Archimedean') imply that the two indifference relations given in the preceding sentence are well posed, with: $u_y, u_x, \left(1-u_y\right)$, $(1-u_x) \in \mathbb{R}$. The statement that: $u_y \geq u_x \Leftrightarrow y \gtrsim x$ follows from a familiar result which states that, for $\kappa, \lambda, (1-\kappa), (1-\lambda) \in [0,1]$, we have: $\lambda x^* + (1-\lambda)x_* \gtrsim \kappa x^* + (1-\kappa)x_* \Leftrightarrow \lambda \geq \kappa$, which is itself implied by those premises and independence (we recall that the generalized concept of independence with which we're working entails the canonical definition, so that the argument just given goes through in the usual fashion). This result also implies that utility is unique.

In the next part of the argument, we want to show how lotteries are evaluated, compared and chosen.

To see how, take the lottery: $L' = \mu'_{\acute{e}}(x_1)x_1 + \mu'_{\acute{e}}(x_2)x_2 + \mu'_{\acute{e}}(x_3)x_3 + \cdots$.

Which is read: x_1 is delivered with likelihood, $\mu'_{\acute{e}}(x_1)$, x_2 is delivered with likelihood $\mu'_{\acute{e}}(x_2)$, etc.

By premise three (independence) and the earlier derivation of utilities, we have:

$\mu'_{\acute{e}}(x_1)x_1 + \mu'_{\acute{e}}(x_2)x_2 + \mu'_{\acute{e}}(x_3)x_3 + \cdots \sim$

$\mu'_{\acute{e}}(x_1)\left[u_{x_1}x^* + \left(1-u_{x_1}\right)x_*\right] + \mu'_{\acute{e}}(x_2)\left[u_{x_2}x^* + \left(1-u_{x_2}\right)x_*\right] + \mu'_{\acute{e}}(x_3)\left[u_{x_3}x^* + \left(1-u_{x_3}\right)x_*\right] \ldots$

This latter expression (i.e., the expression following the indifference sign) can be more conveniently rendered by writing: $u(x_1) = u_{x_1}$, and re-arranging to yield the following statement:

$$L' \sim \left(\sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) \cdot u(x)\right)x^* + \left(\sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) \cdot \left(1-u(x)\right)\right)x_*$$

The expressions in the parentheses define the values, $v', v^{c'} \in E$, spoken of in the fourth premise (attitude to ambiguity):

$$v' = \sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) \cdot u(x) \qquad\qquad v^{c'} = \sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) \cdot \big(1 - u(x)\big)$$

On the supposition stated there, there's a mapping, φ, that sends: $v' \mapsto r'$, $v^{c'} \mapsto 1 - r'$, and which normalizes: $\sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) = 1$, so as to attain: $L' \sim v'x^* + v^{c'}x_* \sim r'x^* + (1 - r')x_*$.

More expansively, we have:

$$r' = \varphi\left(\sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) \cdot u(x)\right)$$

and:

$$L' \sim \varphi\left(\sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) \cdot u(x)\right)x^* + \left(1 - \varphi\left(\sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) \cdot u(x)\right)\right)x_*$$

To complete this phase of the argument, we want to show that comparative decisions are made in the manner proposed. To see that this is so, suppose that there's another lottery, L'':

$$L'' \sim \varphi\left(\sum_{x \in \mathrm{X}} \mu''_{\acute{e}}(x) \cdot u(x)\right)x^* + \left(1 - \varphi\left(\sum_{x \in \mathrm{X}} \mu''_{\acute{e}}(x) \cdot u(x)\right)\right)x_*$$

Then, by the last premise (monotonicity), we have:

$$\varphi\left(\sum_{x \in \mathrm{X}} \mu''_{\acute{e}}(x) \cdot u(x)\right) \geq \varphi\left(\sum_{x \in \mathrm{X}} \mu'_{\acute{e}}(x) \cdot u(x)\right) \Leftrightarrow L'' \gtrsim L'$$

Which is as needed to be shown.

Comment: the argument here demonstrates that decision makers' behaviour can be understood by way of a relatively straightforward extension of the algebra over which they conduct their calculations. Thus, so long as we're willing to grant that decision makers are capable of operating on the ring, E, we can readily account for the Ellsberg paradox and its extension by Machina. Since E is isomorphic to $\mathbb{R}^2$, as we've seen, this seems like a reasonable assumption to make.