Why is Fertility Rising in Developed Economies?  
The Dynamics of Skill Composition,  
Fertility and Economic Growth  

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Abstract

This paper develops an overlapping generations model that incorporates choice of occupation (education), fertility and how to rear children. We examine the dynamic interplay between occupational structure, economic growth and fertility as an economy moves through two phases distinguished by the skill composition of the workforce. The model exhibits the possibility of multiple equilibria and dynamic behaviour in the second phase that is consistent with a unit elastic version of Diamond (1965). The presence of multiple equilibria explains the observation that while fertility has decreased with per capita income in some countries, per capita income remains low and fertility high in others. By introducing child rearing goods and services, as an alternative to parental time, we explain the recent fertility upturn witnessed in some developed economies.

1 Introduction

After several decades of fertility decline, developed economies are experiencing an upturn in fertility. Because this upturn is a very recent phenomenon, the existing theoretical literature focuses on explaining a negative relationship between fertility and economic growth. This paper develops a comprehensive model that provides a consistent and cohesive explanation of both high fertility in low income economies and the downward trend in fertility and its recent reversal in high income economies.

The model in this paper explores the following facts regarding the relationship among education, fertility and economic growth:

- High fertility is observed in the most impoverished economies, with the lowest education levels.
Overall fertility in developed economies has been declining until recently.

Within most countries, on average, better educated parents have less children than low skilled parents.

However, the fertility of better educated parents may rise, bringing about an upturn in fertility in some developed countries.

Referring to Figure 1, notwithstanding differences in family policies, several high income economies are experiencing an upturn in fertility. This recent phenomenon is pervasive in the developed world (Goldstein et al, 2009). There is emerging evidence of a reverse J-shaped relationship between fertility and economic development (Myrskyla et al, 2009; Luci and Thevenon, 2010).


A notable exception, Greenwood et al (2005) show that rapid technological progress in home production generates a Baby Boom to Bust sequence, witnessed post World War II in developed economies. Their model does not allow for a possible upturn in fertility, once an economy enters the Baby-Bust phase. Day (2004) shows that the dramatic rise in female labour force participation generates a Baby Boom, Bust and Bounce-Back sequence. Despite predicting non-monotonicity, both papers overlook the rising educational attainment of parents which characterises the recent fertility upturn. This is what this paper sets out to explore.

We hypothesise that an explanation for nuances in the relationship between overall fertility, education and income per capita, over time and across countries, lies in the combined effects of the differential fertility of high and low skilled workers and the opportunity cost of time spent rearing children. Amongst individuals, there is a evidence of a negative relationship between parental education and fertility (Beenstock, 2007). On average, highly educated individuals in high paying occupations have less children than individuals in low paying occupations, reflected in evidence of a positive link between income inequality and fertility (Guest and Swift, 2008).
As their relative wages increase over time, skilled workers find children more affordable, but more costly to rear using unpaid time. Cette et al (2007) find evidence corroborating the simultaneous and opposing effects of these two effects on fertility.

A priori, as wages increase, we might expect highly educated parents to choose to have more children if the opportunity cost of rearing children is mitigated by, for instance, family friendly policies and the substitution of child rearing goods and services for parental time. There is some evidence that child rearing goods and services have indeed mitigated the rising opportunity cost in developed economies. In the United States, child rearing is a relatively goods intensive activity (Gronau and Hamermesh, 2006).

Kimura and Yasui (2007) focus on the interrelationship between economic growth, parents’ investment in their own education and fertility. Like the early literature, they predict that fertility and economic growth are negatively interrelated, but we find the model of individuals’ choice to become a skilled worker or to remain unskilled, at its core, to be a useful starting point for the analysis of this paper. Apps and Rees (2004) allow parents to minimise the cost of child rearing by switching to goods and services, but confine their analysis to a static model of fertility.

In this paper, we incorporate household choice of education, fertility and child rearing within a model of economic growth to explore possible non-monotonicity in the dynamic path of fertility at various stages of economic development.

