Dynamic optimisation on infinite dimensions: existence of solutions, Euler and transversality conditions

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1 Introduction

In this paper, I study dynamic optimisation problems on infinite dimensional state space. Infinite dimensional states are important when we want to analyse an economy with heterogeneity; heterogeneity can be product heterogeneity or agent heterogeneity. For example, airlines make decisions about what types of different aeroplanes to use for different routes (product heterogeneity). Firms make decisions about where to locate offices and factories for different types of production (spatial heterogeneity). And different types of consumers make different decisions about how much to save (consumer heterogeneity) . In all these cases, a function or distribution living in an infinite dimensional space describes the state of the economy.

It is a truism to say that there has already been much interest in the literature about product differentiation, consumer and firm heterogeneity. The literature recognises also that it is convenient to represent varieties as an infinite dimensional function or distribution on some space of characteristics (see for example, Acemoglu (2009) ch. 12.4, MasCollel (1975)). But we still do not have a complete theory of how to do *dynamic optimisation*, particularly for social planner type problems, across all agents or product varieties. Much of the theory for discrete time dynamic optimisation assumes finite dimensional state spaces (ch.6 Acemoglu (2009), Stachurski (2009), Kamihigashi (2001) and Stokey and Lucas (1989)). Models that assume product varieties (Romer, 1990; Koren and Tenreyro, 2013) do away with infinite state spaces by assuming product symmetry.

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Other heterogeneous agent models, such the traditional Aiyagari-Hugget type models, do not consider the *global optimum* in a way that heterogeneity matters. Sometimes product differentiation takes on a finite number of types – such as clean and dirty (Acemoglu et al., 2012), or high and low skill (Acemoglu, 2002).

A drawback with finite dimensional models is that we cannot study features of entire distributions of, say, income across skill or capital across space. However, recent work suggests important economic insight can be gained from explicitly considering global optima across distributions in areas such as consumer saving decisions (Dávila et al., 2012), skilled bias technological change (Jones and Kim, 2014), as well as economic geography (Brock et al., 2014).

This paper will present results to characterise solutions to discrete time dynamic problems on infinite dimensional state space. The paper's main result verifies existence of solutions to a dynamic unbounded problem on any topological vector space. The existence result shows how the infinite horizon pay-off function can be continuous on a compact feasible space. However, the choice of topology and feasible sets is not trivial. We discuss some feasible sets that modellers can use for applications.

The paper also shows that on a normed vector space if we assume Frechet differentiability – a generalisation of differentiability to normed spaces– of the per-period pay-off, we can characterise solutions using an Euler condition. This vector space Euler condition gives similar insight to the familiar finite dimensional Euler equation. The Euler condition will also give us a system of difference equations in a normed space. For simple models, we may be able to solve the difference equations. For more complex models, we may have to characterise behaviour rather than solve the system.

The paper demonstrates the main results with a simple example – a spatial endogenous growth model. Firms accumulate capital on a flat surface. Firms receive spillovers from each other depending on how far away they are, so the firms located towards the centre of the surface receive the most spillovers. Individual firms prefer to spread out from each other chasing higher returns on the periphery, but the planner wants to cluster production together in the centre. There are no dynamics and we can compute the solution explicitly as a sequence of parallel spatial distributions.

With regard to related theoretical results, dynamic optimisation on general vector spaces has received some recent attention. Cosar and Green (2014) show sufficiency and necessity of a transversality condition under general conditions. However, Cosar and Green (2014) do not discuss under what conditions a solution exists, and how to characterise economic problems using Euler conditions as we do here. Brock et al. (2014) also gives results and solution techniques for a variational problem for a specific model on Hilbert spaces, though the results there apply to continuous time bounded problems.

