Samuelson's Correspondence Principle Reassessed

Andrew McLennan
University of Queensland*

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Abstract

Samuelson's correspondence principle is reassessed in the light of modern mathematical tools and views concerning economic dynamics. The principle, as it was described by Samuelson, is thought of as consisting of three parts: a) if an isolated equilibrium is dynamically stable, then its fixed point index is +1; b) if the index of an isolated equilibrium is not +1, it is empirically irrelevant; c) an index of +1 has implications for comparative statics. Both in general equilibrium and in game theory, stability with respect to naive adjustment dynamics implies the index condition, but models with sophisticated agents do not provide the same sort of support. We show that b) can be phrased in a quite general manner, as befits a supposedly universal principle, and we argue that it is indeed a very reasonable hypothesis. The usefulness of the index condition is perhaps more limited than Samuelson had hoped, but nonetheless extends far beyond Samuelson's applications to comparative statics.

Keywords: Paul Samuelson, correspondence principle, fixed point index, vector field index, dynamical system, asymptotic stability, adjustment to equilibrium, Lyapunov function, converse Lyapunov theorem.

1 Introduction

The correspondence principle was described by Paul Samuelson in two articles (Samuelson (1941, 1942)) and his famous *Foundations of Economic Analysis* (Samuelson (1947))

*Email: a.mclennan@economics.uq.edu.au. I would like to thank Federico Echenique, Klaus Ritzberger, Glen Weyl, John Wooders, and audiences at the 2015 Australian Economic Theory Conference and the University Technology Sydney for helpful comments.

after being stated informally by Hicks (1939). The idea can be illustrated in a two good exchange economy. Figure 1 shows the excess demand for the second good as a function of the second good's price, when the first good is the numeraire. There are three equilibria, two of which are stable relative to price dynamics that increase (decrease) the price of the second good when it is in excess demand (supply).

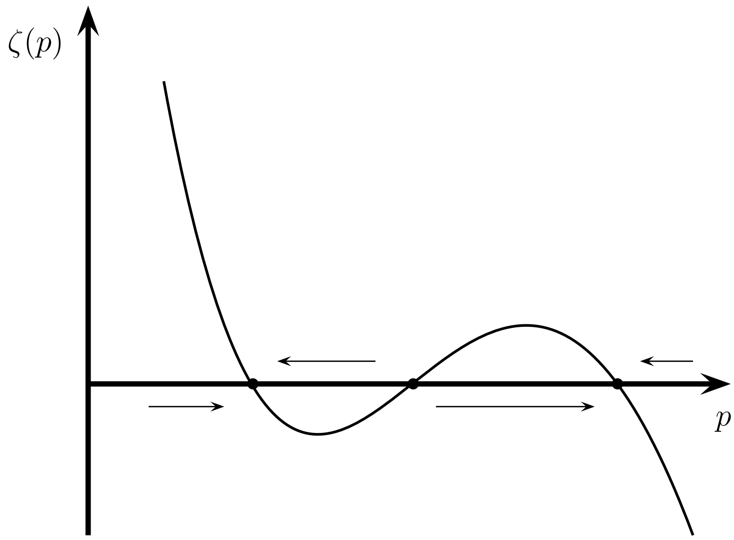

Figure 1

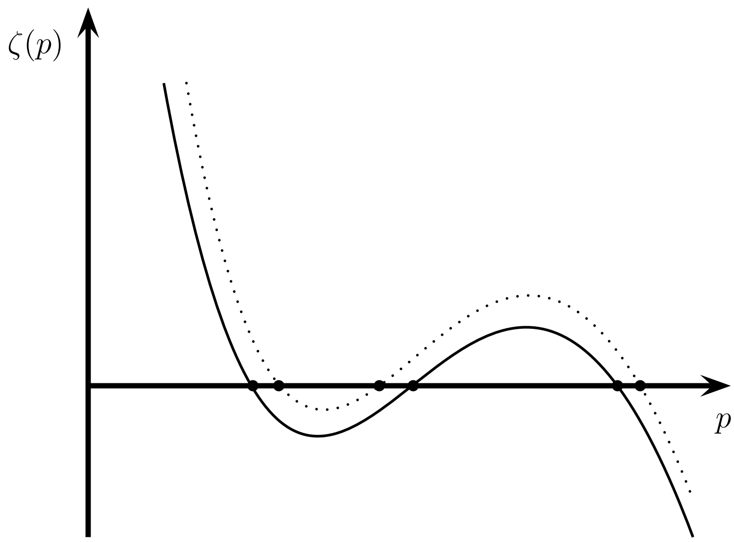

Figure 2

Figure 2 shows the effect of changing a parameter in a way that increases demand for the second good. This has the expected effect of increasing the second good's equilibrium price for the two stable equilibria, but it leads to a price decrease in the unstable equilibrium. In a nutshell, Samuelson's understanding of the correspondence principle was that dynamic stability had implications for comparative statics.

Samuelson saw the correspondence principle as an initial insight gleaned from the development of a dynamic approach to economic analysis, whose relationship to static models was comparable to the relationship between static and dynamic analysis in physics: "An understanding of this principle is all the more important at a time when pure economic theory has undergone a revolution of thought—from statical to dynamical modes." (*Foundations of Economic Analysis*, p. 284). He held it in quite high regard (*Foundations of Economic Analysis*, p. 351):

> Broadly speaking, the development of analytic economics has proceeded in a natural evolutionary order. First, in Walras we have the final culmination of the notion of *determinateness* ... However, Pareto took a second, further step. He laid the basis for a theory of *comparative statics* ... Pareto ... rarely concerned himself with the secondary inequalities ... [and it] was left for W. E. Johnson, Slutsky, Hicks and Allen, Georgescu-Roegen, Hotelling, and other modern writers to begin to make progress along this third line ... Where the interactions between individuals are concerned, the scope of fruitful *comparative statics* may be greatly extended by a fourth advance, the apprehension of the *correspondence principle* ...

Researchers in general equilibrium theory (e.g., Arrow and Hurwicz (1958); Arrow et al. (1959)) found some special cases in which some equilibria are necessarily stable with respect to Walrasian tatonnement dynamics, but examples developed by Scarf (1960) showed that this phenomenon is restricted to very small numbers of goods or agents. After summarizing this line of research, Arrow and Hahn expressed the following negative assessment (*General Competitive Analysis*, p. 321):

> Thus what the "correspondence principle" amounts to is this: Most of the restrictions on the form of the excess-demand functions that are at present known to be sufficient to insure global stability are also sufficient to allow certain exercises in comparing equilibria. It should be added that these same conditions also turn up in the discussion of the uniqueness of a competitive equilibrium. All these restrictions share the characteristic that they are not necessary for the task for which they were invented; they are only sufficient and this explains why the correspondence principle "isn't."

Over the last half century the correspondence principle has appeared only rather infrequently in theoretical economics. The most important work is due to Echenique (2002, 2004) (see also Echenique (2008)) who works with games with strategic complementarities (Topkis (1979); Vives (1990); Milgrom and Roberts (1990)) that satisfy very strong order theoretic conditions, but are otherwise technically quite general. We provide brief sketches of his results and arguments in Section 4.

The purpose of this paper is to give a modern assessment of the logic underlying Samuelson's development of the correspondence principle. The goal is not to judge which authors (if any) were correct, but rather to recast the correspondence principle in modern terms that allow us to see it as a conjunction of several ideas, in the hope of assessing the continuing value of each of its elements. We will apply more advanced mathematical methods than were available at the time the correspondence principle was formulated. In addition, economists now think about dynamics in a quite different manner, leading to new perspectives.

Samuelson's writings consider a host of specific models, but he did not formulate the correspondence principle as a definite and general theorem, so it is first of all necessary to give a precise interpretation. For us the central concept is the fixed point index. The fixed point index of a regular fixed point of a smooth function f is +1 or -1 according to the sign of the determinant of the matrix of partial derivatives of $\mathrm{Id} - f$. (A fixed point is *regular*, by definition, if this determinant does not vanish.) The fixed point index can be extended to functions that are merely continuous, to suitable upper semicontinuous correspondences, and to a very general class of spaces, which includes some infinite dimensional spaces. The extended index assigns an index to each subset of the set of fixed points that is both open and closed in the relative topology of this set, and it has an axiomatic characterization which is presented in Section 2.

