

Matched-Sign Discounting and Neoclassical Consistencies

Prapas Bawornbancha*

Thammasat University

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Abstract

This research proposes a theory of matched-sign discounting which can reconcile a fundamental inconsistency in neoclassical macroeconomics. The proposed theory bases on a mathematical fact that a consumer who prefers present over future consumption has negative discount rate if his future utility being discounted is represented by a negative number. Three important neoclassical macroeconomic models are reassessed by replacing the positive discount rate condition by matched-signed-discounting (MSD) condition. The reassessment results turn out to be remarkable: first, the neoclassical growth (RCK) model is found fitting all historical stylized facts very well, second, Mehra and Prescott's equity premium puzzle is resolved by the model itself, and third, Lucas' welfare gain from eliminating consumption risk is found to be not trivial, it is more than four-fifths of the welfare gain from ending ten percent inflation.

JEL Classification: D90, D91, E13, E21, O40, O41

Key Word: subjective rate of time preference, intertemporal macroeconomics, neoclassical growth theory, equity premium puzzle, welfare gain from eliminating consumption risk.

* PhD candidate, faculty of economics Thammasat University, Bangkok Thailand. Email: prapas.ba@gmail.com. This paper is from my dissertation.

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1. Introduction

Intertemporal macroeconomics typically employs some variation of the discounted utility (DU) model introduced by Samuelson (1937), which discounts the utility function as

$$U = \int_0^T e^{-\rho t} u(c(t)) dt ,$$

in which the discount rate ρ is used to reflect time preference¹ and the positive discount rate condition $\rho > 0$ is required. The positive discount rate condition has been firmly held in macroeconomics since about twenty years after the inception of the DU model. From then on it was mandatory in theoretical setups and prerequisite in empirical studies. Despite such a firm and widely held belief, and the fact that consumers prefer present to future consumption, the positive discount rate condition may not be true.

Since a consumer's utilities can be represented by either positive or negative numbers², this paper argues that to reflect a correct sense of discounting the sign of the discount rate is depending on the sign of the numbers representing the utilities. If one uses a positive discount rate to discount a future utility which is represented by a negative number, the act will cause a wrong sense of discounting, i.e., it will increase (rather than decrease) the utility being discounted. Hence, the sensible way to discount a negative-sign utility is to use a negative discount rate. Moreover, in the case of discounting the whole utility function in the DU model, it can be shown that if the utility function is negative-sign³, negative discount rate can correctly shift the relevant utility stream downward while positive discount rate cannot⁴. So, in utility discounting both positive and negative discount rates are necessary, if the correct sense of discounting is to be always preserved⁵.

¹ There are two different concepts here. (1) The time preference, which is people's attitude toward consumption time, e.g. when a consumer prefers present over future consumption it is said that he has premium on present utilities, is impatient, or myopia. (2) The discount rate, which indicates how severe future utility is discounted. Conventionally, the two concepts are combined as a condition $\rho > 0$ in which the sign of ρ also indicates time preference (positive rate means premium on present), and thus the term 'discount rate', 'rate of time preference' and 'time preference' are used interchangeably.

² We consider numerical representations of utility only; disutility is excluded throughout our discussion. Ordinal preferences can be represented by positive-, negative-, or mixed-sign utility functions. Many standard utility functions in macroeconomics today yield negative values; e.g. the most prevalent class of utility function the CRRA class, $u(c) = \frac{c^{1-\theta}}{1-\theta}$, usually yields negative values, because, according to Lucas (2003), the risk aversion coefficient θ in use in macroeconomics and public finance application today ranges from 1 to 4.

³ Generally, negative-sign utility function is the utility function that yields only negative values. However, in some exception cases a more precise definition is required, see section 3.4.

⁴ For the correct sense of discounting a utility stream see section 3.2, 3.3 and Ex.3 in section 3.4.

⁵ Conventionally, negative discount rate is reserved for premium on future utilities which is possible in some occasions, e.g. see Loewenstein and Prelec (1991). But, in this paper (section 3) the discount rate will be defined as excluding premium time, so that it can be used to discount both positive- and negative-sign utility function. Thus, in here, negative discount rate does not mean premium on future, the premium time is handled separately.

This paper presents a theory of matched-sign discounting which consists of two features: (1) simple-matched-sign-discounting (SMSD) condition, and (2) matched-sign-discounting (MSD) condition. These conditions can identify a correct sense of utility discounting, thus can direct macroeconomic models to a more sensible solution. The theory can also help explain macroeconomic theory more appropriately.

Macroeconomic models are usually about some sort of DU optimization problem, the output of which is typically some equilibrium conditions such as the Euler equation. There are many ways to interpret model's equilibrium conditions; each way assumes different sign for each subjective parameter in the model⁶, which leads to different kinds of optimal solution or path. Thus, it is crucial that the interpretation method is chosen appropriately otherwise a mistaken solution may result. Matched sign discounting theory is a new interpretation method that takes both the sign of the discount rate and the sign of the utility function into account, such that the correct sense of discounting is a rule. This is different from the conventional interpretation method which uses positive discount rate condition as a rule and ignores the sign of the utility function being discounted; the conventional method thus could lead the model to a solution that has a wrong sense of discounting.

To illustrate how MSD condition is used and how accurate it can lead a model's solution to, three important neoclassical macroeconomic models are reassessed. By replacing the positive discount rate condition by MSD condition, the reassessment results are compared to the existing ones. It turns out that under MSD condition all the three models can explain historical data remarkably well, with a correct sense of discounting. This is in sharp contrast to the existing cases of positive discount rate condition where the three models cannot explain historical data and their solutions often exhibit a wrong sense of discounting.

Section 2 investigates the origin of positive discount rate condition. Section 3 explains the two features of matched-sign-discounting theory. Section 4 reassesses three important macroeconomic models and section 5 is the conclusion.

2. Historical Development of the Positive Discount Rate Condition⁷

There are at least seven reasons that support the positive discount rate condition: (1) as in money discounting the sign of discount rate reflects premium time, (2) people have premium on present utilities, (3) insignificant of cardinal utility representation, (4) consequence of

⁶ Detail discussion about the subjective parameters and their effects on optimal solution is provided in section 3.

⁷ See also Frederick et al. (2002)

axiomatic proof⁸, (5) convergence of the discounted-utility integral over an infinite future, (6) avoidance of infinite deferral of consumption, and (7) discount rate is equal to interest rate. All these reasons allow economists to confidently apply the positive discount rate condition. However, this section will show, through historical development, why these reasons cannot appropriately support the positive discount rate condition.

There are two important periods regarding to the development of the positive discount rate condition in the DU model. These two brief periods defined the positive discount rate condition as it is known today.

2.1 The Early Period of Utility Discounting (1928-1937)

It was well understood that the rate of discount for money, or the interest rate, is positive, because money has premium on present time. On the other hand, negative interest rate, if exist, means money has premium on future time. In the time before the DU model, the idea of discounting future sums of money was well known, but the idea of discounting future utilities was not yet well recognized. The natural practice of the time was, therefore, to follow the logic of money discounting when contemplating utility discounting. One of the earliest direct comparisons between the interest rate and the discount rate appeared in Ramsey (1928). When arguing that the money discount rate r is variable but utility discount rate ρ is constant Ramsey wrote

“This rate of discounting future utilities must, of course, be distinguished from the rate of discounting future sums of money. If I can borrow or lend at a rate r I must necessarily be equally pleased with an extra £1 now and an extra £(1+r) in a year’s time, since I could always exchange the one for the other. My marginal rate of discount for money is, therefore, necessarily r , but my rate of discount for utility may be quite different, since the marginal utility of money to me may be varying by my increasing or decreasing my expenditure as time go on.

In assuming the rate of discount constant, I do not mean that it is the same for all individuals, since we are at present only concerned with one individual or community, but that the present value of an enjoyment at any future date is to be obtained by discounting it at the rate ρ . Thus, taking it to be about $\frac{3}{4}$ per cent., utility at any time would be regarded as twice as desirable as that a hundred years later, four times as valuable as that two hundred years later and so on at a compound rate. This is the only assumption we can make, without contradicting our fundamental hypothesis that successive generations are actuated by the same system of preferences. ...” (Ramsey 1928, p.553)

⁸ see Koopmans (1960)

In here, Ramsey explained utility discounting process comparatively to money discounting process and also in the same positive sense, i.e., he also used positive discount rate and referred to utility as a positive value.

The idea of maximizing the sum of discounted future utilities was used indirectly by Ramsey (1928). But Samuelson (1937) was the person who explicitly formalized the idea of exponential utility discounting. However, this was not the main purpose of Samuelson's paper; its objective was to find a way to measure utility, that is, to determine the form of the unknown utility function, by means of maximizing the sum of discounted utility stream. He, therefore, assumed that individual maximizes the integral $\int_0^b U(x)e^{-\pi t} dt$,

“where π bears the following familiar relationship to the rate of discount (positive or negative) p , here assumed to be constant:

$$\pi = \log_e(1 + p) \quad \text{” (Samuelson 1937, p.156)}$$

Samuelson assumed that the discount rate is constant and can have positive or negative sign, he then explained the meaning of the sign of the discount rate.

“3. *The individual discounts future utilities in some simple regular fashion which is known to us.* For simplicity, we assume in the first instance that the rate of discount of future utilities is a constant. This constant might of course be such that there is no time preference whatsoever, or even a premium on future utilities. ” (Samuelson 1937, p156)

In here π was used to reflect time premium, it suggests that $\pi = 0$ means no time premium, $\pi < 0$ means premium on future utility, and thus $\pi > 0$ means premium on present utility. So, in Samuelson view, the sign of the discount rate reflects premium time: positive (negative) discount rate means premium on present (future) utility.

This way of thinking about the sign of utility discount rate is the same as that of money discount rate explained at the beginning. However, we view that the analogy between utility discount rate and money (or goods) discount rate is not in exact correspondent, because utility can be represented by a positive or a negative number but money, likes goods, is always represented by a positive number. So, to preserve the correct sense of discounting it is necessary to deal with utility discounting in a more general, sign sensitive, way.

For the utility function, both Ramsey and Samuelson did not specify any particular sign, form, or restriction other than its basic curvature: marginal utility never increasing ($u'' \leq 0$). In conclusion, Ramsey and Samuelson viewed that positive discount rate is normal but negative discount rate is not impossible. Negative discount rate is possible if people have premium on future utilities. And they were aware that negative-sign utility function is also

possible, but did not link it to the discount rate, because they viewed that the sign of discount rate ties to premium time the same way the sign of the interest rate does.

2.2 The Emerging Period of Utility Discounting (1955-1981)

About twenty years after the great depression, from the mid-1930s to mid-1950s, the studies of macroeconomics were dominated by Keynesian macroeconomics. During which period the neoclassical optimization approach to macro-level study, like that of Ramsey (1928), was silent. However, many influential papers in the mid-1950s, including the seminal Solow (1956) neoclassical growth theory, have awakened neoclassical optimal utility approach to reemerge in a big way.

Both Ramsey (1928) and Samuelson (1937) assumed that (1) the utility discount function, like compound-interest discount function, is in the form of $e^{-\rho t}$ and (2) the rate of time preference ρ is constant. But, Strotz (1955) was the one who derived, by mean of calculus of variation, the discount function. It turns out that the discount function follows the form of k^t , where k is a positive constant, and the two assumptions Ramsey and Samuelson made are necessary if the original consumption plan is to be adhered to forever. In other words, if the discount function does not have exponential form with constant discount rate, the consumer will find a better optimal plan in some future dates and will discard the original one.

Mathematically, Strotz specified neither the form of utility function nor the range of the discount factor k , which implied that the discount rate could be either positive or negative. However, he argued that people usually over-values the more proximate satisfactions relative to the more distance ones and sketched their discount function as a distorted decaying curve, which means that Strotz viewed the discount factors as some values between zero and one. So Strotz, like other economists, viewed that people normally have positive discount rates⁹.

Tinbergen (1956) was the first to use explicit forms of a utility function to determine the optimal saving rate through the DU model. He used the log utility function and a (primitive CRRA) utility function of the form $u_t = \left(1 - \frac{c_0}{c_t^\epsilon}\right) u_0$. The method Tinbergen used to solve the problem was initiative and noteworthy. When κ is the ratio of investment to change in

⁹ In section 4, neoclassical models and historical data imply that people normally have negative discount rate, negative-sign utility function, and premium on present utilities.

output, c is the consumption rate, and both are assumed to be constant, the DU optimization model is written as:

$$\max \int_0^{\infty} \frac{1}{(1+\rho)^t} \left(1 - \frac{C_0}{C_t^\epsilon}\right) u_0 dt \quad \text{s.t.} \quad C_t = c \cdot Y_t = Y_t - \kappa \dot{Y}_t.$$

From the constraint he derived the consumption path,

$$\dot{Y}_t = \frac{1-c}{\kappa} Y_t \quad \rightarrow \quad Y_t = Y_0 e^{\left(\frac{1-c}{\kappa}\right)t} \quad \rightarrow \quad C_t = c Y_0 e^{\left(\frac{1-c}{\kappa}\right)t},$$

where $\left(\frac{1-c}{\kappa}\right)$ is the consumption growth rate. Substitute C_t into the objective function,

$$\max \int_0^{\infty} e^{-\rho t} \left(1 - C_0 (c Y_0)^{-\epsilon} e^{-\left(\frac{1-c}{\kappa}\right)\epsilon t}\right) u_0 dt.$$

By FOC, the optimal (constant) consumption rate can be found as

$$c = \frac{\kappa\rho + \epsilon}{\epsilon + 1} \quad \rightarrow \quad \rho + \left(\frac{1-c}{\kappa}\right)\epsilon = \frac{c}{\kappa} > 0.$$

And the predicted saving rate $(1 - c)$ turned out to be fairly accurate. However, the more important lesson learnt from Tinbergen's paper is the convergence condition, as Chakravarty (1962) pointed out:

“Even here, the functional is bounded above provided the combined effects of diminishing marginal utility and time preference relative to the rate of growth of consumption are such as to satisfy the convergence condition. ...” (Chakravarty 1962, p.185)

In other words, the above integral is convergent only when the term $e^{-\rho t} e^{-\left(\frac{1-c}{\kappa}\right)\epsilon t}$ approaches zero as t go to infinity, i.e., the convergence condition is $\rho + \left(\frac{1-c}{\kappa}\right)\epsilon > 0$.