2 Basic structure of the model

We consider an overlapping generations model in which people live for three periods. In childhood, an agent consumes time, as well as goods and services, from their parents. In adulthood, an agent supplies labor, raises children, and may receive further education. The agent decides whether or not to undertake training to become a skilled worker, how many children to have and how to rear them. In old age, an agent is retired from the labor force and consumes the proceeds of their savings from the previous period.

The closed economy identity of savings and investment provides the link with growth in the capital stock, productivity and wages, which in turn influence fertility. Rising skilled wages are a consequence of economic growth. Skilled labor is complementary to physical capital. Combined with the neoclassical capital intensity effect, this final feature generates a feedback loop between growth in output per worker and fertility.
Taking the lead of Kimura and Yasui (2007) and Chen (2010), we model the economy as it moves through two phases, distinguished by the skill composition of the labour force.\footnote{Like Chen (2010), although our model is based on Kimura and Yasui (2007), the focus of this paper differs from theirs. Kimura and Yasui (2007) are concerned with the negative relationship between fertility and education. Chen (2010) explores the effect of longevity on education. We extend the Kimura and Yasui (2007) by incorporating household choice of child rearing inputs in order to explore possible non-monotonic behaviour of fertility and, as we will see, our modifications alter the dynamic analysis. For instance, the law of motion of capital per household will depend on the per unit cost of child rearing.} We set out to explore the behaviour of fertility, and its implications for growth, during an initial phase when the labour force comprises skilled and unskilled workers, before entering a second phase when every agent chooses to become skilled.

\subsection*{2.1 Production of final output}

Physical capital ($K$), unskilled labor ($L_u$) and skilled labor ($L_s$) are factors of production, all with non-increasing marginal products. The greater the capital-labor ratio in the economy, the more highly rewarded is skilled labor relative to unskilled labor. This is consistent with the relative rise in rewards to skilled labor characterizing economic development.

The production function is given by

\[ Y_t = A \left[ K_t^\alpha (L_s^t)^{1-\alpha} + bL_u^t \right]; \quad A > 0; \quad b > 0; \quad \alpha \in (0, 1) \quad (1) \]

where the separability captures the assumption that, whereas capital complements skilled labor, unskilled labor is a perfect substitute for other factors of production.\footnote{Galor and Weil (1996) first introduced such a production function to an endogenous fertility model to capture the rise in female relative wages which characterised the rise in services and decline in manufacturing.}

Each adult (household) is endowed with a unit of time, which can be allocated to child rearing, labor force participation and education (training). Each agent chooses whether to become a skilled worker or an unskilled worker. To supply skilled labor, an agent must spend a fraction of their time endowment, $\tau \in (0, 1)$, to acquire higher education.

Each agent chooses the number of children, $n_i^t (i = s, u)$. To raise one child, an agent purchases goods and services and employs a fraction of their time endowment, denoted $\hat{x}$ and $\hat{z}$, respectively.

Let $N_t$ be the number of working age agents and $\varphi$ be the ratio of skilled workers to all workers. The aggregate supply of skilled labor and unskilled
labor at time $t$ is $L_t^s = (1 - \tau - \tilde{z}n_t^s) \varphi N_t$ and $L_t^u = (1 - \tilde{z}n_t^u) (1 - \varphi) N_t$, respectively.

Perfectly competitive factor markets imply

$$w_t^s = \frac{\partial Y_t}{\partial L_t^s} = A(1 - \alpha) [k_t/h_t \varphi]^\alpha$$

$$w_t^u = Ab$$

where $k_t \equiv K_t/N_t$ and $h_t \equiv L_t^s/\varphi N_t$ denote physical capital per working age agent and skilled labor per skilled working age agent, respectively. Ceteris paribus, an increase in capital intensity will therefore raise the wage\(^3\) for skilled labor ($w_t^s$) while the wage for unskilled labor ($w_t^u$) is constant.