We will argue that the most valuable aspect of the correspondence principle is the hypothesis that if the index of an isolated equilibrium is not +1, then it is empirically irrelevant. We call this *the index +1 principle*. Samuelson's understanding of the correspondence principle can be thought of as consisting of the index +1 principle and two additional components, namely the passage from dynamic stability of an equilibrium to its index being +1, and the derivation of consequences for comparative statics.

Section 3 assesses the consequences of an equilibrium having an index of +1. We will see that in order for there to be implications for the signs of comparative statics, quite demanding additional conditions need to be satisfied. These are much more likely to hold when the dimension of the model is quite small, and for this reason contemporary economists are perhaps more inclined than Samuelson to regard them as quite special. At the same time an index of +1 is a piece of mathematical information with many more possible applications than Samuelson had in mind.

Samuelson did not consider dynamic adjustment processes beyond tatonnement and its most obvious generalizations, and for the most part this is true of other authors studying general equilibrium. In biological applications of game theory strategies are identified with genotypes, and payoffs with inclusive reproductive fitness, so naive or mechanistic adjustment processes have a plausible motivation. This, together with the continuing search for useful refinements of the Nash equilibrium concept, has recently generated a large literature on evolutionary game theory (e.g., Samuelson (1997), Fudenberg and Levine (1998)) with many results concerning dynamic stability. The dynamic processes studied by Echenique (2002, 2004) are among the most general studied in this literature, encompassing fictitious play and iterative best response dynamics, Demichelis and Ritzberger (2003) provides pointers to a rather large literature in which evolutionary processes are studied, with various consequences for Nash equilibria and sets of equilibria that are, in various senses, stable.

Section 4 studies the connection between an index of +1 and dynamic stability. We focus on the result of Demichelis and Ritzberger (2003) which establishes a connection between stability and the index that is quite general and makes minimal assumptions on the dynamic process: if an isolated Nash equilibrium is stable with respect to *some* adjustment dynamics that is "payoff consistent," then its index is +1. (More generally, if a connected component of the set of Nash equilibria is stable with respect to "natural" adjustment dynamics, then its index agrees with its Euler characteristic). In order to illustrate the relationship between dynamics and the fixed point index, and the application of the axiomatic formulation of the index, we transport one of their arguments to a general equilibrium setting.

Samuelson thought of dynamic price adjustment as something that could be modelled and perhaps observed directly, but upon reflection, and in the light of extensive experience, the current prevailing view is that price adjustment cannot be predictable and must therefore be instantaneous if the eventual equilibrium prices can be predicted. (More generally, absence of arbitrage implies that the price process, say of a financial asset, must be at least roughly a martingale.) Similarly, a process of continuous strategic adjustment of mixed strategies cannot describe the behavior of sophisticated agents who understand the process and best respond to it.

Intuitively one might regard a price equilibrium as the "shadow" of a complex dynamic interaction between arbitrageurs, speculators, and other traders, adjusting to the arrival of new information, which might be observed by some but not others. Might one be able to "justify" the restriction to index +1 equilibria by arguing that these are the only equilibria supported by equilibria of the more complicated model with dynamically rational agents? Section 5 provides a result that suggests that this program is unlikely to succeed: under natural assumptions, the set of equilibria of the complex game lying above an equilibrium of the simple model has the same index as the equilibrium of the simple model. If the index of the simple model is -1, then there will be equilibria of the complex game that "support" it, and if there is a single equilibria, its index will be -1. For this reason it may be regarded as implausible, but this is just another appeal to the index +1 principle, and not a "justification" of it. This result does provide some assurance that we will not run into situations that render the commitment to index +1 equilibria unworkable. For example, it cannot happen that an index -1 equilibrium of the simple model lies below the unique equilibrium of the complex game, which necessarily has index +1.

Thus, the restriction to index +1 equilibria is an *hypotheses* rather than a consequence of other principles. As such, it can be compared with experience and intuition. Counterexamples—for instance a mixed equilibrium of a battle-of-the-sexes that persisted over time—are at least very rare, if any have ever been reported. The supporting intuition seems (to me at least) much stronger than the theoretical motivations based on dynamic stability that have been developed to date. Section 6 elaborates and describes an experimental study supporting the hypothesis.

Section 7 summarizes and concludes.

2 The Fixed Point Index

A fundamental principle of economic analysis should be valid in great generality. In this section we present the axiomatic characterization of the fixed point index, emphasizing that it is applicable to quite general spaces and correspondences, and is not restricted by assumptions of smoothness or genericity.

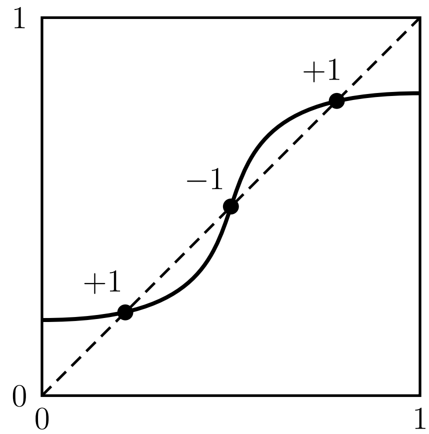

Figure 3

The key ideas are already illustrated by Figure 3, which shows the graph of a continuous function $f : [0, 1] \to [0, 1]$ that has three fixed points. At two of these the graph of f goes from above the diagonal to below as we go from left two right. Each of these has index +1. At the third fixed point the graph of f goes from below the diagonal to above, and this fixed point has index -1. Since f is differentiable, and each of its fixed points is "regular," in the sense that the derivative of $\mathrm{Id}_{[0,1]} - f$ does not vanish there, we can think of the index of a fixed point as the sign of the derivative of $\mathrm{Id}_{[0,1]} - f$ at that point. Note that the sum of the indices is +1 for all C^1 functions whose fixed points are all regular; as we will see below, this phenomenon is completely general.

The axiomatic characterization of the index extends the concept to higher dimensions (in fact even to infinite dimensional settings) and to correspondences, which need not satisfy any analogue of regularity, and which can have infinite sets of fixed points. It is based on two properties. First, the index of a set of fixed points is a local property, insofar as it is determined by the restriction of the function or correspondence to an arbitrarily small neighborhood of the set. Second, small perturbations of the function or correspondence do not change the index of the set of fixed points that lie in some set whose boundary contains no fixed points.

Historically the fixed point index evolved out of the general development of algebraic topology in the first half the 20$^{\text{th}}$ century. Beginning with the work of Poincaré and Brouwer. Lefschetz and Hopf brought the fixed point theorem to the context of manifolds in the 1920's, and Leray and Schauder provided the initial formulation of the index during the next two decades. In his Ph.D. thesis Browder (1948) extended the index to very general spaces, and the index was given an axiomatic formulation by O'Neill (1953). Book length treatments include Brown (1971) and Dugundji and Granas (2003), which study the subject from the point of view of pure mathematics, and McLennan (2012), which emphasizes correspondences rather than functions, with an eye to economic applications.

The fixed point index for regular economies was introduced in general equilibrium theory by Dierker (1972, 1974). It plays a role in the analysis of the Lemke-Howson algorithm in Shapley (1974). Hofbauer (1990) applies the vector field index (defined below) to dynamic issues in evolutionary game theory, and Ritzberger (1994) applies it to game theory systematically.

The most general setting for the fixed point index depends on advanced topological concepts. If Z is topological space and $X \subset Z$, a *retraction of* Z *onto* X is a continuous function $r : Z \to X$ such that $r(x) = x$ for all $x \in X$. We say that X is a *retract* of Z. A topological space X is an *absolute neighborhood retract* (ANR) (for metric spaces) if, whenever Z is a metric space and (a homeomorphic image of) X is a closed subspace, X is a retract of some neighborhood $U \subset Z$ of X. A retract of a convex subset of a locally convex topological vector space is an ANR, and any ANR has a homeomorphic image of this sort. (E.g., Section 7.4 of McLennan (2012).) An important intuition is that each point of an ANR has a neighborhood that retains some of the simplicity of the neighborhood in the topological vector space. Manifolds, simplicial complexes, and convex subsets of normed linear spaces are ANR's, so from the point of view of economic modelling this is a quite general class of spaces.

A topological space X is *contractible* if the identity function is homotopic to a constant function: there is a continuous function $c : X \times [0,1] \to X$ such that $c(\cdot, 0) = \mathrm{Id}_X$ and $c(\cdot, 1)$ is a constant function. Any star shaped subset of a topological vector space is contractible, and in particular convex sets are contractible. The circle is an example of a space that is not contractible.