Tinbergen's paper also showed that for the case of log utility function ϵ is equal to zero. Thus, its corresponding convergence condition is reduced to $\rho > 0$, independent from the consumption growth rate. This creates a perception that, for the DU optimization model to converge, the assumption of positive rate of time preference is necessary. And this belief was made secure by Koopmans' (1960) converse proof.

Koopmans (1960) has shown that given that the optimal consumption path (program) exists, impatience ($\rho > 0$) is a logical consequence. He defined ${}_1x \equiv (x_1, x_2, x_3, \dots, x_t, \dots)$ as an infinite consumption program. And the utility function $U({}_1x)$ was defined axiomatically under five postulates: continuity, sensitivity, stationary of utility function, the absent of intertemporal complementarity, and the existence of a best and worst program. These postulates assure from the beginning that $U(\cdot)$ has a best program, which means that $U(\cdot)$ is convergent. Given these premises Koopmans managed to show that impatience is a logical

consequence. And this result eventually leads to a formula which is equivalent to the DU model, plus an impatience condition:

$$U({}_1x) = \sum_{t=1}^{\infty} \alpha^{t-1} u(x_t), \quad 0 < \alpha < 1.$$

Koopmans' proof was based on ordinal utility function:

"Moreover, if the idea of preference for early timing is to be expressed independently of assumptions that have made the construction of cardinal utility possible (...) it will be necessary to express it in terms of an ordinal utility function, that is, a function that retains its meaning under a monotonic (increasing) transformation." (Koopmans 1960, p.287)

"... Since we are dealing with ordinal utility, there is still freedom to apply separate increasing transformations to $u(x_1)$ and to $U({}_2x)$, with corresponding transformations of $V(u, U)$, so as to make both I_u and I_U coincide with the unit interval extending from 0 to 1." (Koopmans 1960, p.294-5)

However, it should be pointed out that although ordinal preferences can be represented by positive-sign, negative-sign, or mixed-sign utility functions, ordinal utility function must be positive-sign utility functions only; because negative-sign and mixed-sign utility functions may not be able to preserve their meaning under monotonic (increasing) transformation¹⁰. So, Koopmans' axiomatic proof is valid for the case of positive-sign utility function only, and thus the conclusion that the discount factor α is between zero and one is not generally true.

In Koopmans (1965) and Chakravarty (1962), both argued that if $\rho < 0$, the integral had no upper bound, and hence no optimal consumption path existed. Their reason was that when $\rho < 0$ the act of saving (sacrificing current consumption) is more profitable than consuming, because it provides a higher sum of discounted utilities, therefore the consumer views infinite deferral of consumption as the strategy that provides more total utility in the end. However, there are two reasons, they argued, that explain why optimality cannot be attained under such scenario. (1) The fact that the future rewards never come to be realized, while present consumption is close to zero, the plan could not yield maximum utility practically. (2) Every period's saving will be multiplied exponentially into the infinite future so the total amount of discounted utilities is incalculable. Thus, under negative discount rate, many candidate consumption paths which yield infinite total discounted utilities cannot be compared, therefore, the best path cannot be distinguished.

However, we view that their arguments against $\rho < 0$ is correct only when the utility function being discounted has positive sign. Had the utility function negative sign, and $\rho < 0$,

¹⁰ For example, the CRRA utility function $u(c) = \frac{c^{1-\theta}}{1-\theta}$ with risk aversion coefficient θ greater than one cannot preserve its order of preference ranking under a monotonic transformation such as $T(x) = x^2$.

the act of saving would not be more profitable than consuming (because now the sense of discounting is correct, i.e., future utilities are reduced by the discount rate), hence, infinite deferral of consumption would not arise and the integral could be bounded¹¹. But, Koopmans (1960) and Chakravarty (1962) did not explain the case in which the utility function has negative sign. In Koopmans (1965), he did not restrict the sign of the utility function but argued for $\rho > 0$ from the positive utility point of view.

The emerging period of the DU model has set a very strong argument for the positive discount rate assumption. The assumption that finally has appeared as a condition alongside the DU optimization model since many economists began to write:

$$V(\rho) = \int_0^{\infty} e^{-\rho t} u(x_t) dt, \quad \text{where } \rho > 0.$$

Up to this point, the idea that the sign of the rate of time preference reflects premium time from Samuelson (1937), the common belief that positive discount rate is the norm, and the concept that people prefer present to future consumption¹² (or impatience) were merged into a positive discount rate condition by Koopmans' (1960) converse proof.

The positive discount rate condition was confirmed again by Olson and Bailey (1981). Let's discuss Olson and Bailey's paper in detail because it can be linked to negative discount rate and negative-sign utility function. By assuming that $v(C)$ is a time separable utility function and η is a constant discount rate under positive time preference¹³, Olson and Bailey derived a household equilibrium condition from which a new meaning of time preference can be defined:

$$\begin{aligned} \text{"} \quad U &= \sum_{t=0}^T \frac{v(C_t)}{(1+\eta)^t} \\ &\dots \\ \left(\frac{dC_1}{dC_0}\right)_{U=\text{const.}} &= -\frac{v'(C_0)}{v'(C_1)}(1+\eta), \end{aligned} \quad (3)$$

The household will demand an increase in C_1 greater than the reduction in C_0 , that is, $\left|\frac{dC_1}{dC_0}\right| > 1$, to hold utility constant, if either of two things is true:

$$C_1 > C_0 \quad \text{with } \eta = 0, \quad (4)$$

$$\text{or} \quad \eta > 0 \quad \text{with } C_1 = C_0 \quad (5)$$

” (Olson and Bailey 1981, p.4)

When household has (positive) time preference, it demands an increase in C_1 greater than the reduction in C_0 , i.e., it requires a positive interest rate, $r > 0$. Equation (3) in the

¹¹ See Ex.3 in section 3.4.

¹² According to Olson and Bailey (1981), this concept was first clearly stated by Bohm-Bawerk (1889, 1959 translation).

¹³ (Positive) time preference means people prefer present to future consumption.

quote is household's equilibrium condition which is another view of the Euler equation where $MRS_{C_0, C_1} = - \left(\frac{dC_1}{dC_0} \right)_{U=const.} = 1 + r$. In order to facilitate the discussion, equation (3) may be rewritten as

$$1 + r = \left(1 + \frac{v'(C_0) - v'(C_1)}{v'(C_1)} \right) (1 + \eta) \quad (3)'$$

From (3) or (3)', Olson and Bailey pointed out two independent causes of positive time preference:

- When $\eta = 0$, $C_1 > C_0$ will cause $r > 0$. (because of diminishing marginal utility)
- When $C_1 = C_0$, $\eta > 0$ will cause $r > 0$. (and $\eta = r$ is a direct consequence).

Olson and Bailey (1981) chose to define (positive) time preference via the second cause: i.e., time preference is a positive value of the constant η . They chose this definition because (1) the ratio of marginal utility, $\frac{v'(C_0)}{v'(C_1)}$, depends not on time but on levels of consumption in the two periods, thus time preference should be defined by excluding it, and (2) the definition allows time preference to be observable through $\eta = r$.

Given the fact that U.S. consumption growth is positive (about 2%) for a long time and similarly for most economies, a more relevant case is thus $C_1 > C_0$. But, the definition Olson and Bailey suggested is for a level consumption path, $C_1 = C_0$, only. So, the definition cannot be useful generally. In our view, searching for a practical definition of time preference, under $C_1 > C_0$, is necessary and can be done. By looking closer into equation (3)', the combined cause of C and η that makes the interest rate positive can be found as

- $C_1 > C_0$ and $\eta > - \frac{v'(C_0) - v'(C_1)}{v'(C_0)}$ will cause $r > 0$.

Since $v'(C_0) > v'(C_1)$, the discount rate η can be negative, zero, or positive. It means that the Euler equation allows the discount rate to be negative even under a positive time preference ($r > 0$). Thus, the practical definition of positive time preference cannot exclude negative and zero discount rates. Because of this and other reasons, in section 3.1, a more appropriate definition of time preference is proposed.

Another interesting case Olson and Bailey (1981) have insightfully pointed out is the one that escapes their definition of time preference.

"... there appears to be a positive rate of time preference, that is, in our formulation an $\eta > 0$. (We discuss later a possible escape from this conclusion if there is both an infinite time horizon and what we called "drastically diminishing marginal utility.")" (Olson and Bailey 1981, p.12)

"

$$v(C) = AC^\alpha + B \quad (13)$$

for $\alpha < 0, A < 0$, and $B > 0$, ...

... if the elasticity of the utility function (13) is minus 5 corresponding to $\alpha = -4$. The function has what we shall label “drastically diminishing marginal utility.” We do not argue that utility functions of this character are the norm but know of no data that rule them out and think it is not without interest that (when exogenous increases in endowments are expected and time horizons are also infinite) the facts of saving force us to choose between positive time preference and drastically diminishing marginal utility of income.” (Olson and Bailey 1981, p19)

“In summary, the case for positive time preference is absolutely compelling, except in the interesting case where there is drastically diminishing marginal utility. Utility functions of this sort have interesting implications of their own which need to be tested separately.” (Olson and Bailey 1981, p.24)

The utility function in equation (13) is essentially a negative sign utility function¹⁴, and its example which is labeled “drastically diminishing marginal utility” is basically a CRRA utility function with coefficient of risk aversion = 5. Olson and Bailey’s exception case is in fact the case of negative discount rate together with negative-sign utility function. Today, negative-sign utility functions are commonly used in macroeconomics, but negative discount rate is never used¹⁵. Olson and Bailey’s exception case is indeed the problem this paper is addressing: how to maintain a correct sense of discounting under a negative-sign utility function. As will be seen in section 4, this exception case is the crucial ingredient that, once appropriately addressed, can allow neoclassical macroeconomic models to accurately explain historical data.

3. Theory of Matched-Sign Discounting

3.1 Simple-Matched-Sign-Discounting (SMSD) Condition

We have discussed before that if the consumers prefer present to future consumption then¹⁶: (1) to preserve the correct sense of discounting, the discount rate has to be negative if the utility function is negative-sign, (2) the sign of the discount rate should not be used to indicate premium time as in the case of the interest rate, because utility and money (or goods) are different in term of numerical representability, (3) Koopmans’ axiomatic proof is valid for the case of positive-sign utility function only, thus the positive discount rate condition cannot be generally true, (4) negative discount rate need not cause the divergence of DU integral and the infinite deferral of consumption, had the utility function negative sign, and (5) the Euler

¹⁴ See the definition of negative-sign utility function in section 3.4.

¹⁵ Because negative discount rate is always interpreted as premium on future utility.

¹⁶ See section 1, 2.1, 2.2, 2.2, and 2.2 respectively.

equation allows the discount rate to be negative, zero, or positive, and thus the practical definition of positive time preference cannot exclude negative and zero discount rates. From these reasons, it can be concluded that if negative-sign utility functions are to be allowed, negative discount rates which are not implying premium on future are necessary, and such discount rates would allow us to explain macroeconomic theory more appropriately.

Therefore, in order to avoid conflict of using discount rate's sign by the two variables (premium time and discount rate) this paper redefines the term time preference¹⁷ as consisting of two distinct elements: (1) the sign of time preference (\overline{TP}) which tells premium time, and (2) the rate of time preference (ρ) which tells severity of discounting.

$$\overline{TP} = \text{sgn}(TP) \equiv \begin{cases} + & \text{if present is preferred to future} \\ - & \text{if future is preferred to present} \\ 0 & \text{if time preference is neutral} \end{cases}$$

The rate of time preference or the discount rate is defined as a real number in $(-1, \infty)$ and

$$\text{sgn}(\rho) \equiv \begin{cases} + & \text{if } \rho \in (0, \infty) \\ - & \text{if } \rho \in (-1, 0) \\ 0 & \text{if } \rho = 0 \end{cases}$$

Since a future utility can be represented by a zero, positive, or negative number, it can be written as $u_0 \in \mathbb{R}$. When discounting u_0 , the depleted utility is defined as the discount (DC).

$$DC \equiv u_0 - \frac{1}{1+\rho}u_0 = \frac{\rho}{1+\rho}u_0$$

The discount is a real number which can be described as consisting of two distinct elements: (1) the sign of discount and (2) the amount of discount, as follows.

$$\text{sgn}(DC) = \text{sgn}\left(\frac{\rho}{1+\rho}u_0\right) = \text{sgn}(\rho) \cdot \text{sgn}(u_0),$$

$$\text{amt}(DC) = \left|\frac{\rho}{1+\rho}u_0\right|.$$

The first equation is true because $1 + \rho > 0$. Since $DC = u_0 - \frac{1}{1+\rho}u_0$, a positive sign of DC means future utility is reduced by the act of discounting, which implies that the consumer prefers present to future consumption. While negative discount ($DC < 0$) means future utility is increased by the act of discounting, which means that the consumer prefers future to present consumption. So, the sense of discounting is defined as follows.