### 2.2 Household optimisation

Each agent (household) derives utility directly from the number of children. Skilled and unskilled agents are assumed to have the same preferences. The household utility function\(^4\) is

$$u_t = \gamma \ln(n_t) + (1 - \gamma) \ln(c_{t+1}) ; \gamma \in (0, 1)$$

where $c_{t+1}$ is consumption in retirement and $n_t$ denotes pairs of children (since the couple is the basic unit of analysis), both chosen by the household at time $t$.

In this paper, we want to explore the effects of the availability of bought in child care, as an alternative input to parental time. To incorporate this into the analysis, we assume a general form “production function” for child rearing, which implicitly defines the combinations of parental time and goods and services required to raise $n_t$.

$$n_t = f(z_t, x_t)$$

where $x$ and $z$ denote total child rearing goods and services and total time input, respectively. The production is homogeneous of degree one, continuously differentiable and concave.

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\(^3\) This is the real wage, with the price of the aggregate good normalized to 1.

\(^4\) This simple preference structure appears in Galor and Weil (1996) and, more recently, Kimura and Yasui (2007). Consumption in the second period of life is assumed to be zero. Galor and Weil (1996) note that if couples had log utility from $c_t$, as analysed by Chen (2010), the equation of motion would be altered only by a multiplicative constant.

\(^5\) For the moment we specify a general functional form, as per Apps and Rees (2004), who introduced such a function a model of static fertility. In the vein of Day (2004), we incorporate child rearing production to the analysis of the dynamics of fertility and economic growth.
In the special case where child rearing uses only a fixed fraction of parental time per child, the household optimisation problem is confined to maximizing utility subject to a budget constraint.

In the general case, where child rearing requires not only parental time but also goods and services, because the child-rearing production function is homogeneous of degree one, the household optimization problem can be solved in two stages. The household first chooses, for a given $n_t$, the cost minimizing input mix and then chooses $n_t$, given the efficient input mix, so as to maximize utility subject to a budget constraint.

**Cost minimisation**
Allowing for the possibility of a government subsidies per unit of bought in goods and services used, $\beta \in (0, 1)$, the total cost of rearing children for agent $i$ is

$$C^i_t = w^i_t z^i_t + (1 - \beta) x^i_t$$  \hspace{1cm} (6)

The household first chooses the input mix, for a given $n_t$, so as to minimize (6) subject to (5). Input demands for time and goods are, respectively,

$$z^i_t = \hat{z}(w^i_t, \beta) n^i_t$$  \hspace{1cm} (7a)

$$x^i_t = \hat{x}(w^i_t, \beta) n^i_t$$  \hspace{1cm} (7b)

The per unit cost function is

$$p(w^i_t, \beta)$$  \hspace{1cm} (8)

which is increasing and concave in the input prices, where $\partial p(.) / \partial w^i_t = \hat{z}$.

**Utility maximization**
Each agent faces the second period budget constraint

$$c_{t+1} = s_t (1 + r_{t+1})(1 - T_{t+1})$$  \hspace{1cm} (9)

where $r_{t+1}$ denotes the rate of return on savings, $s_t$, $T_{t+1}$ denotes the rate of taxation on old age consumption.

A skilled worker faces the first period budget constraint

$$p(w^s_t, \beta)n_t + s_t \leq w^s_t (1 - \tau)$$  \hspace{1cm} (10)

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6 The price of goods and services is normalized to 1. By implication, the goods and services component of child rearing costs is denominated in terms of goods (final output). The subsidies are financed by a tax on old age consumption (see second period budget constraint below).