2 Infinite horizon optimisation problem: main results

2.1 The optimisation problem

We introduce our infinite horizon problem by describing elements of a tuple $(X,\tau,S,\Gamma,\beta,u)$. Let X be a topological vector space endowed with topology τ. Our problem's state space will be $S\subset X$. $\Gamma\colon S\twoheadrightarrow S$ is a time invariant feasibility correspondence and $u\colon \operatorname{Gr}\Gamma\to\mathbb{R}_+$ [1]is a per-period payoff function and β is a discount factor.

We seek a sequence in S to maximise the discounted infinite sum of per-period utilities

$$\max_{(x_t)} U\left((x_t)\right) := \sum_{t=0}^{\infty}\beta^t u\left(x_t,x_{t+1}\right) \tag{1}$$

subject to

$$x_{t+1}\in\Gamma\left(x_t\right) \qquad \forall t$$

$$x_0\in S \text{ given}$$

We call a sequence feasible if $x_{t+1}\in\Gamma\left(x_t\right)$ for all t. And we call a feasible sequence optimal if it attains the maximum of (1).

We now collect the assumptions required for the results in this paper.

Assumption 1. *If $A\subset X$ is compact then $\Gamma(A)$ is compact*

Assumption 2. *The relative topology $\tau_t\colon =\Gamma^t(x_0)\cap\tau$ on $\Gamma^t(x_0)$ is metrizable for all $t\geq 1$ and $x_0\in S$*

Assumption 3. *The function u is Frechet differentiable on* $\operatorname{int}\operatorname{Gr}\Gamma$

Assumption 4. $\operatorname{Gr}\Gamma$ *is convex and u is concave*

Assumption 5. *If (x_n,y_n) is a convergent sequence in* $\operatorname{Gr}\Gamma$*, then $u(x_n,y_n)\to u(x,y)$ in $\mathbb{R}$*

Note the above assumption is not equivalent to continuity as $\operatorname{Gr}\Gamma$ may not be first countable.

Assumption 6. *Let $u_1(x,y)$ be the partial (Frechet) derivative of $u(x,y)$ with respect to x, where $(x,y)\in\operatorname{int}\operatorname{Gr}\Gamma$. Then $u_1(x,y)(h)\geq 0$ for any $h\in S$*

[1] we have ruled out negative infinity pay-offs. why?

In this assumption, we know the partial derivatives exist as u is Frechet differentiable on $\operatorname{int}\operatorname{Gr}\Gamma$. See proposition **??** in the appendix for a proof.

Assumption 7. $\operatorname{Gr}\Gamma$ *is closed*

Definition 2.1. (Feasible two period deviation) For any path (x_t), define

$$G_t := \Gamma(x_t) \cap \Gamma^{-1}(x_{t+2}) \tag{2}$$

Note if x_t is feasible, $\Gamma(x_t)$ and $\Gamma^{-1}(x_{t+2})$ will overlap so G_t will be non-empty

Assumption 8. *If $(x_t) \in X^{\mathbb{N}}$ is a solution to (1), then $x_t \in \operatorname{int} G_t$ for all t*

Assumption 9. *Let (x_t) be feasible, then*

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1})$$

exists in $\mathbb{R}$.

Assumption 10. *Fix x_0, there exists $M\colon \mathbb{N} \times X \to \mathbb{R}$ such that*

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t M(x_0)_t < \infty \tag{3}$$

and

$$|u(x_t, x_{t+1})| \leq M(x_0)_t \ \ (\forall t \in \mathbb{N})$$

for all feasible (x_t)

2.2 Results

We now state our main result.

Theorem 2.1. *Let X be a topological vector space with topology τ. If assumptions 1, 2, 5, 7 and 10 hold, then given an initial value $x_0 \in S$ there exists (x_t) that solves (1)*

When we want to solve an economic problem, we need necessary and sufficient conditions for a solution path. Now let X be a normed space.