¹⁷ Conventionally, the term time preference and rate of time preference convey the same meaning.

The sense of discounting is correct, if $sgn(DC) = \overline{TP}$.

The sense of discounting is wrong, if $sgn(DC) \neq \overline{TP}$.

When discounting a future utility value, the only role of a discount rate is to reduce the future utility value if the consumer has $\overline{TP} = +$ and to increase the future utility value if the consumer has $\overline{TP} = -$. Hence, the role of the discount rate is to change a future utility value in such a way that the sign of discount $sgn(DC)$ is always equal to the sign of time preference \overline{TP} . As the following picture exemplifies:

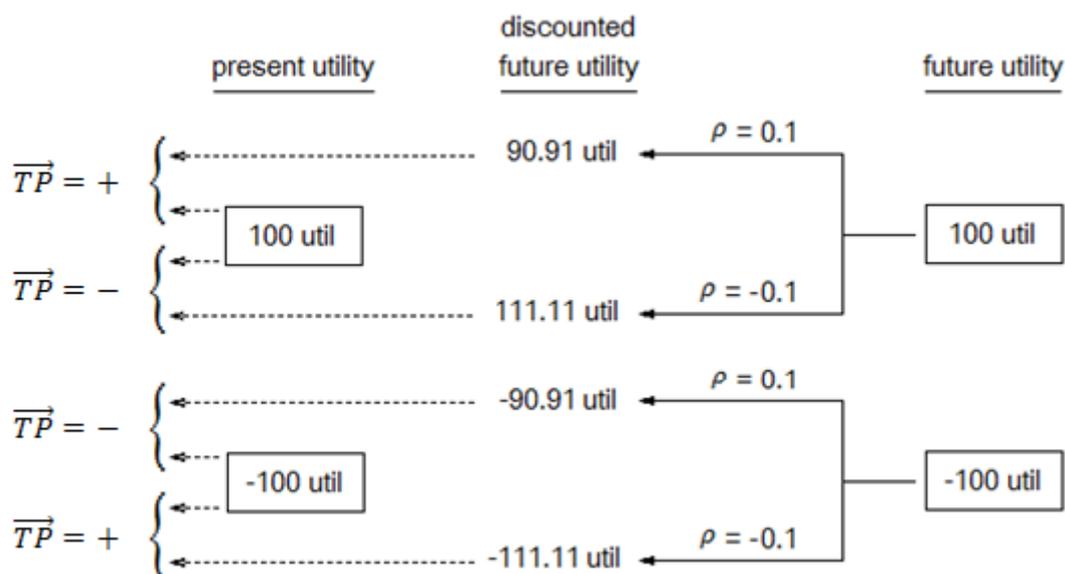


Figure 1: An example of simple discount possibilities

Figure 1 also implies that if $u_0 \in \mathbb{R}$ is a future utility value, the sense of discounting is correct if the following SMSD condition holds¹⁸

$$sgn(\rho) \cdot sgn(u_0) = \overline{TP} \quad (1)$$

3.2 The Discount

This section defines many basic concepts that relate to the discount and the utility functions, which are necessary for the discussion in subsequent sections. Given a utility function $u(c)$ with $u'(c) > 0$, define two basic types of utility function $u(c)$:

Positive-number utility function is the utility function that produces only zero or positive numbers,

Negative-number utility function is the utility function that produces only zero or negative numbers.

¹⁸ Here, the $sgn(\rho)$ is to ensure a correct sense of discounting, so it must be employed fittingly. For more detail about SMSD condition see Appendix A.1.

So, the sign of each utility function can be unambiguously defined as

$$\begin{aligned} \text{sgn}(u(c)) &\equiv +, \text{ if } u(c) \text{ is a positive-number utility function,} \\ \text{sgn}(u(c)) &\equiv -, \text{ if } u(c) \text{ is a negative-number utility function.} \end{aligned}$$

Given a discount rate $\rho \in (-1, \infty)$, define following terms.

$$\text{The discount at } c: \quad DC(c) = u(c) - \frac{1}{1+\rho} u(c) = \frac{\rho}{1+\rho} u(c)$$

$$\text{The sign of discount at } c: \quad \text{sgn}(DC(c)) = \text{sgn}(\rho) \cdot \text{sgn}(u(c))$$

$$\text{The amount of discount at } c: \quad \text{amt}(DC(c)) = \left| \frac{\rho}{1+\rho} u(c) \right|$$

$$\text{The pattern of discount amount} \equiv \begin{cases} \text{increasing} & \text{if } \frac{d}{dc} \text{amt}(DC(c)) > 0, \forall c \in \mathbb{R}_+, \rho \neq 0 \\ \text{decreasing} & \text{if } \frac{d}{dc} \text{amt}(DC(c)) < 0, \forall c \in \mathbb{R}_+, \rho \neq 0 \end{cases}$$

If $u(c)$ is a positive-number utility function, the sign of discount is always the same as the sign of the discount rate. And, if $u(c)$ is a negative-number utility function, the sign of discount is always opposite to the sign of the discount rate. Thus, to reflect a correct sense of discounting, if a consumer has $\overline{TP} = +$ he always uses positive (negative) discount rate with positive-number (negative-number) utility function, and always uses opposite sign discount rate if he has $\overline{TP} = -$.

For a constant discount rate $\rho \neq 0$, consider the discount amounts of a utility function. Since a positive-number utility function produces a monotone sequence of numbers starting from a small positive number to a large positive number, the amount of discount is increasing with c . But, since a negative-number utility function produces a monotone sequence of numbers starting from a large negative number to a small negative number, the amount of discount is decreasing with c . So, the two types of utility function possess different property¹⁹

Positive-number utility functions have an increasing pattern of discount amount,

Negative-number utility functions have a decreasing pattern of discount amount.

As will be seen subsequently, because of their difference in pattern of the discount amount, the two types of utility function affect model's optimal solutions differently²⁰.

3.3 Discounting Effects in DU Optimization Model

¹⁹ For the proof see Appendix A.2.

²⁰ Conventional economists tend to pay less attention to negative-number utility function, partly because it also has positive and decreasing marginal utility, and thus seems to affect the Euler equation the same way the positive-number utility function does. However, we will see in the next section that this is not the case.

This section studies how the act of discounting affects the solutions of a DU optimization model via different types of utility function and discount rate; such that both the relevant and irrelevant components of a utility function can be identified and the role of time preference can be understood. For simplicity we will use a simple two-period DU optimization model as a mean of study²¹.

Consider a simple no-interest consumption problem that has one consumer who was born with an endowment E , lives for two periods, and has a CRRA utility function of the form $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$. The consumer also has a constant discount rate (i.e., no retraction of decision made), and assume that $\rho \neq 0$. Thus, the consumer's problem is

$$\max \sum_{t=1}^2 \frac{1}{(1+\rho)^{t-1}} \frac{c_t^{1-\theta} - 1}{1-\theta} \quad s. t. \quad c_1 + c_2 = E.$$

Since the utility function can be written as $u(c) = \frac{c^{1-\theta}}{1-\theta} + a$, it is viewed as consisting of two constituent parts: a consumption dependent utility $\frac{c^{1-\theta}}{1-\theta}$, and a constant utility $a = \frac{-1}{1-\theta}$. The problem thus can be rewritten as

$$U^* = \max_{c_1, c_2} \left\{ \frac{c_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{c_2^{1-\theta}}{1-\theta} + \left(a + \frac{a}{1+\rho} \right) \right\} \quad s. t. \quad c_1 + c_2 = E$$

The nature of the problem is to search for a consumption amount (c_1, c_2) that makes the value in the curly braces maximum. Since the value in the parenthesis does not depend on c , it does not involve in the search process. Thus, the constant utility a in $u(c)$ has no role and is always ignored by the choice process. To find the optimal solution, the constant utility can be omitted from the beginning, but we will leave it there so that its behavior can be seen. This section uses two methods of problem solving as vehicles to explain the behaviors of all parties involved in the discount and choice process.

Method 1: Lagrange Multiplier Method

$$\mathcal{L} = \frac{c_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{c_2^{1-\theta}}{1-\theta} + \left(a + \frac{a}{1+\rho} \right) + \mu(E - c_1 - c_2)$$

By FOC, the model's equilibrium condition (the Euler equation) is found as

$$c_2 = (1+\rho)^{-1/\theta} c_1$$

And the model's optimal solutions are

²¹ Appendix A.3 shows that a two-period decision model can be viewed as a part of an n-period decision model.

$$c_2^* = \frac{E}{1 + (1 + \rho)^{1/\theta}}, \quad c_1^* = \frac{(1 + \rho)^{1/\theta} E}{1 + (1 + \rho)^{1/\theta}} = E - c_2^*$$

How should these outputs be interpreted? Conventionally, the discount rate ρ is viewed as a single-thread parameter starting from zero to infinity, and the risk aversion coefficient θ as another single-thread parameter starting continuously from zero to infinity²². Thus, the Euler equation and the optimal solutions appear as if being single-pattern. However, this is hard to be true, because ρ can be any values from -1 to ∞ , and when $\theta \in (0, 1)$ the consumption dependent utility $\frac{c^{1-\theta}}{1-\theta}$ produces positive numbers but when $\theta \in (1, \infty)$ it produces negative numbers. Thus, we need to think about the two subjective parameters in a more subtle ways.

When $\rho \in (-1, 0)$ and $\rho \in (0, \infty)$, they affect the Euler equation and the optimal solutions in an opposite sense. When $\theta \in (0, 1)$ and $\theta \in (1, \infty)$, the sign of the consumption dependent utility turns opposite which, as will be seen shortly, causes an opposite effects on the optimal solutions. Thus, both ρ and θ are double-thread parameters, together they establish four combinations of parameter region which affect the Euler equation and optimal solutions in four different ways (or patterns). To see these concretely, let $E=100$ and (θ, ρ) to be some numbers representing each of the four parameter regions, as shown in the Table 1.

Representative Parameters	$c_2^* = \frac{100}{1+(1+\rho)^{1/\theta}}$	$U^* = \frac{(c_1^*)^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{(c_2^*)^{1-\theta}}{1-\theta} + \left(a + \frac{a}{1+\rho}\right)$
1) $\theta = 0.5, \rho = 0.2$	$c_1^* = 59.02, c_2^* = 40.98$	$U^* = 15.36 + \frac{1}{1+0.2} \cdot 12.80 + (-3.67) = 22.36$
2) $\theta = 0.5, \rho = -0.2$	$c_1^* = 39.02, c_2^* = 60.98$	$U^* = 12.49 + \frac{1}{1-0.2} \cdot 15.62 + (-4.50) = 27.52$
3) $\theta = 1.2, \rho = 0.2$	$c_1^* = 53.79, c_2^* = 46.21$	$U^* = -2.25 + \frac{1}{1+0.2} \cdot (-2.32) + 9.17 = 4.99$
4) $\theta = 1.2, \rho = -0.2$	$c_1^* = 45.36, c_2^* = 54.64$	$U^* = -2.34 + \frac{1}{1-0.2} \cdot (-2.25) + 11.25 = 6.10$

Table 1: Optimal choice and discounting behavior in different parameter regions

For each case, it can be checked that U^* is the maximum total utility, by supplying some (c_1, c_2) other than (c_1^*, c_2^*) . One will also see that when (c_1, c_2) changes the third term on the RHS of U^* remains constant. This implies that, for each case, the maximum total utility U^* depends on the first two terms of its RHS only. In other words, the optimal solutions depend

²² The original form of CRRA utility function $\frac{c^{1-\theta}}{1-\theta}$ is not defined at $\theta=1$, because its limit is undefined there. Since $\lim_{\theta \rightarrow 1} \frac{c^{1-\theta}-1}{1-\theta} = \ln c$, it creates a perception that for θ to be single-thread, the CRRA utility function should be written in the form of $\frac{c^{1-\theta}-1}{1-\theta}$. However, this cannot change the nature of the CRRA utility function, because the added constant (-1) cannot induce any effects on the Euler equation and optimal solutions, if $\theta \neq 1$.

on the consumption dependent utility $\frac{c^{1-\theta}}{1-\theta}$ and the discount rate ρ only. Thus, the discounting behavior of each case can be studied from the second term of the RHS of U^* :

Case 1: the consumption dependent utility $\frac{c^{1-\theta}}{1-\theta}$ yields positive numbers and the discount rate is positive so the sign of the discount is positive, i.e. $sgn\left(\frac{0.2}{1+0.2} \cdot 12.80\right) = +$, which means that the consumer has premium on present utility. So, this is the case of

$$\overline{TP} = +, \quad sgn\left(\frac{c^{1-\theta}}{1-\theta}\right) = +.$$

Case 2: the utility is positive but the discount rate is negative, so $sgn\left(\frac{-0.2}{1-0.2} \cdot 15.62\right) = -$.

$$\overline{TP} = -, \quad sgn\left(\frac{c^{1-\theta}}{1-\theta}\right) = +.$$

Case 3: in this case $\frac{c^{1-\theta}}{1-\theta}$ yields negative number, and the sign of the discount is negative.

$$\overline{TP} = -, \quad sgn\left(\frac{c^{1-\theta}}{1-\theta}\right) = -.$$

Case 4: negative discount rate and negative-sign future utility, so $sgn\left(\frac{-0.2}{1-0.2} \cdot (-2.25)\right) = +$.