A topological space X has *the fixed point property* if every continuous function from X to itself has a fixed point. Whether the hypothesis of convexity in Brouwer's fixed point theorem could be weakened to contractibility was unknown for several years, but Kinoshita (1953) gave an example of a compact contractible subset of $\mathbb{R}^3$ that is not an ANR and does not have the fixed point property. On the other hand Eilenberg and Montgomery (1946) showed that every compact contractible ANR has the fixed point property. In view of these results, ANR's seem to be the most general setting in which fixed point theory is well behaved.

In general the graph of a set valued function $F : X \to Y$ is $\mathrm{Gr}(F) = \{\, (x, y) \in X \times Y : y \in F(x) \,\}$. If X is a space, $C \subset X$, and $F : C \to X$ is a set valued mapping, the set of *fixed points* of F is

$$\mathcal{F}(F) = \{\, x \in C : x \in F(x) \,\}.$$

A *correspondence* is a nonempty valued set valued function. (A function is treated as a singleton valued correspondence.) If X and Y are topological spaces, a correspondence $F : X \to Y$ is *upper hemicontinuous* if it is compact valued and, for each $x \in C$ and each neighborhood V of $F(x)$, there is a neighborhood $U \subset C$ of x such that $F(x') \subset V$ for all $x' \in U$.

Fix an ANR X. If $C \subset X$, a contractible valued upper hemicontinuous correspondence is *index admissible* if it is upper hemicontinuous and contractible valued, C is compact, and $\mathcal{F}(F)$ is contained in the interior $\mathrm{int}\,(C) = C \setminus \overline{(X \setminus C)}$ of C. Let $\mathcal{I}_X(C)$ be the set of index admissible correspondences $F : C \to X$, and let $\mathcal{I}_X = \bigcup \mathcal{I}_X(C)$ where the union is over all compact $C \subset X$.

Theorem 1. *There is a unique function $\Lambda_X : \mathcal{I}_X \to \mathbb{Z}$ satisfying:*

(I1) (Normalization) If $c : C \to X$ is a constant function whose value is in $\mathrm{int}\, C$, then

$$\Lambda_X(c) = 1.$$

(I2) (Additivity) If $F : C \to X$ *is an element of* $\mathcal{I}_X$, $C_1, \ldots, C_r$ *are pairwise disjoint compact subsets of* C, *and* $\mathcal{F}(F) \subset \operatorname{int} C_1 \cup \ldots \cup \operatorname{int} C_r$, *then*

$$\Lambda_X(F) = \sum_i \Lambda_X(F|_{C_i}).$$

(I3) (Continuity) For each $F : C \to X$ *in* $\mathcal{I}_X$ *there is a neighborhood* $U \subset C \times X$ *of* $\mathrm{Gr}(F)$ *such that* $\Lambda_X(\hat{F}) = \Lambda_X(F)$ *for every* $\hat{F} \in \mathcal{I}_X(C)$ *with* $\mathrm{Gr}(C) \subset U$.

Additivity implies that the fixed point index is a *local property* in the sense that $\Lambda_X(F)$ agrees with $\Lambda_X(F|_{C'})$ for any compact neighbrhood $C' \subset C$ of $\mathcal{F}(F)$. When $\mathcal{F}(F)$ is a finite union of connected components, the decomposition provided by Additivity allows us to speak of the index (for F) of each of these components. The index +1 principle can now be stated: *in order to be empirically relevant, the index (relative to the excess demand or best response correspondence) of an isolated equilibrium must be +1.* If X is compact, $\Lambda_X(\mathrm{Id}_X)$ is the *Euler characteristic* of X. The connection with dynamics investigated in Section 4 suggests that the following is the most general formulation of the index +1 principle: *if the set of equilibria has finitely many connected components, each of which is an ANR, in order for a connected component to be empirically relevant, its index must agree with its Euler characteristic.*

The Continuity axiom is closedly related to invariance under homotopy. Let $C \subset X$ be compact. A *homotopy* is a upper u.h.c.c.v. correspondence

$$H : C \times [0, 1] \to X.$$

We usually think of H as a continuous (in the appropriate topology) function taking each t to a correspondence $H_t = H(\cdot, t)$ "at time t." An *index admissible homotopy* (IAH) is a homotopy H such that each H_t is index admissible. If H is an IAH, then $\Lambda(H_t)$ is a (locally constant, hence) constant function of t, so

$$\Lambda(H_0) = \Lambda(H_1).$$

Most applications of Continuity invoke this fact.

We mention two other properties of the index.

(I4) (Commutativity[1]) If X and Y are compact ANR's, and $f : X \to Y$ and $g : Y \to X$ are continuous functions, then

$$\Lambda_X(g \circ f) = \Lambda_Y(f \circ g).$$

[1]This is actually a special case that is adequate for our applications. The general version of Commutativity is: if X and $\hat{X}$ are ANR's, C, D, $\hat{C}$, and $\hat{D}$ are compact sets with $D \subset C \subset X$ and $\hat{D} \subset \hat{C} \subset \hat{X}$, $g : C \to \hat{X}$ and $\hat{g} : \hat{C} \to X$ are continuous with $g(D) \subset \operatorname{int} \hat{C}$ and $\hat{g}(\hat{D}) \subset \operatorname{int} C$, $\hat{g} \circ g|_D$ and $g \circ \hat{g}|_{\hat{D}}$ are index admissible, and $g(\mathcal{F}(\hat{g} \circ g|_D)) = \mathcal{F}(g \circ \hat{g}|_{\hat{D}})$, then $\Lambda_X(\hat{g} \circ g|_D) = \Lambda_{\hat{X}}(g \circ \hat{g}|_{\hat{D}})$.

(I5) (Multiplication) If X and Y are ANR's, $F : C \to X$ is an element of $\mathcal{I}_X$, $G : D \to Y$ is an element of $\mathcal{I}_Y$, and $F \times G$ is the correspondence taking (x, y) to $F(x) \times G(y)$, then

$$\Lambda_{X \times Y}(F \times G) = \Lambda_X(F) \cdot \Lambda_Y(G).$$

In view of the uniqueness asserted in Theorem 1, these properties are in principle consequence of axioms (I1)-(I3), but no direct proof of them is known. They are established in the most elementary cases, and then treated as axioms.

The proof of Theorem 1 is quite lengthy and has several stages. First, the theorem is established for smooth index admissible functions $f : C \to \mathbb{R}^m$ (where C is a compact subset of $\mathbb{R}^m$) whose fixed points are all regular, using the methods of differentiable topology. The second stage uses the fact that continuous functions can be approximated by smooth functions to extend the index to continuous index admissible functions on Euclidean spaces. Using the fact that compact subsets of ANR's can be approximated, in a certain sense, by simplicial complexes (Theorem 7.6.3 of McLennan (2012)) the index is then extended to continuous functions on ANR's. Commutativity, which was introduced by Browder (1948), is already a nontrivial fact of linear algebra for regular fixed points, and is the key tool used to achieve the generalization at this stage. Finally the index is extended from functions to contractible valued correspondences using a suitable generalization (see Ch. 9 of McLennan (2012)) of the following result. The finite dimensional version of this result for convex valued correspondences was the method Kakutani (1941) used to prove his extension of Brouwer's theorem. It was extended to contractible valued correspondences by Mas-Colell (1974) and to ANR's by the author in McLennan (1991).

Proposition 1. *Suppose that X and Y are ANR's, $C \subset X$ is compact, and $F : C \to Y$ is an upper semicontinuous contractible valued correspondence. Then for any neighborhood U of $\mathrm{Gr}(F)$ there is a continuous function $f : C \to Y$ such that $\mathrm{Gr}(f) \subset U$.*

3 Comparative Statics and Other Consequences

Samuelson wrote during the heyday of logical positivism, which attempted to understand science as a collection of testable hypotheses. While economics clearly provided some large scale overview of its subject matter, the discipline was somewhat embarassed to find itself with a paucity of concrete predictions that could be applied to available data sets. For this reason great conceptual importance was attached to the method of comparative statics, which can sometimes predict the sign of the change of an endogenous variable as an exogenous parameter changes. The importance Samuelson attached to the correspondence principle is largely a matter of its crucial role in these comparative statics exercises. In this section we review the basic computations, following Samuelson's presentation, then discuss the significance of the index +1 principle more generally.