Proposition 2.1. *Let assumptions 3 and 8 hold. If (x_t) is an optimal sequence, then*

$$u_2(x_t, x_{t+1})(h) + \beta u_1(x_{t+1}, x_{t+2})(h) = 0 \quad (\forall t), (\forall h \in X) \tag{4}$$

Theorem 2.2. *Let assumptions 3, 4, 6 and 8 hold. If (x_t) is feasible, Frechet differentiable at each pair (x_t, x_{t+1}), satisfies (4) and*

$$\lim_{t\to\infty} \beta^t u_1(x_t, x_{t+1})(x_t) = 0 \tag{5}$$

then (x_t) is optimal

Theorem 2.3. *Let Assumptions 3, 4, 6, 8 and 9 hold. Suppose $u(0,0) \in \mathbb{R}$ where $0 \in X$ is the null vector. If (x_t) is optimal then*

$$\lim_{t\to\infty} \beta^t u_1(x_t, x_{t+1})(x_t) = 0$$

3 Applications

3.1 Endogenous growth and economic geography

Let's use the results from section 2 to study the planner's problem of a geographical growth model. Firms live at points in $\mathscr{O} \subset \mathbb{R}^k$ and produce identical goods. Total output for each firm takes two inputs. The first is capital accumulated by the firm, and the second is spillovers as a function of other firms' capital. As in Romer (1986), we assume spillovers are strong enough to generate endogenous growth.

Unlike Brock et al. (2014), we assume boundaries are not periodic, so the distance between firms $i, j \in \mathscr{O}$ is $\|i - j\|$. Central locations receive and generate more spillovers than others towards the periphery of $\mathscr{O}$.

3.2 The planner's problem

Recall the tuple $(X, \tau, S, \Gamma, u, \beta)$ describes the elements of our infinite dimension optimisation problem. Let X be the Hilbert space $L^2(\mathscr{O})$ and equip X with the weak topology (see ch.13 in Aliprantis and Border (2005)). Let the space of positive square integrable functions $L^2(\mathscr{O})_+$ be our state space S. For $x \in S$, the number $x(i)$ tells us the amount of capital accumulated by firm at location $i \in \mathscr{O}$.

To discuss feasibility and pay-offs consider a compact operator $\Omega\colon S \to S$. Think of x as capital, then Ωx takes x and tells us the spillovers each location receives. If $\mathscr{O}$ is a compact space, we can make Ω by using a continuous distance kernel $\omega\colon \mathscr{O} \times \mathscr{O} \to \mathbb{R}$

$$\Omega x(i) = \int_{\mathscr{O}} \omega(\cdot, i) x \, \mathrm{d}\lambda = \int_{\mathscr{O}} \omega(s, i) x(s) \, \mathrm{d}s$$

where λ is the Lebesgue measure.

Firms at all locations produce according to $f\colon \mathbb{R}^2_+ \to \mathbb{R}_+$. Assume f is continuously differentiable on $\mathbb{R}^2_{++}$, concave and strictly quasi-concave. To generate constant growth, also assume f is homogeneous of degree one so $f(\alpha x, \alpha y) = \alpha f(x, y)$ for any scalar α.

Define a function $F\colon S \to \mathbb{R}_+$ giving total output by adding up output at all locations

$$F(x) = \int_{\mathscr{O}} f(x, \Omega x) \, \mathrm{d}\lambda \tag{6}$$

Lemma 3.1. *F is differentiable on* $\operatorname{int} S$*, concave, strictly quasi-concave and homogeneous of degree one. Moreover, $x \mapsto F'(x)$ is homogeneous of degree zero*

Note this implies $F'(\alpha x) = F'(x)$ for all α that are scalars.

Assume the planner can move capital and output around without any cost. This may be unrealistic, but it means we can focus on how spillovers effect the geographical distribution of accumulation. The planner faces the following feasibility correspondence

$$\Gamma(y) = \left\{ x \in S \mid 0 \le \int_{\mathscr{O}} x \, \mathrm{d}\lambda \le F(y) \right\}$$

Now assume consumers have a utility function $v\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the Inada conditions. Define per-period pay-off $u\colon \operatorname{Gr}\Gamma \to \mathbb{R}_+$ as

$$u(x, y) = v\left(F(x) + \int_{\mathscr{O}} x \, \mathrm{d}\lambda - \int_{\mathscr{O}} y \, \mathrm{d}\lambda \right) \tag{7}$$

The planner maximises (1) subject to $x_{t+1} \in \Gamma(x_t)$ for all t and $\beta \in (0, 1)$.