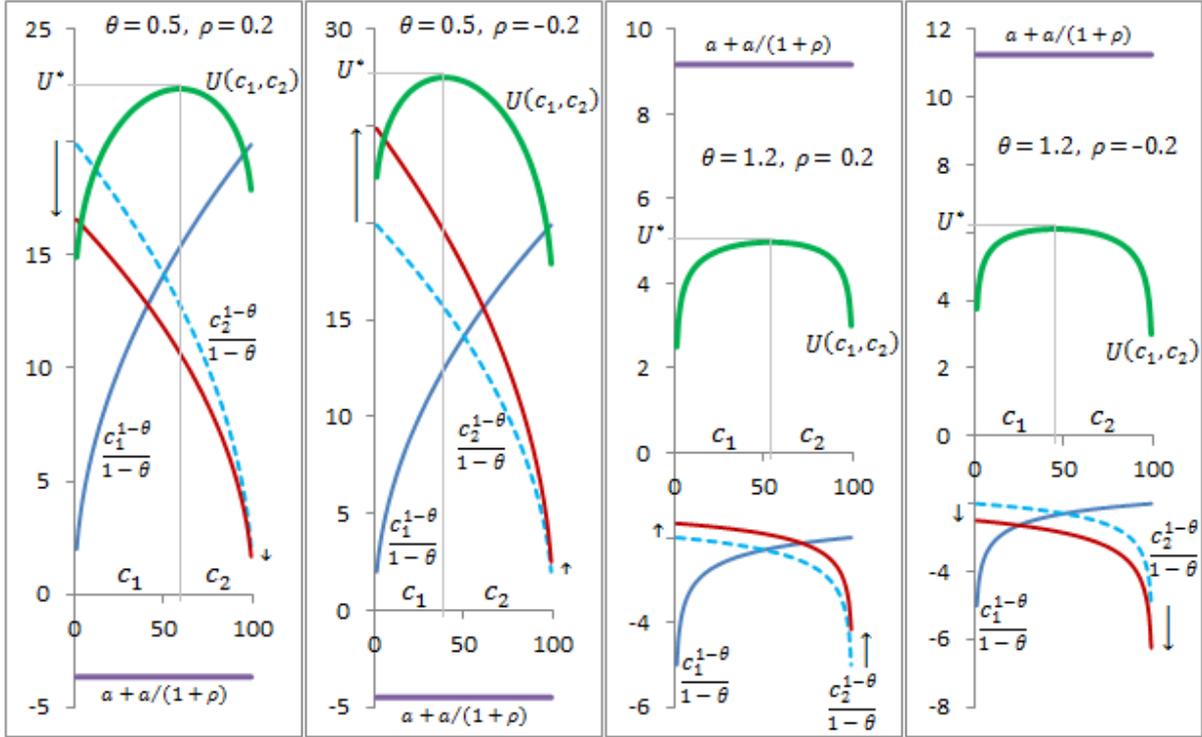
$$\overline{TP} = +, \quad sgn\left(\frac{c^{1-\theta}}{1-\theta}\right) = -.$$

So, the model has shown that each parameter region has a specific discounting configuration (the second term of the RHS of U^*) which is different from other parameter regions'. And each configuration implies a specific combination of the consumer's sign of time preference and sign of consumption dependent utility. As will be seen more clearly via method 2, these two elements are the two basic forces that move the optimal solution.

Method 2: Brute Force Method

$$U^* = \max_{c_1, c_2} \left\{ \frac{c_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{c_2^{1-\theta}}{1-\theta} + \left(a + \frac{a}{1+\rho} \right) \right\} \quad s.t. \quad c_1 + c_2 = E$$

In this method, given a specific parameter point (θ, ρ) we will exhaustively search all (c_1, c_2) for the one that yields the maximum value of total (lifetime) utility $U(c_1, c_2)$. The search will be done once for each of the four parameter regions, because different parameter points in the same parameter region have the same solution pattern. For the sake of comparison, we use the same representative parameters as in Table 1, and all searching results are summarized graphically as shown correspondingly in Figure 2.



(a) $sgn(DC(c)) = +$ and increasing discount amount. (b) $sgn(DC(c)) = -$ and increasing discount amount. (c) $sgn(DC(c)) = -$ and decreasing discount amount. (d) $sgn(DC(c)) = +$ and decreasing discount amount.

Figure 2: Total utility at different (c_1, c_2) , the maximum total utility is moved by two basic forces.

The dotted curve $\frac{c_2^{1-\theta}}{1-\theta}$ in each figure is the (before discounting) second period's consumption dependent utility, calculated at different level of c_2 . The second period's constant utility is combined with that of the first period, and shown as a level line in each figure.

Before discounting, the curve $\frac{c_1^{1-\theta}}{1-\theta}$ and $\frac{c_2^{1-\theta}}{1-\theta}$ are symmetry because they share the same functional form, with the arguments c_1 and c_2 move in opposite directions. The total utility curve $U(c_1, c_2)$ is the additive of the two periods' consumption dependent utility curves (the second period's one may be either discounted or not discounted) and the level line. Thus, before discounting, $U(c_1, c_2)$ is a symmetrical bell-shape curve whose maximum value is located at $c_1 = c_2 = 50$. When the second period's curve is discounted, the total utility curve in each parameter region changes differently:

Figure 2a: $0 < \theta < 1$ and $\rho > 0$

The discounting act shifts the curve $\frac{c_2^{1-\theta}}{1-\theta}$ downward which means that $sgn(DC(c_2)) = +$, i.e. the consumer has premium on present utility. Since the curve $\frac{c_2^{1-\theta}}{1-\theta}$ with θ less than one is a positive-number utility function, it has increasing pattern of discount amount; as shown by

the arrow's size that is increasing from the right to left side of the figure. The two properties, the positive discount and the increasing pattern of discount amount, together pulls the total utility curve downward but harder on the left side. The discounting thus lowers the left side of the total utility curve and leaves the right side higher, that is, it moves the maximum total utility point to the right. Hence, (c_1^*, c_2^*) is moved from (50, 50) to (59, 41) by the two discounting forces, or equivalently by

$$\overline{TP} = +, \quad \text{sgn}\left(\frac{c_2^{1-\theta}}{1-\theta}\right) = +.$$

Figure 2b: $0 < \theta < 1$ and $-1 < \rho < 0$

This case is explained similar, but in opposite direction, to case 2a. The two forces,

$$\overline{TP} = -, \quad \text{sgn}\left(\frac{c_2^{1-\theta}}{1-\theta}\right) = +,$$

move the maximum utility point (c_1^*, c_2^*) to the left, from (50, 50) to (39, 61).

Figure 2c: $\theta > 1$ and $\rho > 0$

In this case, the two discounting forces are the negative discount ($\overline{TP} = -$) and the decreasing pattern of discount amount (the negative-number utility function). Thus, the arrow points up and its size is decreasing from right to left. Because the total utility curve is lifted up harder on the right side the maximum total utility point is moved to the right, by

$$\overline{TP} = -, \quad \text{sgn}\left(\frac{c_2^{1-\theta}}{1-\theta}\right) = -.$$

Figure 2d: $\theta > 1$ and $-1 < \rho < 0$

the act of discounting pulls the total utility curve down and harder on the right side, thus the maximum total utility point is moved to the left, from (50, 50) to (45, 55), by

$$\overline{TP} = +, \quad \text{sgn}\left(\frac{c_2^{1-\theta}}{1-\theta}\right) = -.$$

In this case $\theta = 1.2$, the constant utility $a = \frac{-1}{1-\theta} = 5$. Before discounting the level line is at 10, after discounting the level line shifts up evenly to 11.25. This lifts the total utility curve up by 1.25 evenly throughout the curve, thus the maximum total utility point is moved straight up. So, the optimal consumption remains at the same point, the constant utility a cannot affect the solution when being discounted. This means that the level line has no role and thus the constant utility a (or the constant -1) can be deleted from the given utility function

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta} = \frac{c^{1-\theta}}{1-\theta} + a.$$

However, if one chooses to keep the constant term as it is, it should be noted that the sign of $u(c)$ itself is not the one that influences the choice process; the relevant sign is the sign of its

consumption dependent utility. And how to derive the consumption dependent utility from a given utility function will be discussed in section 3.4.

Figure 2a and 2d are for the economy populated by consumers with $\overrightarrow{TP} = +$ who always choose the sign of discount rate to match the sign of their consumption dependent utility. But, Figure 2b and 2c are for $\overrightarrow{TP} = -$ economy; the consumers always choose the sign of discount rate to be opposite to that of their consumption dependent utility (see also section 3.2).

If the consumers in an economy have $\overrightarrow{TP} = +$, positive consumption dependent utility moves (c_1^*, c_2^*) to the right but negative consumption dependent utility moves it to the left. Thus, the sign of the consumption dependent utility plays a crucial role in determining the solution or path of the economy. Although Figure 2a and 2d can depict this behavior vividly, the Euler equation cannot; because of this reason the sign of the utility function has received virtually no attention from economic literature.

If the consumers have $\overrightarrow{TP} = +$ and the optimal solutions are interpreted under positive discount rate condition $\rho > 0$, only Figure 2a and 2c are applied. However, the two figures should not coexist in the same economy, because they have different sign of discount (which would be interpreted as the consumers' \overrightarrow{TP} is changing or the sense of discounting is wrong).

In conclusion, the two methods have revealed that there are two discounting forces that drive the optimal consumption point sideways²³: (1) the sign of time preference \overrightarrow{TP} (which is indicating the sign of the discount), and (2) the sign of the consumption dependent utility (which indicates the pattern of discount amount). The four combinations of these two forces move the optimal solution differently, and each movement is peculiar to each of the four parameter regions underpinning four kinds of model's optimal solution or path.

3.4 Matched-Sign-Discounting (MSD) Condition

Consider a DU optimization model:

$$\max \int_0^{\infty} e^{-\rho t} u(c_t) dt.$$

Given a utility function $u(c)$, its marginal function is $u'(c)$. $u'(c)$ is the only information from $u(c)$ that the model uses when determining the equilibrium conditions. However, there are many utility functions $v(c)$ which also have marginal function equal to $u'(c)$, and all of them can be written as a family:

²³ Another force that moves the optimal point is the interest-rate force which can be considered separately.

$$v(c) = \int u'(c) dc = v_0(c) + K \quad , K \in \mathbb{R} \text{ is an arbitrary constant of integration.}$$

Every utility function in $v(c)$ family can replace $u(c)$ without affecting the model's solution, because they all yield the same marginal function $u'(c)$. And the purest form of $v(c)$ is $v_0(c)$ which shall be called the natural form of the family. The natural form is also denoted as $u^n(c)$ which is thus defined as

$$u^n(c) \equiv \int u'(c) dc - K.$$

Since $u(c)$ is also a utility function in $v(c)$ family, it can be written as

$$u(c) \equiv u^n(c) + a \quad , a \text{ is a particular constant.}$$

If $a = 0$, $u(c)$ is said to be in its natural form.

If $a \neq 0$, $u(c)$ is in its shifted form.

Now, $u(c)$ is viewed as consisting of two pure parts: (1) $u^n(c)$, the consumption dependent utility, which is the only part of $u(c)$ that plays a role in determining model's solution, and (2) a , the constant utility, which has no role in the choice process. As it has been pointed out before that the sign of $u(c)$ itself is not the one that influences the choice process, but the sign of $u^n(c)$ is. Therefore, throughout the DU optimization domain, the sign of a utility function is defined by its natural part.

$$sgn(u(c)) \equiv \begin{cases} + & \text{if } u^n(c) \geq 0 & \forall c \in \mathbb{R}_+ \\ - & \text{if } u^n(c) \leq 0 & \forall c \in \mathbb{R}_+ \\ mixed & \text{if } u^n(c) \cdot u^n(c') < 0 & \exists c, c' \in \mathbb{R}_+ \end{cases}$$

And the utility function for each case is called positive-sign, negative-sign, and mixed-sign utility function respectively. The positive-sign and negative-sign utility function are also referred to as mono-sign utility functions²⁴.

Since $sgn(DC(c)) = sgn(\rho) \cdot sgn(u(c))$, when the sense of discounting is correct the following MSD condition holds for a mono-sign utility function²⁵.

$$sgn(\rho) \cdot sgn(u(c)) = \overline{TP}. \quad (2)$$

²⁴ In the setup of a DU optimization model, it is more intuitive to replace $u(c)$ by $u^n(c)$. And if there exists a limiting case which allows the constant term to alter the form of the natural part, the new utility function should be considered separately; e.g., since $\lim_{\theta \rightarrow 1} \left(\frac{c^{1-\theta}}{1-\theta} - \frac{1}{1-\theta} \right) = \ln c$ the case of $\theta=1$ should be treated separately.

²⁵ Given \overline{TP} and $sgn(u(c))$, the role of $sgn(\rho)$ is to ensure a correct sense of discounting. Equation (2) is the central message from Figure 2. For more detail about MSD condition see Appendix A.1.

The following three examples illustrate how to apply the three newly defined concepts (the sign of time preference, the rate of time preference, and the sign of utility function) in actual problem. Ex.3 also shows graphically the sense of discounting in different scenarios.

Ex.1 The log-limiting form of the CRRA utility function, $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, can be analyzed as:

The natural part of $u(c)$ is $u^n(c) = \int u'(c) dc - K = \frac{c^{1-\theta}}{1-\theta}$, and the constant term can be found as $a = u(c) - u^n(c) = \frac{-1}{1-\theta}$. The utility function is currently in its shifted form,

$$u(c) = \frac{c^{1-\theta}}{1-\theta} - \frac{1}{1-\theta}.$$

Ex.2 In a DU optimization problem if the representative consumer has $\overline{TP} = +$ and his utility function is $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, what are the correct signs of the discount rate ρ ?