Consider a system of equations $0 = g(x, \alpha) \in \mathbb{R}^n$ where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Here we are thinking of $g(\cdot, \alpha)$ as the function $\mathrm{Id} - f(\cdot, \alpha)$, where $f(\cdot, \alpha)$ is the function whose fixed points are the equilibria for parameter α. Fix an initial value α^* of the parameter and an equilibrium x^* for this parameter, let M be the matrix of partial derivatives $\frac{\partial g_i}{\partial x_j}(x^*, \alpha^*)$, and let Δ be its determinant. The index +1 principle is then the condition $\Delta > 0$. Differentiating the equation $g(x(\alpha), \alpha) = 0$ gives the equation

$$M \cdot \frac{dx}{d\alpha} + \frac{\partial g}{\partial \alpha} = 0$$

where $\frac{dx}{d\alpha}$ is the vector with entries $\frac{dx_j}{d\alpha}(x^*)$ and $\frac{\partial g}{\partial \alpha}$ is the vector with entries $\frac{\partial g_i}{\partial \alpha}(x^*, \alpha^*)$. Let Δ_{ij} be the (i, j)-cofactor of M, which is the determinant of the matrix obtained by eliminating the i^{th} row and the j^{th} column of M. Then the (i, j)-entry of the inverse of M is Δ_{ji}/Δ, so we obtain the formula

$$\frac{dx_j}{d\alpha} = -\frac{\sum_{i=1}^{n} \Delta_{ji} \frac{\partial g_i}{\partial \alpha}}{\Delta}.$$

In order to attribute a definite sign to this quantity one needs to be able to sign Δ, but one must also be able to determine the sign of the numerator. One way in which this can happen is that M is a (positive or negative) definite matrix, so that each Δ_{ij} has a known sign. Another possibility is that for some i (most commonly $i = j$) it is known a priori that $\frac{\partial g_k}{\partial \alpha} = 0$ for all $k \neq i$, and that $\frac{\partial g_i}{\partial \alpha}$ and Δ_{ji} have definite signs. As the dimension n increases, the sorts of assumptions that have such implications become quite complex and restrictive. All of the concrete examples given by Samuelson are low dimensional.

However, there are testable implications of the index +1 principle that are not matters of comparative statics. One of the most obvious and compelling is that we should not observe the mixed equilibrium of the battle of the sexes as a stable, self-replicating pattern of behavior. Concretely, consider what happens when you and a stranger are walking towards each other on a sidewalk. Should you go to your left, expecting the other person to also go to their left, or should you go to the right? It can easily happen that different societies have different equilibria. (In Australia people pass on the left, in the US on the right.) But it would be very surprising to find a society that persistently failed to settle into one convention or the other.

As this example illustrates, game theory has given us a much broader perspective on the sorts of models whose testable predictions might be compared with available data. This is also true of economics, construed more narrowly as the study of markets; the models in use today are much richer and varied than when Samuelson was writing. Experimental methods can provide customized data, allowing a much broader range of hypotheses to become testable. The index +1 principle supplies only one bit of information, but certainly in many cases that bit will be critical.

4 Dynamic Motivation

We now examine the relationship between the index +1 principle and dynamic stability with respect to iterative adjustment processes. Although we will see results according to which the index +1 principle is a consequence of dynamic stability, the support the principle receives from such results is actually quite limited and qualified. As we will explain in greater detail in the next section, such dynamics are "naive" rather than the consequence of optimization by agents who understand their environment. The classes of dynamics that have been considered in the literature are special, insofar as they are implicitly assumed to depend on quite restricted information. (It seems likely that these implicit assumptions can be weakened in many directions.)

Furthermore, in general theory provides no assurance that stability will prevail. Scarf (1960) gave an example of an economy for which the unique Walrasian equilibrium is unstable with respect to tatonnement (the speed of price adjustment of each good is proportional to its excess demand). In fact the Debreu-Mantel-Sonnenschein theorem implies the existence of an exchange economy with a Walrasian equilibrium p^* such that at every price p in some neighborhood of p^* excess demand (normalized to lie in the simplex, as decribed below) is a positive multiple of $p - p^*$, so that the equilibrium is unstable with respect to any price dynamics that is "tatonnement-like" in the weak sense of adjusting prices at p in a direction that increases the value of the excess demand at p. When the number of goods is odd such an equilibrium has index +1, so it can be the economy's unique equilibrium.

A later line of research (Saari and Simon (1978); Saari (1985); Williams (1985); Jordan (1987)) showed that stability is informationally demanding, in the sense that an adjustment process that is guaranteed to return to equilibrium after a small perturbation requires essentially all the information in the matrix of partial derivatives of the aggregate excess demand function.

These considerations cast doubt on attempts to provide a foundation for the index

+1 principle that depends on a particular dynamical system such as tatonnement. In addition, outside of the context of evolutionary biology, it seems unlikely that the actual dynamical process can ever be known with much precision. Instead, following Demichelis and Ritzberger (2003), we argue that an equilibrium (component of the set of equilibria) has index +1 (index equal to its Euler characteristic) if it is *potentially stable*, in the sense of being locally stable with respect to *some* dynamic adjustment process that is not perverse, relative to the direction of excess demand or utility improving adjustment of mixed strategies.

We now lay out the elementary definitions for dynamical systems. Let $C \subset \mathbb{R}^n$ be a C^∞ n-dimensional manifold with boundary. That is, each point $p \in C$ has a neighborhood $U \subset C$ for which there is a C^∞ diffeomorphism between U and an open subset of the half space $\{ x \in \mathbb{R}^n : x_n \geq 0 \}$. The *boundary* ∂C of C is the set of points taken by some such diffeomorphism to a point in $\{ x \in \mathbb{R}^n : x_n = 0 \}$. For each $p \in \partial C$ the *tangent space* of ∂C at p is the set $T_p(\partial C)$ of vectors that are derivatives at p of smooth curves in ∂C that pass through p. Elementary arguments show that $T_p(\partial C)$ is an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$

A *vector field* on C is a continuous function $\nu : C \to \mathbb{R}^n$. Zeros of ν are *equilibria*. We assume that ν is *inward pointing*, which is to say that, for each $p \in \partial C$, $\nu(p) \notin T_p(\partial C)$ (so p is not an equilibrium) and $p + \varepsilon\nu(p) \in C$ for all small $\varepsilon > 0$. If $U \subset C$ is open and there are no equilibria in the topological boundary $\partial U = \overline{U} \setminus U$, then the *vector field index* of $\nu|_{\overline{U}}$ is defined to be $\Lambda_{\mathbb{R}^n}(\tilde{\nu}|_{\overline{U}})$ where $\tilde{\nu} : C \to \mathbb{R}^n$ is the function $\tilde{\nu}(p) = p + \nu(p)$. (With some technical effort this definition can be extended to a vector field index for vector fields on smooth manifolds.)

We always assume that ν is Lipschitz, so that the fundamental results for ordinary differential equations imply that there is a unique continuous *flow* $\Phi : C \times [0, \infty) \to C$ such that for all $p \in C$, $\Phi(p, \cdot)$ is C^1 with $\Phi(p, 0) = p$ and $\frac{\partial \Phi}{\partial t}(\Phi(p,t), t) = \nu(\Phi(p,t))$ for all t. A set $A \subset C$ is *invariant* if $\Phi(A, t) \subset A$ for all $t \geq 0$, and it is *stable* if, for any neighborhood V of A, there is a neighborhood W of A such that $\Phi(p, t) \in V$ for all $p \in W$ and $t \geq 0$. The set A is *uniformly attractive* if there is a neighborhood V of A such that for any neighborhood W of A, there is a $T \geq 0$ such that $\Phi(p, t) \in W$ for all $p \in V$ and $t \geq T$. Finally, the set A is *uniformly asymptotically stable* if it is compact, invariant, stable, and uniformly attractive.

The *domain of attraction of* A is

$$D = \{ p \in C : \limsup_{t \to \infty} d(\Phi(p,t), A) = 0 \}.$$

A *Lyapunov function* for D is a C^1 function $L : D \to [0, \infty)$ such that:

(a) $V^{-1}(0) = A$.