7 The rate of subsidy and taxation are set so as to satisfy the balanced-budget constraint: $\beta_t \hat{x} n_t [n_{t-1} L_{t-1}] = s_{t-1} (1 + r_{t}) T_t L_{t-1}$. Although endogenous at the aggregate level, the rate of subsidy and taxation is treated as exogenous by each individual household.
Maximizing (4) with respect to (9) and (10), it follows that the number of children and the savings of a skilled worker are

\[
\begin{align*}
n_t^s &= \frac{\gamma w_t^s}{p(w_t^s, \beta)} (1 - \tau) \\
s_t^s &= (1 - \gamma) (1 - \tau) w_t^s
\end{align*}
\] (11a) (11b)

An unskilled worker faces the first period budget constraint

\[ p(w_t^u, \beta)n_t + s_t \leq w_t^u \] (12)

Maximizing (4) with respect to (9) and (12), it follows that the number of children and the savings of an unskilled worker are

\[
\begin{align*}
n_t^u &= \frac{\gamma w_t^u}{p(w_t^u, \beta)} \\
s_t^u &= (1 - \gamma) w_t^u
\end{align*}
\] (13a) (13b)

(11a) and (13a) imply that a skilled worker has fewer children than an unskilled worker if the per unit cost of child rearing as a proportion of income is higher for skilled workers. Intuitively, on the one hand, to become a skilled worker, an agent must spend a fraction of their time training, and therefore has less time to rear children. On the other hand, higher wages make children more affordable for a skilled worker. For the purposes of this paper, we assume that the per unit cost of child rearing as a proportion of income is higher for skilled workers,\(^8\) so that the former effect dominates and fertility of a skilled worker is less than fertility of an unskilled worker.

For the special case where time is the sole child rearing input, both skilled and unskilled workers face the same per unit cost of child rearing as a proportion of wages. Substituting \(p(w_t^i) = w_t^i \bar{z}\) into (11a) and (13a),

\[
n_t^s = \frac{\gamma (1 - \tau)}{\bar{z}}; n_t^u = \frac{\gamma}{\bar{z}}
\] (14)

where \(\tau \in (0, 1)\) implies a skilled worker must have fewer children than an unskilled worker.\(^9\)

We summarize this discussion with the following

\(^8\)This assumption holds for plausible values of a skilled wage premium. To illustrate, consider a Cobb-Douglas child rearing production function: \(w_t^s < (1 - \tau)^{-1/(1 - a)} w_t^u \Rightarrow n_t^s < n_t^u\). When \(a = \tau = 0.5\), skilled workers have lower fertility than unskilled workers provided the average skilled wage rate is less than four times the average unskilled wage rate.

\(^9\)Together, (11a), (13a) and (14) imply the introduction of child rearing goods and services reduces the difference in fertility across skilled and unskilled workers.
Remark 1 A skilled worker has fewer children than an unskilled worker.

Since agents choose to become skilled or to remain unskilled, agents are indifferent between becoming skilled and remaining unskilled in an equilibrium where both types of workers exist. Equating the indirect utility of both types of workers yields the arbitrage condition

\[ \frac{w^u_t}{w^s_t} = \left[ \frac{p(w^u_t, \beta)}{p(w^s_t, \beta)} \right]^{\gamma} (1 - \tau) \]  

(15)

The equilibrium wage rate of a skilled worker is higher than that of an unskilled worker, provided an unskilled worker's relative child rearing costs as a share of income are sufficiently small:

\[ w^s_t > w^u_t \iff \frac{p(w^u_t, \beta)}{p(w^s_t, \beta)} > (1 - \tau)^{1/\gamma} \]  

(16)

where, referring to the appendix, \( n^s_t < n^u_t \) implies that the right hand side inequality condition holds.

Intuitively, each agent derives utility from children and old age consumption. A skilled worker has lower fertility than an unskilled worker. Therefore, in order to be indifferent between becoming skilled or remaining unskilled, a skilled worker must earn a higher wage to fund higher old age consumption than their unskilled counterpart.

We may also analyse the condition for a skilled wage premium in terms of the relative costs of child rearing. The inequality condition for a skilled wage premium is satisfied when, as Kimura and Yasui (2007) assume, time is the only child rearing input and unskilled and skilled workers use an identical time input, because the left hand side of the inequality is necessarily unity.