Since the sign of $u(c)$ is determined from its natural part, and the sign of ρ is determined from MSD condition equation (2),

$$\text{sgn}(\rho) \cdot \text{sgn}(u(c)) = +.$$

So,
$$\text{sgn}(\rho) = \text{sgn}(u(c)) = \text{sgn}\left(\frac{c^{1-\theta}}{1-\theta}\right) = \begin{cases} + & \text{if } \theta \in (0,1) \\ - & \text{if } \theta \in (1, \infty) \\ \text{NA.} & \text{if } \theta = 1 \end{cases}$$

When $\theta=1$, $u(c) = \ln c$, the $\text{sgn}(\rho)$ depends on consumption path $c(t)$. (see Appendix A.1)

Ex.3 From Ex. 2, suppose that the optimal consumption path is known as $c^*(t) = e^{gt}$, $g > 0$, analyze and explain the behavior of the DU model's component parts.

Since the relevant utility stream is the consumption dependent utility, $u^n(c) = \frac{c^{1-\theta}}{1-\theta}$, we substitute $u^n(c^*(t))$ into the DU model,

$$U = \int_0^{\infty} e^{-\rho t} \frac{e^{(1-\theta)gt}}{1-\theta} dt.$$

Now, one can plot all the related curves of this discounted utility stream in two cases of risk aversion coefficient θ and three cases of discount rate ρ ; for example Figure 3a uses $\theta = 0.5$, $\rho = (0.15, 0, -0.15)$, and $g = 0.21$, Figure 3b uses $\theta = 5$ and $\rho = (0.8, 0, -0.8)$ and $g = 0.21$.

Note: when $\theta=1$, $u(c^*(t)) = \ln e^{gt} > 0$. This is the case of positive-sign utility function and thus can be analyzed similar to the case of Figure 3a.

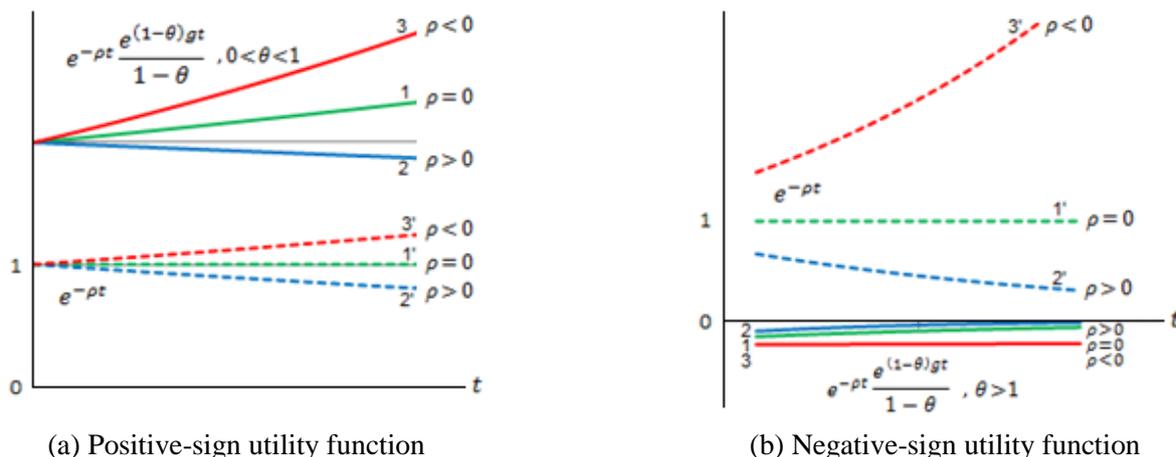


Figure 3: Analysis of discounted utility stream for exponential consumption path

In Figure 3a and 3b, curve 1 is the graph of $\frac{e^{(1-\theta)gt}}{1-\theta}$. Curve 1', 2', and 3' are the graph of the discount function $e^{-\rho t}$ when ρ are equal to, greater than, and less than zero respectively. Curve 1, 2, and 3 are the graph of $e^{-\rho t} \frac{e^{(1-\theta)gt}}{1-\theta}$ which are the multiplication results of two curves: 1'x1, 2'x1, and 3'x1 respectively. Since the consumer has $\overline{TP} = +$ the discounting can be analyzed as follow:

Curve 2 in Figure 3a is the discounted stream when $\rho > 0$. This is the correct discounted utility stream because the curve $\frac{e^{(1-\theta)gt}}{1-\theta}$ is shifted downward by the discount function. The area under curve 2 is the discounted sum U . This area converges by the TVC requirement²⁶.

Curve 3 in Figure 3a is the discounted stream when $\rho < 0$. This is the case that the curve $\frac{e^{(1-\theta)gt}}{1-\theta}$ is shifted upward by the discount function which is a wrong, because the discount function is meant to decrease, not to increase, the utility stream. Hence, the area under this curve diverges.

Curve 2 in Figure 3b is the discounted stream when $\rho > 0$. This is the case in which the curve $\frac{e^{(1-\theta)gt}}{1-\theta}$ is shifted upward by the discount function which is wrong. The negative area between curve 2 and the horizontal axis is the discounted sum U , which is bounded. This case is to show that, when the utility function has negative sign, the positive discount rate even though allows the integral to converge it gives a wrong sense of discounting.

²⁶ The transversality condition is to ensure that the area under the curve converges, i.e., $\rho > (1 - \theta)g$. In this case, $\theta=0.5$, $g=0.21$, and $\rho=0.15$, the TVC is satisfied. For more detail about this form of the TVC, see Barro and Sala-i-Martin (2004) p.101.

Curve 3 in Figure 3b is the discounted stream when $\rho < 0$. This is the correct discounted utility stream because the curve $\frac{e^{(1-\theta)gt}}{1-\theta}$ is shifted downward by the discount function. The negative area between curve 3 and the horizontal axis is the discounted sum U . This area converges by the TVC requirement²⁷. This case is the evidence for showing that, not only the positive discount rate, negative discount rate can allow the integral to converge too.

Therefore, this example has shown that when consumer has $\overline{TP} = +$ the correct sense of discounting is exhibited only when (1) $\rho > 0$ and $0 < \theta < 1$ (positive-sign utility function) and (2) $-1 < \rho < 0$ and $\theta > 1$ (negative-sign utility function). So, the appropriate sign of the discount rate that correctly shifts the utility stream downward, for both cases of the risk aversion coefficient, are consistent with MSD condition $sgn(\rho) \cdot sgn(u(c)) = +$. And, to guarantee the convergence of the discounted sum, in both cases of utility function, the TVC is needed as an additional requirement.

4. Reassessment of Three Neoclassical Models

This section reassesses three important macroeconomic models, in which inconsistencies are obvious and can be directly attributed to the embracing of positive discount rate condition. The first case, section 4.1, is the inability of the well-recognized neoclassical growth (RCK) model to reconcile all historical stylized facts especially the speed of convergence which Barro and Sala-i-Martin (1992) has found to be around 2%. The second case, section 4.2, is Mehra and Prescott's (1985) equity premium puzzle; the inability of the neoclassical pure exchange model to explain why historical equity premium was as high as 6 percent. The third case, section 4.3, is Lucas' (2003) estimation of welfare gain from eliminating consumption risk which turns out to be extremely small. This result is considered weird because consumption fluctuation is obvious in people's daily life yet Lucas's estimation of the effect of its absenteeism is unnoticeably small.

4.1 Reassessment of Neoclassical Growth Model

The major contribution of Barro and Sala-i-Martin's (1991 and 1992) works are lying in their successful measurement of the speed of convergence. The speed of convergence (β) is an important stylized fact that allows the overall assessment of the RCK model possible.

²⁷ In this case, $\theta=5$, $g=0.21$, and $\rho=-0.8$, the TVC $\rho > (1 - \theta)g$ is satisfied.

However, the result of their assessment turns out to be inconsistent; the model cannot reconcile simultaneously all historical stylized facts. Thus, this section will reassesses the RCK model under matched-sign-discounting condition, equation (2), by using the same set of stylized facts employed by Barro and Sala-i-Martin.

Barro and Sala-i-Martin (1992 and 1995) used the RCK model (under Cobb Douglas production and CRRA utility function) as recapped here to fit U.S. historical stylized facts.

$$\begin{aligned} \text{Euler equation: } r &= \rho + \theta g, & \text{TVC: } \rho &> (1 - \theta)g + n, \\ s^* &= \alpha \frac{\delta + n + g}{\delta + \rho + \theta g}, & \frac{K}{Y} &= \frac{\alpha}{\delta + \rho + \theta g}, \\ \beta &= \left[\xi^2 + \frac{(1 - \alpha)(\rho + \theta g + \delta)}{\theta} \left[\frac{\rho + \theta g + \delta}{\alpha} - (g + n + \delta) \right] \right]^{\frac{1}{2}} - \xi, \\ \xi &= [\rho - (1 - \theta)g - n]/2. \end{aligned}$$

By fixing four basic stylized facts (α , δ , n , and g) to some well-accepted values, the objective is to search for the consumer's risk aversion coefficient θ and discount rate ρ , such that these equations simultaneously hold and yield as close as possible the four remaining stylized facts²⁸: $0.055 \leq r \leq 0.09$, $s^* \approx 0.185$, $0.015 \leq \beta \leq 0.03$, $2.1 \leq K/Y \leq 3.3$.

	θ	ρ	α	$1 - \alpha$	δ	n	g	r	s^*	β	K/Y
1) BS92	1	+0.05	0.35	0.65	0.05	0.02	0.02	0.070	<u>0.26</u>	<u>0.126</u>	2.92
2) BS92*	1	+0.05	<u>0.80</u>	<u>0.20</u>	0.05	0.02	0.02	0.070	<u>0.60</u>	0.026	<u>6.67</u>
3) MSD92	17.24	-0.27	0.35	0.65	0.05	0.02	0.02	0.070	<u>0.26</u>	0.022	2.92
4) BS95	17	+0.02	0.30	0.70	0.05	0.01	0.02	<u>0.360</u>	<u>0.06</u>	<u>0.056</u>	<u>0.73</u>
5) BS95*	1.75	+0.02	<u>0.75</u>	<u>0.25</u>	0.05	0.01	0.02	0.055	<u>0.57</u>	0.020	<u>7.14</u>
6) MSD95	15.86	-0.26	0.30	0.70	0.05	0.01	0.02	0.055	0.22	0.022	3.16
7) MSD*	16.75	-0.26	0.30	0.70	0.04	0.01	0.02	0.070	0.19	0.022	2.73

Table 2 RCK model's fitting results, under positive discount rate condition (line 1, 2, 4, and 5) and under MSD condition: $sgn(\rho) \cdot sgn(u(c_t)) = +$ (line 3, 6, and 7).

Note: Underlined numbers are those do not fit stylized facts. Shaded numbers are Barro and Sala-i-Martin's broaden-capital suggestion. Due to round off, verification may not be precise, e.g. in line 7 the more precise values are $\theta=16.747$ and $\rho=-0.26495$.

Line 1 of Table 2 is the result of Barro and Sala-i-Martin's (1992) direct fitting. Since the speed of convergence β is too high they suspected that it is because the capital share in the model is too low. So, they suggested that the capital should be reinterpreted as a broaden capital, in which human capital is included. To reflect this idea the capital share should be adjusted to be 0.8 and labor share to be 0.2, as shown in line 2. This change brings β down to 0.026 which is in the acceptable range, but the saving rate and capital to output ratio response

²⁸ see Appendix A.4 for the sources of these stylized facts

strongly and get out of range (which can be explained as because they are now including human capital).

However, if the positive discount rate condition is abandoned and the MSD condition is used instead, the discount rate ρ need not be positive it can be negative if $\theta > 1$. Thus the application of MSD condition increases the searching range for ρ . And line 3 shows that, without resorting to broaden-capital concept, the speed of convergence can reach 0.022 (the middle of the 0.015-0.03 range) when $\theta = 17.24$ and $\rho = -0.27$. At this point, excepting for a little high saving rate, the RCK model's predictions can reconcile well all the rest of the stylized facts.

Next consider line 4 and 5, which is another Barro and Sala-i-Martin's fittings in their 1995 and 2004 textbooks. This time they used $\alpha = 0.3$ and $n = 0.1$ which are more accurate. Their results and suggestion of using broaden-capital concept are similar to previously. However, from the MSD perspective, there is a problem in this fitting. Since θ was chosen to be greater than one it means that the CRRA utility function has negative sign, hence, the appropriate way to discount this utility function is to use negative discount rates, but positive discount rates are used. This mismatch between the sign of utility function and the sign of discount rate means that the model is predicting the behavior of an economy in which the consumer prefers future over present consumption, a hidden inconsistency.

In contrast, when the positive discount rate condition is replaced by MSD condition as shown in line 6 all predictions are now consistent with the stylized facts, and no hidden inconsistency exists.

According to Mankiw, Romer, and Weil (1992) the depreciation rate δ is about 0.03. Evans (2000) uses $0.024 < \delta < 0.048$. The 0.05 depreciation rate used by Barro and Sala-i-Martin seems a bit too high, so, in line 7 the middle value $\delta = 0.04$ is used. Line 7 shows that the RCK model under MSD condition can fit all historical stylized fact remarkably well. Thus line 7 is regarded as the model's best fit here, and now the two unknown subjective parameters θ and ρ can be reasonably inferred, by basing on all the nine stylized facts and $\overrightarrow{TP} = +$, to be 16.75 and -26.5% respectively.