(b) There is $a : (0, \infty) \to (0, \infty)$ such that $-DL(V(p))\nu(p) \geq a(d(p, A))$ for all $p \in D$.

(c) $V(p_n) \to \infty$ whenever $\{p_n\}$ is a sequence converging to a point outside of D.

It is intuitive and very well known that if A is compact and there is a Lyapunov function for A, then A is asymptotically stable. The converse is a highly nontrivial result with a rather complicated history, that is briefly sketched by Nadzieja (1990). Briefly, a sequence of partial solutions, over several decades, eventually culminated in a complete (in the sense that the Lyapunov function can be required to be C^∞) solution by Wilson (1969).

We illustrate these concepts by applying them to general equilibrium theory, following one of the arguments in Demichelis and Ritzberger (2003). Let $e = (1, \ldots, 1) \in \mathbb{R}^\ell$, and let $H_0 = \{\, p \in \mathbb{R}^\ell : \langle e, p \rangle = 0 \,\}$ and $H_1 = \{\, p \in \mathbb{R}^\ell : \langle e, p \rangle = 1 \,\}$. Let $P = \{\, p \in H_1 : p_i > 0 \text{ for all } i \,\}$ be the open price simplex. Let $\zeta : P \to \mathbb{R}^\ell$ satisfy:

(a) $\langle p, \zeta(p) \rangle = 0$ for all p (Walras' law).

(b) ζ is bounded below.

(c) $\|\zeta(p_n)\| \to \infty$ whenever $p_n \to p \in \overline{P} \setminus P$.

For $p \in P$ let $\tilde{\zeta}(p) = \zeta(p) - \langle e, \zeta(p) \rangle p$. Then $\tilde{\zeta}(p) = 0$ if and only if $\zeta(p) = 0$, and $\tilde{\zeta}(p) \in H_0$, so $\tilde{\zeta}$ is a vector field on P.

Let $C \subset P$ be a compact $(\ell-1)$-dimensional C^∞ manifold with whose interior contains all the equilibria of ζ. A *natural price dynamics* is a Lipschitz vector field $\nu : C \to H_0$ such that:

(a) ν is inward pointing.

(b) $\nu(p) = 0$ for all p such that $\zeta(p) = 0$.

(c) $\nu(p) \notin \{\, \alpha\tilde{\zeta}(p) : \alpha \leq 0 \,\}$ for all p such that $\zeta(p) \neq 0$.

That is, a price dynamics is natural if the Walrasian equilibria are the equilibria of the dynamics and prices never adjust in the direction that is precisely opposite to excess demand.

Theorem 2. *If ν is a natural price dynamics, and A is an asymptotically stable set for ν that is also an ANR, then the index of A (with respect to ζ) is the Euler characteristic χ_A of A.*

Proof. In view of (c) convex combination gives an IAH between $\tilde{\zeta}$ and ν, so (by homotopy) suffices to show that χ_A is the fixed point index of A with respect to the function

$$\beta_\delta : p \mapsto p + \delta\nu(p)$$

for some $\delta > 0$.

Let D be the domain of attraction for A. Since A is asymptotically stable, hence Lyapunov stable, Wilson's theorem gives a Lyapunov function $L : D \to [0, \infty)$. Since A is an ANR, for sufficiently small $\varepsilon > 0$ there is a retraction $r : N \to A$ where $N = L^{-1}([0, \varepsilon])$. We have $\Phi(N \times [0, \infty)) \subset N$. Since ν is inward pointing on ∂N and Lipschitz, there is some $\delta > 0$ such that convex combination gives an IAH between β_δ and $\Phi(\cdot, \delta)|_N$. It suffices to show that χ_A is the fixed point index of A with respect to $\Phi(\cdot, \delta)|_N$.

The function $s \mapsto \Phi(\cdot, s)|_N$ gives an IAH between $\Phi(\cdot, \delta)|_N$ and $\Phi(\cdot, t)|_N$ for any $t > 0$. If t is sufficiently large, then convex combination gives an IAH between $\Phi(\cdot, t)|_N$ and $r \circ \Phi(\cdot, t)|_N$. The function $s \mapsto r \circ \Phi(\cdot, s)|_N$ gives an IAH between $r \circ \Phi(\cdot, t)|_N$ and $r \circ \Phi(\cdot, 0)|_N = r$.

It now suffices to show that χ_A is the fixed point index of A with respect to r. Let $i : A \to N$ be the inclusion. Then $\Lambda(r) = \Lambda(i \circ r) = \Lambda(r \circ i) = \Lambda(\mathrm{Id}_A) = \chi_A$ where the second equality is Commutativity. ☐

The special case of this result for an isolated equilibrium of a dynamical system is well known. (E.g., Krasnosel'ski and Zabreiko (1984), Th. 52.1.) The result for more general asymptotically stable sets was, so far as I know, first established by Demichelis and Ritzberger (2003), even though it should have physocal applications. Since generic payoffs for an extensive form game give rise to associated normal forms with infinitely many Nash equilibria, the additional generality is pertinent to that setting. In addition, due to the polyhedral nature of the set of mixed strategy profiles, one cannot simply assume a sufficiently smooth inward pointing dynamics, as we did in the general equilibrium context.

By way of contrast we now sketch the logic of the results of Echenique (2002). Let X be a complete lattice: X is a set that is partially ordered $\preceq$ such that for any nonempty $S \subset X$, X contains a greatest lower bound $\inf S$ and a least upper bound $\sup S$. We assume that for all $x, x' \in X$, if $x \preceq x'$ and $x' \preceq x$, then $x = x'$, so $\inf S$ and $\sup S$ are unique. For $x \in X$ let $[-\infty, x] = \{\, x' \in X : x' \preceq x \,\}$ and $[x, \infty] = \{\, x' \in X : x' \preceq x \,\}$, and for $x, x' \in X$, let $[x, x'] = [-\infty, x'] \cap [x, \infty]$. We endow X with the *order interval topology*, which is the topology that has the sets of the form $[-\infty, x]$ and $[x, \infty]$ as a subbase of the set of closed sets.

Let $\phi : X \to X$ be a correspondence that is *weakly increasing*: for all x and x', if $x \preceq x'$ then $\inf \phi(x) \preceq \inf \phi(x')$ and $\sup \phi(x) \preceq \sup \phi(t')$. For $x_0 \in X$ let $\mathcal{D}(x_0, \phi)$ be the set of sequences $x_0, x_1, x_2, \ldots$ such that for all $k = 1, 2, \ldots$,

$$\phi(\inf\{x_0, \ldots, x_{k-1}\}) \preceq x_k \preceq \phi(\sup\{x_0, \ldots, x_{k-1}\}).$$

Let $F(x_0, \phi)$ be the set of limits of convergent elements of $\mathcal{D}(x_0, \phi)$. Theorem 3 of Echenique (2002) asserts that if $x \preceq \inf \phi(x)$, then:

(a) $F(x, \phi) \neq \emptyset$,

(b) $\inf F(x, \phi), \sup F(x, \phi) \in F(x, \phi) \cap \mathcal{F}(\phi)$,

(c) $\inf F(x, \phi) = \inf\{\, z \in \mathcal{F}(\phi) : x \leq z \,\}$, and

(d) for all $\{x_k\} \in \mathcal{D}(x, \phi)$, $\inf F(x, \phi) \preceq \liminf x_k \preceq \limsup x_k \preceq \sup F(x, \phi)$.

A fixed point $e \in \mathcal{F}(\phi)$ is *best case stable* if there is a neighborhood V of e such that $e \in F(x_0, \phi)$ for all $x_0 \in V$, and it is *worst case stable* if there is a neighborhood V of e such that for all $x_0 \in V$, e is the limit of every element of $\mathcal{D}(x_0, \phi)$.

Let T be another partially ordered set, again satisfying $t = t'$ whenever $t \preceq t'$ and $t' \preceq t$. Now $\phi : X \times T \to X$ be a correspondence that is weakly increasing in X, as above, and *strongly increasing* in T: for all $t, t' \in T$ and $x \in X$, if $t \prec t'$, then $\sup \phi(t, x) \preceq \inf \phi(t', x)$. For $t \in T$ let $\phi_t = \phi(\cdot, t)$ be the correspondence for the parameter t.

The main results in Echenique (2002) can now be explained. Let $e : T \to X$ be a continuous function with $e(t) \in \mathcal{F}(\phi_t)$ for all t.