When bought in services are also a child rearing input, the inequality condition for a skilled wage premium is necessarily satisfied when an unskilled worker’s child rearing costs as a share of income are less than or equal to a skilled worker’s child rearing costs as a share of income. The inequality condition may still be satisfied, even if the child rearing costs as a share of income for an unskilled worker are greater than that for a skilled worker.

We summarize this discussion with the following

Remark 2 The equilibrium wage rate of a skilled worker is higher than that of an unskilled worker.
2.3 Dynamic system

Capital stock per working age agent fuels growth in this model. The capital stock in each period is determined by the aggregate savings in the previous period

\[ K_{t+1} = s_t N_t \]

\[ = \varphi_t s_t^h + (1 - \varphi_t) s_t^u \]

The number of working age households at time \( t + 1 \) is

\[ N_{t+1} = f_t N_t \]

where the total fertility rate at time \( t \), is

\[ f_t = \varphi_t n_t^h + (1 - \varphi_t) n_t^u \]

Capital stock per household is therefore given by

\[ k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{s_t}{f_t} \]

From (19), an equation of motion \( k_{t+1} = \phi(k_t) \) is obtained, since overall savings and fertility is determined by \( \varphi_t \), which we will see is a function of \( k_t \) during an initial phase, and savings and fertility of a skilled worker are determined by \( w^s_t \), which in turn is a function of \( k_t \) during a second phase.

Capital per household evolves through two phases distinguished by skill composition of the workforce. In the initial phase, the workforce comprises unskilled and skilled workers. With rising capital intensity, the fraction of skilled workers increases. Once capital per household reaches a sufficiently high level, all agents choose to become skilled.

The evolution of capital stock per household is governed by two distinct equations of motion, as the economy moves first through

- Phase 1: workforce comprising unskilled and skilled workers; and, then
- Phase 2: skilled workforce.

The economy enters Phase 2 once a sufficiently high level of capital per working age household has been accumulated, denoted by \( \bar{k} \). That is, from (11a), (13a) and (18)

\[ f_t = \begin{cases} 
\gamma w^s_t (1 - \tau) / p(w^s_t, \beta) & \text{if } k_t \geq \bar{k} \\
\frac{\varphi_t w^s_t (1 - \tau)}{p(w^s_t, \beta)} + (1 - \varphi_t) \frac{Ab}{p(\bar{a} \bar{b}, \beta)} & \text{if } k_t < \bar{k}
\end{cases} \]
where overall fertility is decreasing in the fraction of skilled workers and increasing in the skilled wage rate during Phase 1 and Phase 2, respectively.

**Skilled wages**

It is straightforward that, all else equal, the skilled wage rate is increasing in $k_t$, because capital complements skilled labour. However, derivation of the equation of motion is complicated by the fact that the skilled wage rate and time spent raising children are interdependent. By (2), $w_t^s$ is a function of $h_t$, which in turn is a function of $\hat{n}_t$. That is, the wage paid to skilled labour reflects its marginal product, affected by the supply of skilled labor, which in turn, for a given fraction of skilled workers, $\varphi_t$, depends on the time spent raising children. By (7a) and (8), $\hat{z}$ is a decreasing function of $w_t^s$: a skilled worker spends less (substitutes out) of their time raising each child as the skilled wage rate rises. Thus, we need obtain an implicit function for $\hat{z}_n_t$.