4.2 Reassessment of Neoclassical Pure Exchange Model

Mehra and Prescott (1985) have found that a class of general equilibrium model, essentially the well-known Lucas' asset pricing model, is strongly violated by U.S. historical data during

period 1889-1978. Since their study was carried out under positive discount rate condition, this section will reassess by repeating their study under MSD condition.

U.S. economy during period 1889-1978 had shown that the real return on relatively riskless security $R^f = 0.8\%$, the real return on risky security $R^e = 6.98\%$, and thus the equity premium is 6.18%. To test whether their model can generate this fact, Mehra and Prescott assumed two states Markov chain and restrict the process to follow:

$$\begin{aligned}\lambda_1 &= 1 + \mu + \delta, & \lambda_2 &= 1 + \mu - \delta, \\ \phi_{11} &= \phi_{22} = \phi, & \phi_{12} &= \phi_{21} = 1 - \phi.\end{aligned}$$

Where μ is the average growth rate of consumption, δ is the variability (S.D.) of consumption growth, and ϕ is the serial correlation of the growth rate.

Historical data indicated that $\lambda_1 = 1.054$, $\lambda_2 = 0.982$, and $\phi = 0.43$. By using these parameters and CRRA utility function Mehra and Prescott calculated their model predictions for different values of (θ, β) ²⁹ and reported them in the following picture.

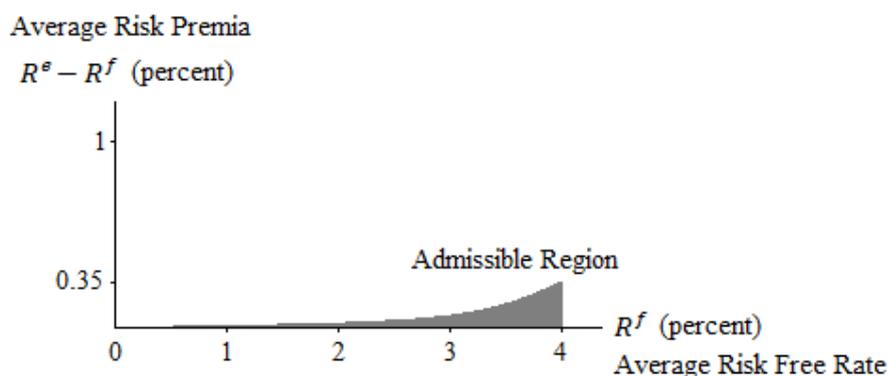


Figure 4: Set of admissible risk premia, from Mehra and Prescott (1985) p.155

Figure 4 shows the risk premia of all the points in the admissible region. The admissible region is defined as the region containing only practical values of R^f and (θ, β) , that is

$$\begin{aligned}R^f &\in (0, 4\%] \text{ and} \\ (\theta, \beta) &\in \{(\theta, \beta): 0 < \theta \leq 10, 0 < \beta < 1, \text{ and the existence condition}^{30} \text{ is satisfied}\}.\end{aligned}$$

Figure 4 indicates that the maximum risk premium that the model can generate is just 0.35 percent, at $R^f = 4\%$. This predicted risk premium is far lower than the observed equity

²⁹ θ is coefficient of risk aversion, β is discount factor which is $1/(1 + \rho)$

³⁰ The existence condition requires that the matrix A with element $a_{ij} \equiv \beta \phi_{ij} \lambda_j^{1-\alpha}$ for $i, j = 1, \dots, n$ is stable; that is, $\lim_{m \rightarrow \infty} A^m$ is zero.

premium of 6.18 percent, at $R^f = 0.8\%$. So, it is a puzzle that the model cannot come even close to reconcile historical data.

Now let's reassess Mehra and Prescott model. By repeating the algorithm described in Mehra and Prescott (1985), Table 3A below shows a big picture's results, in which the four parameter regions can be seen simultaneously; now the discount factor β is no longer capped at one and the coefficient of risk aversion θ is not capped at ten.

Table 3A shows that the tip of Mehra and Prescott's admissible region appears at the upper-left corner, and the maximum risk premium of the region is also confirmed as 0.35%, at $R^f = 4\%$ when $(\theta, \rho) = (2.33, 0.0024)$.

Under MSD condition a consumer exhibits $\overline{TP} = +$ only when (1) $0 < \theta < 1$ and $\rho > 0$ or (2) $\theta > 1$ and $-1 < \rho < 0$. The shaded areas in the tables are where the consumer has $\overline{TP} = +$ and $R^f \in (0, 4\%]$. Note that some part of Mehra and Prescott's admissible region, including the 0.35% risk-premium point, is not in the shaded areas. It means that the consumer, whose subjective parameters are in this part, even though in the admissible region, has premium on future utility.

The striking feature of Table 3A and 3B combined is that when the positive discount rate condition is replaced by the MSD condition, Mehra and Prescott's model can generate a result that exactly fits the historical observation. When $(\theta, \rho) = (18.35, -0.11035)$, the model's outputs match historical values: $R^e = 6.98\%$, $R^f = 0.8\%$, and $RP = 6.18\%$. Since this point is in a shaded area and also satisfies the existence condition it means that the consumer has premium on present utility, the economy is in equilibrium and convergent. Therefore, the historical returns and risk premium are explained by Lucas' pure exchange model with Mehra and Prescott's Markov-chain representation. And since the six percent high equity premium is explicitly explained by the model itself, it should not be any longer considered the model's puzzle.

θ	0.5			1.5			2.33			3			8			13			18			23		
	Re	Rf	RP	Re	Rf	RP	Re	Rf	RP	Re	Rf	RP	Re	Rf	RP	Re	Rf	RP	Re	Rf	RP	Re	Rf	RP
0.020	0.0293	0.0287	0.0006	0.0473	0.0453	0.0020	0.0618	0.0583	0.0035	0.0731	0.0683	0.0049	0.1463	0.1269	0.0194	0.1960	0.1560	0.0400	0.2201	0.1566	0.0636	0.2201	0.1328	0.0873
0.015	0.0242	0.0236	0.0006	0.0422	0.0401	0.0020	0.0566	0.0531	0.0035	0.0679	0.0630	0.0048	0.1407	0.1214	0.0193	0.1902	0.1504	0.0399	0.2144	0.1510	0.0635	0.2144	0.1273	0.0870
0.010	0.0192	0.0186	0.0006	0.0370	0.0350	0.0020	0.0514	0.0479	0.0035	0.0626	0.0578	0.0048	0.1351	0.1159	0.0192	0.1845	0.1447	0.0398	0.2086	0.1453	0.0633	0.2087	0.1217	0.0870
0.005	0.0141	0.0135	0.0006	0.0319	0.0299	0.0020	0.0462	0.0427	0.0035	0.0574	0.0526	0.0048	0.1295	0.1103	0.0192	0.1787	0.1390	0.0397	0.2028	0.1396	0.0632	0.2030	0.1162	0.0868
0.0024	0.0115	0.0109	0.0006	0.0292	0.0272	0.0020	0.0435	0.0400	0.0035	0.0546	0.0498	0.0048	0.1266	0.1075	0.0192	0.1757	0.1361	0.0396	0.1998	0.1367	0.0631	0.2000	0.1133	0.0867
0.000	0.0091	0.0085	0.0006	0.0268	0.0248	0.0020	0.0410	0.0375	0.0034	0.0521	0.0473	0.0048	0.1239	0.1048	0.0191	0.1729	0.1334	0.0396	0.1970	0.1339	0.0631	0.1973	0.1106	0.0867
-0.010	-0.0010	-0.0016	0.0006	0.0165	0.0145	0.0020	0.0306	0.0272	0.0034	0.0416	0.0369	0.0047	0.1128	0.0938	0.0190	0.1614	0.1220	0.0394	0.1854	0.1226	0.0628	0.1858	0.0995	0.0863
-0.015	-0.0061	-0.0066	0.0006	0.0114	0.0094	0.0020	0.0254	0.0220	0.0034	0.0363	0.0316	0.0047	0.1072	0.0882	0.0190	0.1556	0.1164	0.0393	0.1796	0.1169	0.0627	0.1801	0.0939	0.0862
-0.020	-0.0111	-0.0117	0.0006	0.0062	0.0043	0.0020	0.0204	0.0168	0.0034	0.0311	0.0264	0.0047	0.1016	0.0827	0.0189	0.1499	0.1107	0.0392	0.1738	0.1113	0.0626	0.1744	0.0884	0.0860
-0.025	-0.0162	-0.0167	0.0006	0.0011	-0.0008	0.0020	0.0150	0.0116	0.0034	0.0258	0.0211	0.0047	0.0960	0.0772	0.0188	0.1441	0.1050	0.0391	0.1680	0.1056	0.0624	0.1687	0.0828	0.0859
-0.030	-0.0212	-0.0218	0.0006	-0.0040	-0.0060	0.0019	0.0098	0.0064	0.0034	0.0206	0.0159	0.0047	0.0905	0.0717	0.0188	0.1383	0.0994	0.0390	0.1622	0.0999	0.0623	0.1630	0.0773	0.0857
-0.035	-0.0262	-0.0268	0.0006	-0.0092	-0.0111	0.0019	0.0046	0.0012	0.0033	0.0153	0.0107	0.0046	0.0849	0.0661	0.0187	0.1326	0.0937	0.0389	0.1564	0.0943	0.0622	0.1573	0.0717	0.0856
-0.040	-0.0313	-0.0318	0.0006	-0.0143	-0.0162	0.0019	-0.001	-0.0040	0.0033	0.0101	0.0054	0.0046	0.0793	0.0606	0.0187	0.1268	0.0880	0.0388	0.1506	0.0886	0.0620	0.1516	0.0662	0.0854
-0.045	-0.0363	-0.0369	0.0006	-0.0194	-0.0213	0.0019	-0.006	-0.0092	0.0033	0.0048	0.0002	0.0046	0.0737	0.0551	0.0186	0.1211	0.0824	0.0387	0.1448	0.0829	0.0619	0.1459	0.0606	0.0852
-0.050	-0.0414	-0.0419	0.0005	-0.0246	-0.0265	0.0019	-0.011	-0.0143	0.0033	0.000	-0.0050	0.0046	0.0681	0.0496	0.0186	0.1153	0.0767	0.0386	0.1390	0.0772	0.0618	0.1402	0.0551	0.0851
-0.055	-0.0464	-0.0470	0.0005	-0.0297	-0.0316	0.0019	-0.016	-0.0195	0.0033	-0.006	-0.0103	0.0046	0.0625	0.0440	0.0185	0.1095	0.0710	0.0385	0.1332	0.0716	0.0616	0.1345	0.0495	0.0849
-0.060	-0.0515	-0.0520	0.0005	-0.0348	-0.0367	0.0019	-0.021	-0.0247	0.0033	-0.011	-0.0155	0.0045	0.0570	0.0385	0.0184	0.1038	0.0654	0.0384	0.1274	0.0659	0.0615	0.1287	0.0440	0.0848
-0.065	-0.0565	-0.0571	0.0005	-0.0400	-0.0418	0.0019	-0.027	-0.0299	0.0032	-0.016	-0.0208	0.0045	0.0514	0.0330	0.0184	0.0980	0.0597	0.0383	0.1216	0.0602	0.0614	0.1230	0.0384	0.0846
-0.070	-0.0616	-0.0621	0.0005	-0.0451	-0.0470	0.0019	-0.032	-0.0351	0.0032	-0.021	-0.0260	0.0045	0.0458	0.0275	0.0183	0.0922	0.0540	0.0382	0.1158	0.0546	0.0612	0.1173	0.0329	0.0845
-0.075	-0.0666	-0.0671	0.0005	-0.0502	-0.0521	0.0019	-0.037	-0.0403	0.0032	-0.027	-0.0312	0.0045	0.0402	0.0220	0.0183	0.0865	0.0484	0.0381	0.1100	0.0489	0.0611	0.1116	0.0273	0.0843
-0.080	-0.0717	-0.0722	0.0005	-0.0554	-0.0572	0.0019	-0.042	-0.0455	0.0032	-0.032	-0.0365	0.0045	0.0346	0.0164	0.0182	0.0807	0.0427	0.0380	0.1042	0.0432	0.0610	0.1059	0.0218	0.0841
-0.085	-0.0767	-0.0772	0.0005	-0.0605	-0.0623	0.0018	-0.047	-0.0507	0.0032	-0.037	-0.0417	0.0044	0.0290	0.0109	0.0181	0.0749	0.0370	0.0379	0.0984	0.0376	0.0608	0.1002	0.0162	0.0840
-0.090	-0.0817	-0.0823	0.0005	-0.0656	-0.0675	0.0018	-0.053	-0.0558	0.0032	-0.043	-0.0469	0.0044	0.0235	0.0054	0.0181	0.0692	0.0314	0.0378	0.0926	0.0319	0.0607	0.0945	0.0107	0.0838
-0.095	-0.0868	-0.0873	0.0005	-0.0708	-0.0726	0.0018	-0.058	-0.0610	0.0032	-0.048	-0.0522	0.0044	0.0179	-0.0001	0.0180	0.0634	0.0257	0.0377	0.0868	0.0262	0.0606	0.0888	0.0051	0.0837
-0.100	-0.0918	-0.0924	0.0005	-0.0759	-0.0777	0.0018	-0.063	-0.0662	0.0031	-0.053	-0.0574	0.0044	0.0123	-0.0057	0.0180	0.0576	0.0200	0.0376	0.0810	0.0206	0.0604	0.0830	-0.0005	0.0835
-0.105	-0.0969	-0.0974	0.0005	-0.0810	-0.0828	0.0018	-0.068	-0.0714	0.0031	-0.058	-0.0626	0.0044	0.0067	-0.0112	0.0179	0.0519	0.0144	0.0375	0.0752	0.0149	0.0603	0.0773	-0.0060	0.0833
-0.110	-0.1019	-0.1024	0.0005	-0.0862	-0.0880	0.0018	-0.073	-0.0766	0.0031	-0.064	-0.0679	0.0043	0.0011	-0.0167	0.0178	0.0461	0.0087	0.0374	0.0694	0.0092	0.0602	0.0716	-0.0116	0.0832
-0.115	-0.1070	-0.1075	0.0005	-0.0913	-0.0931	0.0018	-0.079	-0.0818	0.0031	-0.069	-0.0731	0.0043	-0.004	-0.0222	0.0178	0.0403	0.0030	0.0373	0.0636	0.0035	0.0600	0.0659	-0.0171	0.0830
-0.120	-0.1120	-0.1125	0.0005	-0.0964	-0.0982	0.0018	-0.084	-0.0870	0.0031	-0.074	-0.0784	0.0043	-0.010	-0.0278	0.0177	0.0346	-0.003	0.0372	0.0578	-0.0021	0.0599	0.0602	-0.0227	0.0829

Table 3A Grand view of model outputs, cover the range of subjective parameters θ from 0.5 to 23 and ρ from -0.12 to 0.02. The area in and above the upper-left corner boundary of the table corresponds to Mehra and Prescott's admissible region where maximum risk premium is 0.55% at 4% risk free rate.