First suppose that $\underline{t} \prec t \prec \overline{t}$ and, for any neighborhood V of $e(t)$, either there is a $t' \in [\underline{t}, t]$ such that $e(t') \in V$ and $e(t') \npreceq e(t)$, or there is a $t' \in [t, \overline{t}]$ such that $e(t') \in V$ and $e(t) \npreceq e(t')$. Theorem 2 of Echenique (2002) asserts that in this circumstance $e(t)$ is not best-case stable for ϕ_t. The contrapositive of this asserts that if e selects stable equilibria, then it must be locally monotone increasing, so this result can be understood as a version of the correspondence principle. The idea of the proof is quite simple: if $t' \in [\underline{t}, t]$ and $e(t') \npreceq e(t)$, then $e(t') \preceq \phi_t(e(t'))$ because $e(t') \in \phi_{t'}(e(t'))$ and ϕ is strongly increasing in T, so $e(t') \preceq \inf F(e(t'), \phi_t)$ because ϕ_t is weakly increasing, and consequently $e(t) \notin F(e(t'), \phi_t)$.

Theorem 4 of Echenique (2002) is a converse. Suppose that X is a subset of a Banach lattice and $e(t)$ is isolated in $\mathcal{F}(\phi_t)$. Consider t_0 and t_1 such that $t_0 \prec t \prec t_1$, $e(t_0) \prec e(t) \prec e(t_1)$, and $[e(t_0), e(t_1)]$ is a neighborhood of $e(t)$. Let $\{y_k\}$ and $\{z_k\}$ be the sequences given by $y_0 = e(t_0)$, $z_0 = e(t_1)$, and $y_k = \inf \phi_t(y_{k-1})$ and $z_k = \sup \phi_t(z_{k-1})$ for $k \geq 1$. The results above imply that these sequences converge to $e_0 = \inf F(e(t_0), \phi_t)$ and $e_1 = \sup F(e(t_1), \phi_t)$ respectively, and that $e_0, e_1 \in \mathcal{F}(t)$. If $x \in [e(t_0), e(t_1)]$, then every sequence $\{x_k\}$ in $\mathcal{D}(x, \phi_t)$ satisfies $y_k \preceq x_k \preceq z_k$ for all k. The idea is that for any neighborhood V of $e(t)$, if $e(t_0)$ and $e(t_1)$ are close enough to $e(t)$, then $[y_k, z_k] \subset V$ for all k and $e_0, e_1 \in V$. (We are eliding many details and lengthy arguments.) If V is small enough, then $e_0 = e(t) = e_1$ because $e(t)$ is isolated, so that necessarily $x_k \to e(t)$.

5 Sophisticated Foundations

After the rational expectations revolution Samuelson's perspective seems problematic: if a continuous adjustment process leads to equilibrium, and the agents in the model understand this, instead of conforming to the process they will exploit it. The nature of the equilibration process is therefore seemingly unknowable in principle, or perhaps not even a meaningful concept. In the face of these concerns, the "justification" of the index +1 principle given by the results in Demichelis and Ritzberger (2003) and the last section seems quite weak.

In an unpublished portion of his Ph.D. dissertation Nash (1950) proposes what has come to be known as the *mass-action interpretation* of a game: a large population of players repeatedly play the game against opponents that are anonymous, in the sense that players are matched randomly in each period, are unable to observe their opponents' past behavior, and do not expect to interact with these individuals in the future. For these reasons players can be expected to maximize expected payoff in each period.

In asset markets there are many traders pursuing various strategies, some based on new information, while others pursue technical analysis or try to spot and exploit arbitrage opportunities. The net effect is a sort of price discovery. If each trader is small, it is reasonable to assume that no one is trying to manipulate future prices.

In both of these examples it seems that equilibration could be the result of the collective behavior of a large number of rational agents. This suggests that we might try to justify the index +1 principle by constructing a game of dynamic strategic adjustment or dynamic trading whose only Nash equilibria "project" onto the index +1 equilibria of the simple underlying model.

We now present a result which casts doubt on this approach, but which is also reassuring, insofar as it gives a sense in which the index +1 will not contradict itself. Let $G = (S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a strategic form game. That is, $S_1, \ldots, S_n$ are finite sets of *pure strategies* and $u_1, \ldots, u_n$ are real valued functions whose domain is the set

$S = S_1 \times \cdots \times S_n$ of *pure strategy profiles.* Let $N = \{1, \ldots, n\}$.

There are the usual associated concepts. For each $i \in N$ a *mixed strategy* for agent i is a function $\sigma_i : S_i \to [0,1]$ such that $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$. Let Σ_i be the set of mixed strategies for agent i, and let $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ be the set of *mixed strategy profiles.* Abusing notation, let u_i also denote the multilinear extension of u_i to Σ:

$$u_i(\sigma) = \sum_{s \in S} \Big(\prod_{h \in N} \sigma_h(s_h) \Big) u_i(s).$$

Agent i's set of *best responses* to $\sigma \in \Sigma$ is

$$B_i(\sigma) = \{\, \tau_i \in \Sigma_i : u_i(\tau_i, \sigma_{-i}) \geq u_i(\tau_i', \sigma_{-i}) \text{ for all } \tau_i' \in \Sigma_i \,\}$$

where (τ_i, σ_{-i}) is the mixed strategy profile obtained from σ by replacing σ_i with τ_i. Let $B(\sigma) = B_1(\sigma) \times \cdots \times B_n(\sigma)$. Then $B : \Sigma \to \Sigma$ is an u.h.c.c.v. correspondence whose fixed points are the Nash equilibria of G.

Let $\tilde{G} = (\tilde{S}_1, \ldots, \tilde{S}_{\tilde{n}}, \tilde{u}_1, \ldots, \tilde{u}_{\tilde{n}})$ be a second normal form game, for which there are surjections $\theta : \tilde{N} \to N$ (where $\tilde{N} = \{1, \ldots, \tilde{n}\}$) and $\pi_j : \tilde{S}_j \to S_{\theta(j)}$ for each $j \in \tilde{N}$. Let $\tilde{\Sigma}_i$, $\tilde{\Sigma}$, $\tilde{B}_i$, and $\tilde{B}$ are defined as above. Abusing notation, for $i \in N$ let π_i also denote the map $\pi_i : \tilde{\Sigma} \to \Sigma_i$ given by

$$\pi_i(\tilde{\sigma})(s_i) = \sum_{j \in \theta^{-1}(i)} \frac{1}{|\theta^{-1}(i)|} \sum_{\tilde{s}_j \in \pi_j^{-1}(s_i)} \tilde{\sigma}_j(\tilde{s}_i),$$

and let $\pi : \tilde{\Sigma} \to \Sigma$ be the map $\pi(\tilde{\sigma}) = (\pi_1(\tilde{\sigma}), \ldots, \pi_n(\tilde{\sigma}))$. We say that $\tilde{G}$ is an *ε-approximation* of G if

$$\big| \tilde{u}_j(\tilde{s}) - u_{\theta(j)}\big(\pi_j(\tilde{s}_j), \pi_{-\theta(j)}(\tilde{s})\big) \big| < \varepsilon$$

for all $j \in \tilde{N}$ and $\tilde{s} \in \tilde{S}$.

Theorem 3. *Let $C \subset \Sigma$ be compact. If the intersection of C with the set of Nash equilibria of G is contained in $\operatorname{int}(C)$, then there is an $\varepsilon > 0$ such that if $\tilde{G}$ is an ε-approximation of G and $\tilde{C} = \pi^{-1}(C)$, then $\tilde{B}|_{\tilde{C}}$ is index admissible and*

$$\Lambda_{\tilde{\Sigma}}(\tilde{B}|_{\tilde{C}}) = \Lambda_{\Sigma}(B|_C).$$

If an equilibrium of G has index -1, then there will be equilibria of $\tilde{G}$ above it. Not all of these equilibria will have index +1, and thus at least some of them may be regarded with suspicion. But our reason for disparaging them is not different from our reason for disparaging the index -1 equilibrium of G, so this maneuver cannot "justify" the index +1 principle.

On the other hand, this result provides some assurance that the index +1 principle is not systematically at odds with itself. For example, it cannot be the case that only one equilibrium of $\tilde{G}$ with index -1 lies above an index +1 equilibrium of G.

The remainder of this section is devoted to the proof of Theorem 3.