Substituting from (2), (11a) and $h_t = 1 - \tau - \hat{z}_n_t^s$ into (7a) where $i = s$,

$$\hat{z}_n_t = f(\hat{n}_t, k_t)$$

Let $G(\hat{z}_n_t, k_t) = \hat{z}_n_t - g(\hat{z}_n_t, k_t) = 0$. Since $G(\hat{z}_n_t, k_t) = 0$ has continuous derivatives, by the Implicit Function Theorem, if $G_{\hat{z}_n_t} \neq 0$ then there is a differentiable and invertible function $\varphi(k_t)$ such that

$$\hat{z}_n_t = \pi(k_t)$$

where

$$\pi'(k_t) = \frac{-G_{k_t}}{G_{\hat{z}_n_t}} = \frac{\partial g/\partial w_t^s \cdot \partial w_t^s / \partial k_t}{[1 - \partial g/\partial w_t^s \cdot \partial w_t^s / \partial h_t \cdot \partial h_t / \partial \hat{z}_n_t]} \leq 0 \text{ if } \partial g/\partial w_t^s \leq 0 \quad (21)$$

With the exception of $\partial g/\partial w_t^s$, the signs of the partial derivatives in (21) are unambiguous. Assigning a negative value to $\partial g/\partial w_t^s$ is tantamount to assuming that the skilled labor supply curve is never backward bending. The possibility of a backward bending supply curve arises in this model, since fertility rises with skilled wages in Phase 2. A backward bending labor supply curve would occur if the proportionate rise in the number of children exceeds the proportionate fall in the time input per child. For the purposes of this paper, $\partial g/\partial w_t^s \leq 0$ is assumed.\textsuperscript{10}

We summarize this with the following\textsuperscript{11}

\textsuperscript{10}A backward bending skilled labor supply, although an interesting proposition in itself, adds an unnecessary layer of complication to the model.

\textsuperscript{11}This occurs in Phase 2 when all workers are skilled. In Phase 1, when the workforce comprises skilled and unskilled agents, capital accumulation raises the fraction of skilled workers, but the equilibrium skilled wage is unchanged, as analysed by Kimura and Yasui (2007). Accordingly, a skilled worker alters neither fertility nor time spent in child rearing.
Remark 3 As skilled wages increase, a skilled worker has more children, but total time spent raising children is non-increasing in skilled wages.

Fraction of skilled workers

We now analyse how the fraction of skilled workers depends on \( k_t \). It follows from (2), (3), (21) and (15) that the fraction of skilled workers, \( \varphi \), is a function of \( k_t \), as implied by the following expression for Phase 1

\[
\frac{b}{(1-\alpha)(1 - \tau - \pi(k_t))}\varphi^\alpha = \frac{p(w_t^s, \beta)}{p(w_t^u, \beta)}(1-\tau)
\]

(22)

For general functional forms, we establish that fraction of skilled workers is increasing in \( k_t \) in Phase 1.\(^{12}\) The fraction of skilled workers and the skilled wage rate are interdependent in Phase 1. Referring to (22), the fraction of skilled workers is a function of the skilled wage rate, which in turn is a function of \( k_t \). By (2), the skilled wage rate reflects the marginal product, which in turn depends on the supply of skilled workers. Substituting from (2) and denoting \( l_t^s \equiv h_t\varphi \), we obtain \( \varphi = j(\varphi_t, k_t) \). Let \( J(\varphi_t, k_t) = \varphi_t - j(\varphi_t, k_t) = 0 \). By the Implicit Function Theorem, there is a function such that \( \varphi_t = \varphi(k_t) \) where

\[
\varphi'(k_t) = \frac{-J_{k_t}}{J_{\varphi_t}} = \frac{\partial j/\partial w_t^s \partial w_t^s/\partial k_t}{1 - \partial j/\partial w_t^s \partial w_t^s/\partial l_t^s \partial l_t^s/\partial \varphi_t} > 0
\]

(23)

where the signs of the partial derivatives are unambiguous. Thus, the fraction of skilled workers is increasing in \( k_t \) in Phase 1.