Note: Shaded areas in both tables are where the outputs satisfy MSD condition with $\bar{r}^p = +$ and $r^f \in (0, 4\%]$

θ	18			18.25			18.35			18.5		
	Re	Rf	RP									
ρ	0.0984	0.0376	0.0608	0.0990	0.0370	0.0620	0.0992	0.0367	0.0625	0.0995	0.0363	0.0632
-0.085	0.0926	0.0319	0.0607	0.0932	0.0313	0.0619	0.0934	0.0311	0.0623	0.0937	0.0307	0.0631
-0.095	0.0868	0.0262	0.0606	0.0874	0.0256	0.0617	0.0876	0.0254	0.0622	0.0879	0.0250	0.0629
-0.100	0.0810	0.0206	0.0604	0.0816	0.0200	0.0616	0.0818	0.0197	0.0621	0.0821	0.0193	0.0628
-0.105	0.0752	0.0149	0.0603	0.0758	0.0143	0.0615	0.0760	0.0141	0.0619	0.0763	0.0137	0.0626
-0.110	0.0694	0.0092	0.0602	0.0700	0.0086	0.0613	0.0702	0.0084	0.0618	0.0705	0.0080	0.0625
-0.11035	0.0690	0.0088	0.0602	0.0696	0.0082	0.0613	0.0698	0.0080	0.0618	0.0701	0.0076	0.0625
-0.115	0.0636	0.0035	0.0600	0.0642	0.0030	0.0612	0.0644	0.0027	0.0617	0.0647	0.0024	0.0624

Table 3B Closer view of table 3A, at $\theta \approx 18$ and $\rho \approx -0.11$. It shows that Mehra and Prescott's model can generate the exact fit at $\theta = 18.35$ and $\rho = -0.11035$.

4.3 Reassessment of Neoclassical Welfare Model

Lucas (2003) has proposed a simple formula to evaluate welfare gain from eliminating consumption risk. However, when applying the formula to estimate welfare gain, he assumed a low value of risk-aversion coefficient which led to the conclusion that the potential gains from improved stabilization policies are extremely small.

By assuming CRRA utility function, Lucas' formula for calculating welfare gain from eliminating consumption risk (λ) is

$$\lambda \cong \frac{1}{2} \theta \sigma^2.$$

Where θ is the coefficient of consumer's risk aversion and σ^2 is consumption risk or the variance about consumption trend.

U.S. historical data for the period 1947-2001 exhibits $\sigma \cong 0.032$. Lucas used $\theta = 1$ and estimated that the welfare gain from eliminating consumption risk is $\frac{1}{2} \cdot 1 \cdot 0.032^2 = 0.05\%$ of consumption. He argued that, for the welfare gain to have a significant meaning, it should be not too much lower than 1%. This one percent benchmark is the welfare gain from reducing the annual inflation rate from 10 to 0 percent. Since λ is just one-twentieth of a percent, he concluded that welfare gain from eliminating consumption risk is trivially small.

Lucas' conclusion based essentially on the low value of θ he had chosen. His argument for $\theta = 1$ is as follows. Since household optimal consumption requires that the Euler equation $r = \rho + \theta g$ must be satisfied, it destines the coefficient of risk aversion θ to be at or near to one, because of the following two reasons.

$$\theta = \frac{r - \rho}{g}$$

(1) U.S. historical consumption growth and interest rate are 2% and 5% respectively, and since the discount rate ρ must be positive (the positive discount rate condition), implying that the value of θ must be lower than 2.5.

(2) If θ is as high as 2.5, the interest rate of a country with high growth will be far too high comparing to that of the low growth country. For example, if the fast-growing economy like Taiwan grows at a rate of 10%, its interest rate needs to be at least 25%. Such a rate is far too high comparing to the 5% interest rate of the mature economy like the U.S. This argument also suggests that if $\theta = 1$ the high growth country will have a proportionately high interest rate.

Now, let's reassess Lucas' welfare gain formula by starting from a basic question; what are the consequences of replacing positive discount rate condition by MSD condition? There are at least two consequences here.

The first consequence is that the coefficient of risk aversion θ needs not be near to one. Under MSD condition with $\overline{TP} = +$, the risk aversion coefficient and the discount rate must coincide in two possible ways: (1) $0 < \theta < 1$ and $\rho > 0$, or (2) $\theta > 1$ and $-1 < \rho < 0$. These are in harmony with the Euler's equation,

$$\theta = \frac{r - \rho}{g}$$

If $r=5\%$, $g=2\%$, and $-1 < \rho < 0$ then θ will lie between 2.5 and 52.5.

If $r=5\%$, $g=2\%$, and $\rho > 0$ then θ will lie between 0 and 1.

Comparing to the previous case where $0 < \theta < 2.5$, here θ spans rationally and is much less restrictive. And for the concern about large interest rate difference across countries if θ is high, it is not true either. For example, suppose $\theta = 2.5$ and a fast-growing economy grows at a rate of 10%, then its interest rate can be anywhere between -75% and 25%, because in this case $-1 < \rho < 0$. Therefore, it is possible that the fast-growing economy like Taiwan and the mature economy like the U.S. may have equal interest rate.

The second consequence is that MSD condition yields a more widespread consistency. When positive discount rate condition is used together with Lucas' suggested parameters ($\theta=1$, $r=5\%$, and $g=2\%$, which imply $\rho=3\%$) it cannot reconcile all historical stylized facts, as shown in line 2 and 3 of Table 4. But when MSD condition is used all stylized facts can be simultaneously reconciled, as shown in line 7 of Table 2.

	θ	ρ	α	$1 - \alpha$	δ	n	g	r	s^*	β	K/Y
1) BS92	1	+0.05	0.35	0.65	0.05	0.02	0.02	0.070	<u>0.26</u>	<u>0.126</u>	2.92
2) Lucas	1	+0.03	0.35	0.65	0.05	0.02	0.02	0.050	<u>0.31</u>	<u>0.108</u>	<u>3.50</u>
3) Lucas*	1	+0.03	0.30	0.70	0.04	0.01	0.02	0.050	0.23	<u>0.111</u>	<u>3.33</u>

Table 4 RCK model's fitting results, under subjective parameters and rate of return on capital suggested by Lucas (2003). Line 1 is from Table 2, line 3 is when using the more suitable basic stylized facts.

Note: Underlined numbers are those do not fit stylized facts.

Table 2 has shown that the reasonable value of θ that fits historical data is about 16.75. Therefore, Lucas' welfare formula may be reassessed as

$$\lambda \cong \frac{1}{2} \theta \sigma^2 = \frac{1}{2} \cdot 16.75 \cdot 0.032^2 = 0.86\% .$$

The welfare gain from eliminating consumption fluctuation now increases from 0.05% to 0.86%, more than an order of magnitude higher. So, the implication is that welfare gain from eliminating consumption fluctuation is more than four-fifths of welfare gain from ending ten percent inflation, the amount that is fair and significant enough to not rejecting and to reconsidering the better countercyclical macroeconomic policies.

Now it has been shown that the three neoclassical models (the RCK model, the Lucas' pure exchange model, and the Lucas' welfare formula) can reconcile historical data very well under MSD condition. The RCK model and the Lucas' pure exchange model estimate the two unknown consumer subjective parameters (θ, ρ) to be (16.75,-26.5%) and (18.35,-11.035%) respectively. Theoretically, the two models are different by nature and thus take different sets of historical data into account. Yet they predict quite similar pattern of (θ, ρ) , the coefficient of risk aversions in particular are nearly the same. Since all other measurable parameters have already been explained by their respective models, the difference in their predictions of the subjective discount rate is thus likely to stem from the theoretical difference.

5. Conclusion

This research has argued that the current coexistence between negative-sign utility functions and positive discount rate condition in macroeconomics causes a wrong sense of discounting. If negative-sign utility functions are to be allowed, the concept of time preference and discount rate used conventionally should be redefined. Deeper investigation about the current definition of time preference and discount rate reveals that a redefinition is beneficial, due to many reasons:

- The use of the sign of discount rate to indicate time preference has no substantiated support, and can cause conflict if the utility function is negative-sign.
- In DU optimization models, the positive discount rate condition is not generally true (Koopmans' proof of $\rho > 0$ is true for the case of positive-sign utility function only).
- The Euler equation allows the discount rate to be positive, zero, or negative. So, practical definition of positive time preference cannot exclude negative and zero discount rates.
- The coexistence between the negative discount rate and negative-sign utility functions is both logical and practical, as was pointed out before by Olson and Bailey (1981) as their exception case.

Therefore, a new definition that separates the concept of time preference into the sign of time preference (\overrightarrow{TP}) and the rate of time preference (ρ) is made. Under new definition, the sign of discount rate ρ no longer indicates premium time; the premium time is independently handled by \overrightarrow{TP} .

When discounting a utility stream, the sign of time preference moves the utility stream up or down, the sign of utility function moves the optimal consumption point sideways, and the sign of discount rate ensures a correct sense of discounting. All these discounting effects are executed through the working of the sign of discount and the pattern of discount amount, which can be seen graphically. The new definition of time preference and the study of discounting behaviors lead to two matched sign discounting conditions (SMSD and MSD) which can ensure a correct sense of utility discounting.

MSD condition is used to reassess three important neoclassical models: the RCK model, the Lucas' pure exchange model, and the Lucas' welfare formula. The reassessment process for the first two models can be viewed as two steps screening process. All possible solutions permitted by models' equilibrium conditions are screened for a correct sense of discounting by MSD condition, the outcomes are then further screened for the one that fit historical data the most. The models' solutions produced by this process are very close to historical fact. Contrasting to the existing cases, where positive discount rate condition is used in the place of MSD condition, both models' solutions cannot fit historical data and often have a wrong sense of discounting. The results of the first two models are thus summarized as follow.

	positive-sign utility function	negative-sign utility function
positive discount rate condition	solutions do not fit historical data, correct sense of discounting	solutions do not fit historical data, wrong sense of discounting
MSD condition	solutions do not fit historical data, correct sense of discounting	solutions fit historical data, correct sense of discounting

The reassessment of Lucas' welfare formula is based mainly on how to choose a more appropriate value of the risk aversion coefficient. It turns out that, comparing to the existing result under the positive discount rate condition, MSD condition can lead to a more practical values of both the possible interest rates across economies and the welfare gain from eliminating consumption risk.

The fact that the three reassessment results can harmoniously reconcile all stylized facts and historical data very well while the sense of all discounting are also correctly preserved should not be viewed as a coincident, but more likely a result of sound and consistent underlying theories.

Appendix

A.1 Theory of Matched-Sign Discounting

Simple-Matched-Sign-Discounting (SMSD) Condition

If \overrightarrow{TP} is the sign of time preference, ρ is the discount rate, and u is a utility value then the sense of discounting is correct if the following simple-matched-sign-discounting (SMSD) condition holds,

$$\text{sgn}(\rho) \cdot \text{sgn}(u) = \overrightarrow{TP} \quad (1)$$

Proof: Figure 5 shows all possible ways to correctly discount a future utility value which may be represented by a positive or negative number by a positive, zero, or negative discount rate.