Lemma 1. *Suppose that X and Y are ANR's, $C \subset X$ and $D \subset Y$ are compact, $r : C \to D$ and $i : D \to C$ are continuous, and $r \circ i = \mathrm{Id}_D$. If $F : C \to D$ is an u.h.c.c.v. correspondence, then $i \circ F$ is an u.h.c.c.v. correspondence and*

$$\Lambda_X(i \circ F) = \Lambda_Y(F \circ i).$$

Proof. Since i is continuous and F has compact values, $i \circ F$ has compact values. Fix $x \in C$. If W is a neighborhood of $i(F(x))$, then $i^{-1}(W)$ is a neighborhood of $F(x)$, and the upper hemicontinuity of F gives a neighborhood V of x such that $F(x') \subset i^{-1}(W)$ and thus $i(F(x'))$ for all $x' \in V$. Thus $i \circ F$ is upper hemicontinuous. If $c : F(x) \times [0,1] \to F(x)$ is a contraction, then $c' : i(F(x)) \times [0,1] \to i(F(x))$ given by $(x', t) \mapsto i(c(r(x'), t))$ is also a contraction, so $i \circ F$ has contractible values.

By Continuity there are neighborhoods U of $\mathrm{Gr}(i \circ F)$ and V of $\mathrm{Gr}(F \circ i)$ such that $\Lambda_X(f) = \Lambda_X(i \circ F)$ for every continuous function $f : C \to C$ such that $\mathrm{Gr}(f) \subset U$ and $\Lambda_Y(g) = \Lambda_Y(F \circ i)$ for every continuous function $g : D \to D$ such that $\mathrm{Gr}(g) \subset V$. Since $(i \times i)(V)$ is the intersection of $i(D) \times i(D)$ with $(r \times r)^{-1}(V)$, it is relatively open in $i(D) \times i(D)$. By taking U sufficiently small we can insure that

$$U \cap (i(D) \times i(D)) \subset (i \times i)(V).$$

(If, for every $\varepsilon > 0$, there was a point in the intersection of $(i(D) \times i(D)) \setminus (i \times i)(V)$ and the ε-neighborhood of the graph of $i \circ F$, a limit point of a sequence of such points would be a point in $\mathrm{Gr}(i \circ F) \cap (i(D) \times i(D)) \setminus (i \times i)(V)$ that was not in $(i \times i)(\mathrm{Gr}(F \circ i))$, which is impossible.) In particular, if $f : C \to C$ is continuous and $\mathrm{Gr}(i \circ r \circ f) \subset U$, then

$$\mathrm{Gr}(r \circ f \circ i) = (r \times r)((i(D) \times i(D)) \cap \mathrm{Gr}(i \circ f \circ r)) \subset (r \times r)((i(D) \times i(D)) \cap U)$$
$$\subset (r \times r)(i \times i)^{-1}(V) = V.$$

Since $i \circ r$ is continuous and its restriction to $i(D)$ is the identity, there is a neighborhood $U' \subset U$ of $\mathrm{Gr}(i \circ F)$ such that if $f : C \to C$ is continuous with $\mathrm{Gr}(f) \subset U'$, then $\mathrm{Gr}(i \circ r \circ f) \subset U$. For such an f we have

$$\Lambda_X(i \circ F) = \Lambda_X(f) = \Lambda_X(i \circ r \circ f) = \Lambda_Y(r \circ f \circ i) = \Lambda_Y(F \circ i)$$

where the third equality is from Commutativity. □

Proof of Theorem 3. If $\tilde{G}$ is an ε-approximation of G and $\tilde{\sigma}$ is a Nash equilibrium of $\tilde{G}$, then $\pi(\tilde{\sigma})$ is an ε-approximate equilibrium of G in the sense that each agent is achieving an expected utility that is within ε of the optimum. If, for arbitrarily small ε, there was an ε-approximate equilibrium of G in the topological boundary $\partial C = C \setminus \text{int}\,(C)$ of C, then a limit point of a suitable sequence of such strategy vectors would be a Nash equilibrium in ∂C, contrary to assumption. Therefore there is an $\overline{\varepsilon}$ such that if $0 \le \varepsilon < \overline{\varepsilon}$ and $\tilde{G}$ is an ε-approximation of G, then $\tilde{G}$ has no Nash equilibria in $\pi^{-1}(\partial C)$. Continuity implies that $\pi^{-1}(\text{int}\,(C))$ is open, so $\pi^{-1}(\partial C)$ contains the topological boundary of $\tilde{C}$, and consequently $\tilde{B}|_{\tilde{C}}$ is index admissible.

Fix a particular ε-approximation $\tilde{G} = (\tilde{S}_1, \dots, \tilde{S}_{\tilde{n}}, \tilde{u}_1, \dots, \tilde{u}_{\tilde{n}})$ where $0 \le \varepsilon < \overline{\varepsilon}$. For $t \in [0,1]$ let $\tilde{G}^t = (\tilde{S}_1, \dots, \tilde{S}_{\tilde{n}}, \tilde{u}_1^t, \dots, \tilde{u}_{\tilde{n}}^t)$ where

$$\tilde{u}_j^t(\tilde{s}) = (1-t)\tilde{u}_j(\tilde{s}) + t u_{\theta(j)}(\pi_j(\tilde{s}_j), \pi_{-\theta(j)}(\tilde{s})),$$

and let $\tilde{B}^t$ be defined as above. Then $\tilde{G}^t$ is an $(1-t)\varepsilon$-approximation of G, so $\tilde{B}^t|_{\tilde{C}}$ is index admissible, and $t \mapsto \tilde{B}^t|_{\tilde{C}}$ is an IAH. By Homotopy it suffices to show that $\Lambda_{\tilde{\Sigma}}(\tilde{B}^0|_{\tilde{C}}) = \Lambda_\Sigma(B|_C)$, which is to say that it suffices to establish the claim when $\tilde{G}$ is a 0-approximation, which we now assume.

For each $j \in \tilde{N}$ fix a map $\hat{s}_j : S_{\theta(j)} \to \tilde{S}_j$ such that $\pi_j \circ \hat{s}_j = \text{Id}_{S_{\theta(j)}}$, and let $\hat{S}_j$ be the image of $\hat{s}_j$. Let $\hat{\Sigma}_j$ be the set of $\tilde{\sigma}_j \in \tilde{\Sigma}_j$ such that $\tilde{\sigma}_j(\tilde{s}_j) = 0$ whenever $\tilde{s}_j \notin \hat{S}_j$. For $\tilde{\sigma} \in \tilde{\Sigma}$ let $\hat{B}_j(\tilde{\sigma}) = \tilde{B}(\tilde{\sigma}) \cap \hat{\Sigma}_j$, and let $\hat{B}(\tilde{\sigma}) = \hat{B}_1(\tilde{\sigma}) \times \cdots \times \hat{B}_{\tilde{n}}(\tilde{\sigma})$. There is an obvious homotopy between $\tilde{B}$ and $\hat{B}$ that restricts to an IAH between $\tilde{B}|_{\tilde{C}}$ and $\hat{B}|_{\tilde{C}}$, so $\Lambda_{\tilde{\Sigma}}(\tilde{B}|_{\tilde{C}}) = \Lambda_{\tilde{\Sigma}}(\hat{B}|_{\tilde{C}})$.

Let $\iota : \Sigma \to \tilde{\Sigma}$ be the linear extension of the map $s \mapsto (\hat{s}_j(s_{\theta(j)}))$. Evidently $\pi \circ \iota = \text{Id}_\Sigma$. Let $i = \iota|_C$ and $r = \pi|_{\tilde{C}}$. Then $B|_C = B|_C \circ r \circ i$ and $\hat{B}|_{\tilde{C}} = i \circ B|_C \circ r$, so Lemma 1 (with $F = B|_C \circ r$) implies that $\Lambda_{\tilde{\Sigma}}(\hat{B}|_{\tilde{C}}) = \Lambda_\Sigma(B|_C)$. ☐

6 An Hypothesis

We have seen that it is difficult to regard the index +1 principle as a consequence of stability of dynamic adjustment of a model populated by sophisticated agents. The results displaying it as a consequence of the stability of "naive" adjustment dynamics are quite particular, and seem (at least to the author) to do a highly imperfect job of expressing the nature and strength of the intuition supporting it.