We can now verify existence and uniqueness for the planner's problem; see the full paper for details. The optimal path will agree with the Euler condition (proposition 2.1) almost everywhere

$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta}\beta^{-1}-1=f_1\left(x_{t+1},\Omega x_{t+1}\right)+\Omega f_2\left(x_{t+1}\right)=F'\left(x_{t+1}\right)\quad\forall t \tag{8}$$

$$c_t=F\left(x_t\right)+\int_{\mathcal{O}}x_t\,\mathrm{d}\lambda-\int_{\mathcal{O}}x_{t+1}\,\mathrm{d}\lambda \tag{9}$$

The Euler condition says that the marginal benefit of increasing the capital stock at location i tomorrow equals the marginal product of capital at location i plus the weighted sum of marginal products of increased spillovers at all other locations.

3.3 Computing a solution

To compute a solution, I assume a CES functional form for production, and use a method called projected decent. The details and proof for this method can be found in Ulbrich (2009).

The solution of the model will be a sequence of capital distributions parallel to the one shown in figure 1. The planner's growth rate (g) is not only higher, but we can see that the planner prefers to concentrate capital at more central locations. The gini coefficient tells us the inequality of capital accumulation across space, and as expected the planner prefers a more unequal distribution.

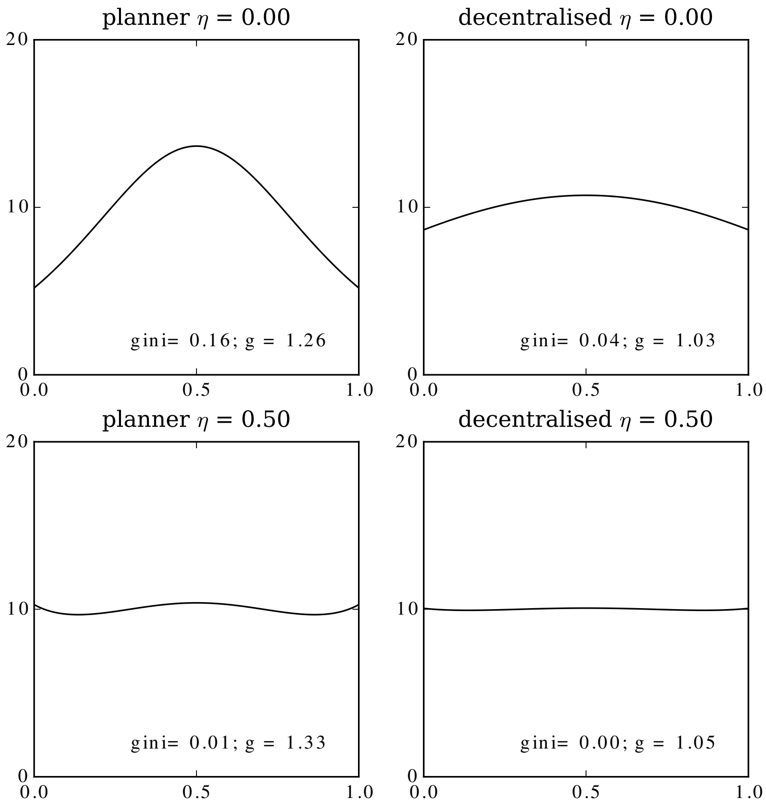

Figure 1: Planner and decentralised distribution distribution of capital

4 Conclusion

In this paper, we present necessary and sufficient for solutions and existence on solutions for dynamic optimisation models on infinite dimensions. We demonstrated our results with a model of spatial heterogeneity.

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