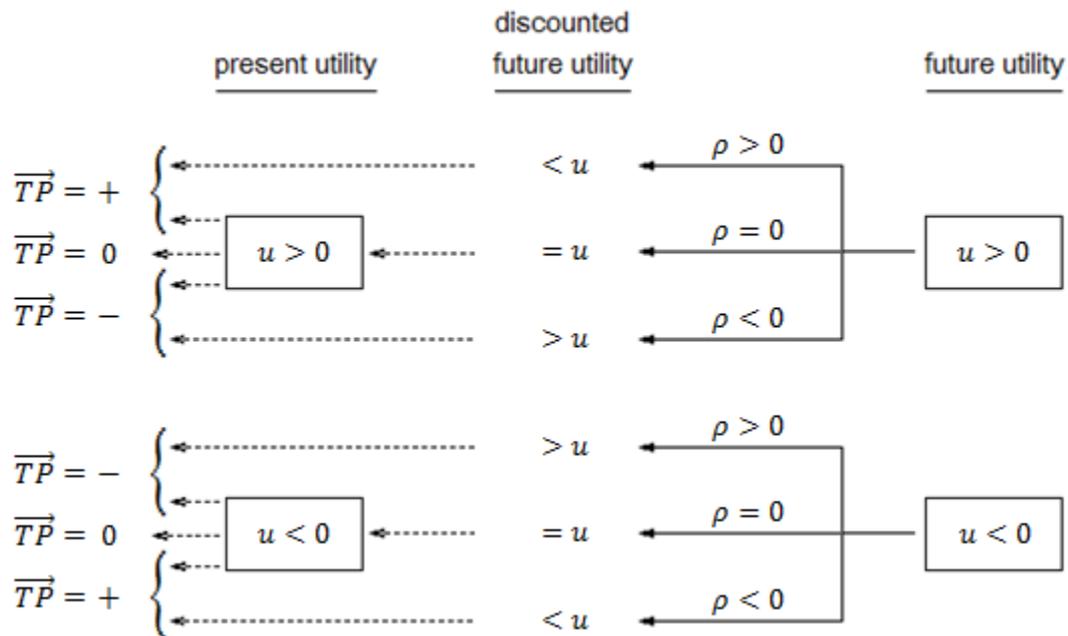


Figure 5: Six possible ways of discounting a utility value with a correct sense

All the cases of correct discounting can be grouped into three conditions by \overrightarrow{TP} :

$$\begin{aligned} \overrightarrow{TP} = + & \quad \Leftrightarrow \quad \rho u > 0 \\ \overrightarrow{TP} = 0 & \quad \Leftrightarrow \quad \rho = 0 \\ \overrightarrow{TP} = - & \quad \Leftrightarrow \quad \rho u < 0. \end{aligned}$$

The three conditions can be further grouped into a SMSD condition:

$$\text{sgn}(\rho) \cdot \text{sgn}(u) = \overrightarrow{TP} \quad \blacksquare$$

Matched-Sign-Discounting (MSD) Condition

If \overrightarrow{TP} is the sign of time preference and ρ is the rate of time preference, then the sense of discounting a mono-sign utility function $u(c)$ in a DU optimization model is correct if the following matched-sign-discounting (MSD) condition holds,

$$\text{sgn}(\rho) \cdot \text{sgn}(u(c)) = \overrightarrow{TP} \quad (2)$$

Proof: For a given sign of time preference \overrightarrow{TP} and a discount rate ρ , the sense of discounting a utility value u is correct if the SMSD condition (1) holds,

$$\text{sgn}(\rho) \cdot \text{sgn}(u) = \overrightarrow{TP}.$$

Now, let the utility value $u = u^n(c_0)$, where c_0 is a fixed value in \mathbb{R}_+ and $u^n(c)$ is the natural form of a utility function $u(c)$, the SMSD condition becomes

$$\text{sgn}(\rho) \cdot \text{sgn}(u^n(c_0)) = \overrightarrow{TP}.$$

For other value of $c \in \mathbb{R}_+$ which is different from c_0 , as long as every $u^n(c)$ has the same sign as $u^n(c_0)$ the rest of the equation remains the same for $u^n(c)$ too. Therefore, if $u(c)$ is a positive-sign or negative-sign utility function (i.e., $u^n(c)$ is mono sign) the SMSD condition holds for the whole utility function, and thus becomes the MSD condition:

$$\text{sgn}(\rho) \cdot \text{sgn}(u(c)) = \overrightarrow{TP}. \quad \blacksquare$$

Composite Mono-Sign Utility Function

If a consumption path $c_0(t)$ is given and the utility function $u(c)$ in a DU optimization model is mono-sign, then the sign of the composite mono-sign utility function $u(c_0(t))$ will be the same as the sign of $u(c)$.

Proof: If $u(c)$ is positive-sign its natural form $u^n(c)$ yields positive sign every time a positive argument is supplied. And since consumption in every time period must be positive, i.e. $c_0(t) > 0, \forall t > 0$, the natural form of composite utility function, $u^n(c_0(t))$, always yields positive sign, and thus $u(c_0(t))$ is a positive-sign utility function too. The same argument is also true for negative-sign utility function. So, if $u(c)$ is mono-sign then

$$\text{sgn}(u(c_0(t))) = \text{sgn}(u(c)). \quad \blacksquare$$

Composite Mixed-Sign Utility Function

If a consumption path $c_0(t)$ is given and the utility function $u(c)$ in a DU optimization model is mixed-sign, then the sign of the composite mixed-sign utility function $u(c_0(t))$ may or may not be mixed-sign.

Proof: If $u(c)$ is mixed-sign its natural form yields positive sign in some ranges and negative sign in other ranges every time a positive argument is supplied. Suppose $u^n(c)$ yields positive sign when $c \in [a, b]$ and yields negative sign when $c \in (e, f)$, given that a, b, e, f are some positive numbers such that the two ranges are not overlapping. Since $c_0(t) > 0, \forall t > 0$, and it is possible that $c_0(t) \in [a, b], \forall t > 0$ as well, in which case $u^n(c_0(t))$ will yield only positive sign and thus $u(c_0(t))$ becomes a positive-sign utility function. Similar argument can be made for the negative-sign and mixed-sign cases. So, if $u(c)$ is mixed-sign

$$\text{sgn}(u(c_0(t))) = \begin{cases} + & \text{if } u^n(c_0(t)) \geq 0 & \forall t \in \mathbb{R}_+ \\ - & \text{if } u^n(c_0(t)) \leq 0 & \forall t \in \mathbb{R}_+ \\ \text{mixed} & \text{if } u^n(c_0(t)) \cdot u^n(c_0(t')) < 0 & \exists t, t' \in \mathbb{R}_+ \end{cases}$$

■

Example Suppose a mixed-sign utility function is $\ln c$, and given three consumption paths: $c_1(t) = e^{gt}$, $c_2(t) = e^{-gt}$, and $c_3(t) = t$, where $t, g \in \mathbb{R}_+$. What are the signs of the three composite utility functions?

The three composite utility functions are $\ln e^{gt}$, $\ln e^{-gt}$, and $\ln t$. Thus, their signs are positive, negative, and mixed respectively.

A.2 Discount Amount of Two Types of Utility Function

This section will show that a positive-number utility function has the discount amount that increases with the consumption c , but a negative-number utility function has the discount amount that decreases with the consumption c .

If $u(c)$ is a positive-number utility function then $u(c) \geq 0$ and $u'(c) > 0$ for all $c \in \mathbb{R}_+$.

If $u(c)$ is a negative-number utility function then $u(c) \leq 0$ and $u'(c) > 0$ for all $c \in \mathbb{R}_+$.

Case 1: $u(c)$ is a positive-number utility function and $\rho > 0$,

$$\begin{aligned} \text{amt}(DC(c)) &= \left| \frac{\rho}{1+\rho} u(c) \right| = \frac{\rho}{1+\rho} u(c) \\ \frac{d}{dc} \text{amt}(DC(c)) &= \frac{\rho}{1+\rho} u'(c) > 0 \end{aligned}$$

Case 2: $u(c)$ is a negative-number utility function and $-1 < \rho < 0$,

$$\begin{aligned} \text{amt}(DC(c)) &= \left| \frac{\rho}{1+\rho} u(c) \right| = -\frac{\rho}{1+\rho} u(c) \\ \frac{d}{dc} \text{amt}(DC(c)) &= -\frac{\rho}{1+\rho} u'(c) > 0 \end{aligned}$$

Case 3: $u(c)$ is a negative-number utility function and $\rho > 0$,

$$\begin{aligned} amt(DC(c)) &= \left| \frac{\rho}{1+\rho} u(c) \right| = -\frac{\rho}{1+\rho} u(c) \\ \frac{d}{dc} amt(DC(c)) &= -\frac{\rho}{1+\rho} u'(c) < 0 \end{aligned}$$

Case 4: $u(c)$ is a negative-number utility function and $-1 < \rho < 0$,

$$\begin{aligned} amt(DC(c)) &= \left| \frac{\rho}{1+\rho} u(c) \right| = \frac{\rho}{1+\rho} u(c) \\ \frac{d}{dc} amt(DC(c)) &= \frac{\rho}{1+\rho} u'(c) < 0 \end{aligned} \quad \blacksquare$$

A.3 Two-Period Representation of N-Period DU Optimization Model

This section will show that a two-period DU optimization model can be considered as a part of an N-period DU optimization model. At first, consider the two problems in isolation:

(1) Consider a DU optimization problem, in which a representative consumer lives for $N = n+1 > 1$ periods and his lifetime endowment is \mathbb{E} . Suppose the consumer has a CRRA utility function and his per period discount rate is constant at ρ .

$$\begin{aligned} \max_{c_0, \dots, c_n} \sum_{t=0}^n \frac{1}{(1+\rho)^t} \frac{c_t^{1-\theta}}{1-\theta} \quad & s. t. \quad \sum_{t=0}^n c_t = \mathbb{E}. \\ \mathcal{L} &= \sum_{t=0}^n \frac{1}{(1+\rho)^t} \frac{c_t^{1-\theta}}{1-\theta} + \mu \left[\mathbb{E} - \sum_{t=0}^n c_t \right] \end{aligned}$$

By FOC, the solutions are

$$c_i^* = \frac{(1+\rho)^{\frac{n-i}{\theta}} \mathbb{E}}{\sum_{t=0}^n (1+\rho)^{\frac{t}{\theta}}}, \quad c_{i+1}^* = \frac{(1+\rho)^{\frac{n-i-1}{\theta}} \mathbb{E}}{\sum_{t=0}^n (1+\rho)^{\frac{t}{\theta}}}, \quad , i = 0, 1, \dots, n-1.$$

(2) Consider another DU optimization problem, in which a representative consumer lives for two periods and his lifetime endowment is E . The consumer has a CRRA utility function and his per period discount rate is constant at ρ .

$$\begin{aligned} \max_{c_A, c_B} \left\{ \frac{c_A^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{c_B^{1-\theta}}{1-\theta} \right\} \quad & s. t. \quad c_A + c_B = E \\ \mathcal{L} &= \frac{c_A^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{c_B^{1-\theta}}{1-\theta} + \mu [E - c_A - c_B] \end{aligned}$$

By FOC, the solutions are

$$c_A^* = \frac{(1+\rho)^{\frac{1}{\theta}} E}{1 + (1+\rho)^{\frac{1}{\theta}}}, \quad c_B^* = \frac{E}{1 + (1+\rho)^{\frac{1}{\theta}}}$$

Next, to show that the solutions of the second problem can be viewed as a part of the solutions of the first problem, we set the endowment of the second problem equal to the combined consumption of any two adjacent periods i and $i + 1$ in the first problem.

$$E = c_i^* + c_{i+1}^* = \frac{\left[(1+\rho)^{\frac{n-i}{\theta}} + (1+\rho)^{\frac{n-i-1}{\theta}} \right] \mathbb{E}}{\sum_{t=0}^n (1+\rho)^{\frac{t}{\theta}}}$$

By doing so the second problem will yield the solutions that equal to that of period i and $i + 1$ of the first problem:

$$c_A^* = \frac{(1+\rho)^{\frac{1}{\theta}}}{1+(1+\rho)^{\frac{1}{\theta}}} \cdot \frac{\left[(1+\rho)^{\frac{n-i}{\theta}} + (1+\rho)^{\frac{n-i-1}{\theta}} \right] \mathbb{E}}{\sum_{t=0}^n (1+\rho)^{\frac{t}{\theta}}} = \frac{(1+\rho)^{\frac{n-i}{\theta}} \mathbb{E}}{\sum_{t=0}^n (1+\rho)^{\frac{t}{\theta}}} = c_i^*,$$

$$c_B^* = \frac{1}{1+(1+\rho)^{\frac{1}{\theta}}} \cdot \frac{\left[(1+\rho)^{\frac{n-i}{\theta}} + (1+\rho)^{\frac{n-i-1}{\theta}} \right] \mathbb{E}}{\sum_{t=0}^n (1+\rho)^{\frac{t}{\theta}}} = \frac{(1+\rho)^{\frac{n-i-1}{\theta}} \mathbb{E}}{\sum_{t=0}^n (1+\rho)^{\frac{t}{\theta}}} = c_{i+1}^*. \quad \blacksquare$$

A.4 Sources of Some Stylized Facts

The four less-known stylized facts r , s^* , β , and K/Y are drawn from various sources:

(1) U.S. rate of return on capital (r) for period 1889-1978 (from Mehra and Prescott, 1985) and for period 1950-98 (from Evans, 2000) are 0.0698 and 0.09 respectively. Barro and Sala-i-Martin (1992) and (1995) uses $r = 0.07$ and 0.055 respectively. Whereas Lucas (2003) uses after-tax return on capital = 0.05. Since the RCK model assumes no government there should be no tax in the economy, thus the before-tax figure corresponding to Lucas' 0.05 is about 0.07. So, the return on capital from these five sources is in the range of 0.055-0.09.

(2) U.S. investment rate (s^*) for period 1960-90 (from Barro and Sala-i-Martin, 1995), period 1959-96 (from Barro, 1997), and period 1959-1996 (from Evans, 2000) are 0.21, 0.203, and 0.14 respectively. So the average of the three sources is $s^* \approx 0.185$.

(3) Barro and Sala-i-Martin (1995, 2004) suggested that, for the U.S., a convergence coefficient β in the range of 1.5-3.0 percent per year fit better with the data.

(4) U.S. capital to output ratio (K/Y) before and after the great depression are about 3.3 and 2.1 respectively. The figures are from Evans (2000) and D'Adda and Scorcu (2003). So this paper uses $2.1 < K/Y < 3.3$.

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