Both logically and intuitively, it seems best to regard the index +1 principle as an *hypothesis*. While it has some loose connection with dynamic stability, it is in fact quite distinct, with perhaps stronger intuitive support than it receives from dynamic considerations. On the whole it does not seem to be less well supported than a number of other hypotheses (e.g., rational expectations) that are standard parts of the analytic toolkit.

In addition, it is an hypothesis that is susceptible to empirical and experimental tests. We have already observed that the mixed strategy equilibrium of the battle of the sexes does not seem to be observed in practice, as a stable, self replicating mode of playing the game. Cox and Walker (1998) is an experimental study of two Cournot duopoly games, one of which has a unique equilibrium in which both firms' quantities are positive, while the other has three equilibria, two of which are monopolistic, while the third, in which both firms have positive production, has index -1. In all but one experiment 20 subjects were divided into two groups, and in each of 30 market periods there was a random matching of the members of the two groups, with each pair playing the duopoly game. Play generally converged to one of the equilibria (even though the data was inconsistent with the particular learning models discussed in the paper) but the index -1 equilibrium was not observed. "The results of the experiments ... strongly suggest that the theoretical stability properties of a Nash equilibrium can serve as an effective 'refinement,' distinguising equilibria that subjects can be expected to play from those that they generally will not play." There seems to be considerable scope for additional experimental work on this issue.

7 Concluding Remarks

Samuelson's vision of a theory of economic dynamics that would provide additional insights in connection with static analysis did not come to pass. Dynamic models have proliferated, but for the most part those that are thought to respect the agents' rationality are not models of strategic adjustment to equilibrium, but are instead models of essentially static equilibria that play out over time. We propose a decomposition of Samuelson's correspondence as a conjunction of three ideas: a) stability of an equilibrium implies that its index is +1; b) only index +1 equilibria are observed empirically; c) having index +1 has (possibly in conjunction with additional hypotheses) implications for comparative statics.

Only "naive" dynamics provides any support for the first of these, and that support is quite limited. While such dynamics are well motivated in some evolutionary contexts, sophisticated agents would generally anticipate the system's evolution, and then undermine the presumed dynamics. In models with sophisticated agents it is still the case that equilibria that violate the index +1 principle are, perhaps, quite implausible, but existing theoretical results do a surprisingly poor job of explaining this.

At least tentatively, the index +1 principle seems like a respectable tool of economic analysis, even if the reasons for this are still somewhat mysterious. Instead of regarding it as a consequence of stability, we suggest regarding as an hypothesis, and indeed there are concrete ways to bring the index +1 principle to data. The one experimental study we know of that has done this supports the principle, but much more is possible.

The in vivo data available to economists is much more varied than when Samuelson wrote, experiments allow data to be customized, and game theory has brought many new perspectives into economic analysis. Consequently the empirical and theoretical implications of the index +1 principle go far beyond the specific applications noted by Samuelson, though perhaps not as far as he had hoped would eventually be the case. The index +1 principle is, in a sense, only one bit of information, but it can be applied across the entire range of equilibrium models, and surely has many interesting consequences.

References

Arrow, K. J., Block, H. D., and Hurwicz, L. (1959). On the stability of the competitive equilibrium, II. *Econometrica*, 27:82–109.

Arrow, K. J. and Hurwicz, L. (1958). On the stability of the competitive equilibrium, I. *Econometrica*, 26:522–552.

Browder, F. (1948). *The Topological Fixed Point Theory and its Applications to Functional Analysis*. PhD thesis, Princeton University.

Brown, R. (1971). *The Lefschetz Fixed Point Theorem*. Scott Foresman and Co., Glenview, IL.

Cox, J. C. and Walker, M. (1998). Learning to play Cournot duopoly strategies. *Journal of Economic Behavior and Organization*, 36:141–161.

Demichelis, S. and Ritzberger, K. (2003). From evolutionary to strategic stability. *Journal of Economic Theory*, 113:51–75.

Dierker, E. (1972). Two remarks on the number of equilibria of an economy. *Econometrica*, 40:951–953.

Dierker, E. (1974). *Topological Methods in Walrasian Economics*. Lecture Notes in Economics and Mathematical Systems, No. 72. Springer Verlag, Berlin-Heidelberg-New York.

Dugundji, J. and Granas, A. (2003). *Fixed Point Theory.* Springer-Verlag, New York.

Echenique, F. (2002). Comparative statics by adaptive dynamics and the correspondence principle. *Econometrica*, 70:833–844.

Echenique, F. (2004). A weak correspondence principle for models with complementarities. *Journal of Mathematical Economics*, 40:145–152.

Echenique, F. (2008). The correspondence principle. In Durlauf, S. and Blume, L., editors, *The New Palgrave Dictionary of Economics (Second Edition).* Palgrave Macmillan, New York.

Eilenberg, S. and Montgomery, D. (1946). Fixed-point theorems for multivalued transformations. *American Journal of Mathematics*, 68:214–222.

Fudenberg, D. and Levine, D. (1998). *The Theory of Learning in Games.* MIT Press, Cambridge.

Hicks, J. R. (1939). *Value and Capital.* Clarendon Press, Oxford.

Hofbauer, J. (1990). An index theorem for dissipative semiflows. *Rocky Mountain Journal of Mathematics*, 20:1017–1031.

Jordan, J. S. (1987). The informational requirement of local stability in decentralized allocation mechanisms. In Groves, T., Radner, R., and Reiter, S., editors, *Information, Incentives, and Economic Mechanisms: Essays in Honor of Leonid Hurwicz*, pages 183–212. University of Minnesota Press, Minneapolis.

Kakutani, S. (1941). A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, 8:457–459.

Kinoshita, S. (1953). On some contractible continua without the fixed point property. *Fundamentae Mathematicae*, 40:96–98.

Krasnosel'ski, M. A. and Zabreiko, P. P. (1984). *Geometric Methods of Nonlinear Analysis.* Springer-Berlin, Berlin.

Mas-Colell, A. (1974). A note on a theorem of F. Browder. *Mathematical Programming*, 6:229–233.

McLennan, A. (1991). Approxiation of contractible valued correspondences by functions. *Journal of Mathematical Economics*, 20:591–598.

McLennan, A. (2012). *Advanced Fixed Point Theory for Economics.* `http://cupid.economics.uq.edu.au/mclennan/Advanced/advanced.html`.

Milgrom, P. and Roberts, J. (1990). Rationalizability, learning and equilibrium in games with strategic complementarities. *Econometrica*, 58:1255–1277.

Nadzieja, T. (1990). Construction of a smooth Lyapunov function for an asymptotically stable set. *Czechoslovak Mathematical Journal*, 40:195–199.

Nash, J. (1950). *Non-cooperative Games.* PhD thesis, Mathematics Department, Princeton University.

O'Neill, B. (1953). Essential sets and fixed points. *American Journal of Mathematics*, 75:497–509.

Ritzberger, K. (1994). The theory of normal form games from the differentiable viewpoint. *International Journal of Game Theory*, 23:201–236.

Saari, D. G. (1985). Iterative price mechanisms. *Econometrica*, 53:1117–1133.

Saari, D. G. and Simon, C. P. (1978). Effective price mechanisms. *Econometrica*, 46:1097–1125.

Samuelson, L. (1997). *Evolutionary Games and Equilibrium Selection.* MIT Press, Cambridge.

Samuelson, P. (1947). *Foundations of Economic Analysis.* Harvard University Press.

Samuelson, P. A. (1941). The stability of equilibrium: Comparative statics and dynamics. *Econometrica*, 9:97–120.

Samuelson, P. A. (1942). The stability of equilibrium: Linear and nonlinear systems. *Econometrica*, 10:1–25.

Scarf, H. (1960). Some examples of global instability of the competitive equilibrium. *International Economic Review*, 1:157–172.

Shapley, L. S. (1974). A note on the Lemke-Howson algorithm. *Math. Programming Stud.*, 1:175–189. Pivoting and extensions.

Topkis, D. M. (1979). Equlibrium points in nonzero-sum n-person submodular games. *SIAM Journal of Control and Optimization*, 17:773–787.

Vives, X. (1990). Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics*, 19:305–321.

Williams, S. R. (1985). Necessary and sufficient conditions for the existence of a locally stable message process. *Journal of Economic Theory*, 35:127–154.

Wilson, F. W. (1969). Smoothing derivatives of functions and applications. *Transactions of the American Mathematical Society*, 139:413–428.