Specific functional form for child rearing

For tractability, we analyse the case where the child rearing function is of Cobb-Douglas form

\[
n_t = (z_t)^a(x_t)^{1-a}; \quad a \in (0,1)
\]

In this case, as the skilled wage rate rises, a skilled worker substitutes to bought in child care in Phase 2, reducing time per child, but not total time spent raising children (i.e. \( \pi'(k_t) = 0 \)). It follows from (22), substituting from the appendix for \( \pi \equiv \hat{z}n^s \) and \( p(w_t^s, \beta) \), that the fraction of skilled workers, \( \varphi_t \), is expressed as a function of \( k_t \):

\[
\varphi(k_t) = \frac{(1-\tau)^{a(1-a)\gamma^{-1}}}{(1-a\gamma)} \left( \frac{1-\alpha}{b} \right)^{1/\alpha} k_t \equiv \theta k_t
\]

(24)

\(^{12}\)The fraction of skilled workers has an upper bound of 1. Once this is met, the economy enters Phase 2.
The fraction of skilled workers is

\[
\varphi_t = \begin{cases} 
1 & \text{if } k_t \geq \bar{k} \\
\theta k_t & \text{if } k_t < \bar{k}
\end{cases}
\]  
(25)

where \( \bar{k} = \theta^{-1} \), yielding the following

**Remark 4** The fraction of skilled workers is non-decreasing in the capital stock per household, and strictly increasing in the capital stock per household for \( k_t < \bar{k} \).

### 3 Equation of motion

Substituting into (19) from (17) and (20), the equation of motion for the system is derived from

\[
k_{t+1} = \frac{(1 - \gamma)}{\gamma} \begin{cases} 
p(w_t^s, \beta) & \text{if } k_t \geq \bar{k} \\
\frac{\varphi_t w_t^s (1 - \tau) + (1 - \varphi_t) w_t}{\varphi_t w_t^s (1 - \tau) + (1 - \varphi_t) w_t / p(w_t^s, \beta)} & \text{if } k_t < \bar{k}
\end{cases}
\]  
(26)

Capital stock per couple evolves from a historically given initial level according to \( k_{t+1} = \phi(k_t) \), which can be derived from (26) by substituting \( \varphi_t \) from (25) and

\[
w_t^s = A(1 - \alpha) \begin{cases} 
[k_t / (1 - \tau - \pi(k_t))]^\alpha & \text{if } k_t \geq \bar{k} \\
[k_t / (1 - \tau - \pi(k_t)) \varphi(k_t)]^\alpha & \text{if } k_t < \bar{k}
\end{cases}
\]  
(27)

where, by diminishing marginal returns, \( w_t^s \) is an increasing, non-linear function of \( k_t \) during Phase 2. It follows that \( k_{t+1} \) is an increasing, non-linear function of \( k_t \) during Phase 2. Since the reduced form \( w_t^s \) is not a function of \( k_t \) during Phase 1\( ^{13} \), substituting for \( \varphi(k_t) \), from (25) or (23), as an implicit or explicit function of \( k_t \) respectively, in the second line of (26) yields \( k_{t+1} \) is an increasing, non-linear function of \( k_t \) in Phase 1. Thus, the equation of motion is a first order non-linear difference equation throughout both phases.

For child rearing of Cobb-Douglas form, the equation of motion is derived from

\[
k_{t+1} = B \frac{(1 - \gamma)}{\gamma} \begin{cases} 
(w_t^s)^\alpha (1 - \beta)^{1-\alpha} & \text{if } k_t \geq \bar{k} \\
\theta k_t w_t^s (1 - \tau) + (1 - \theta k_t) w_t^s & \text{if } k_t < \bar{k}
\end{cases}
\]  
(28)

\( ^{13} \)See (29) below.
where
\[
w_t^* = A(1 - \alpha) \begin{cases} 
  [k_t/(1 - a\gamma)(1 - \tau)]^{\alpha} & \text{if } k_t \geq \bar{k} \\
  [(1 - a\gamma)(1 - \tau)^{\alpha}]^{-\alpha} & \text{if } k_t < \bar{k}
\end{cases}
\] (29)

Referring to the Appendix, capital per household evolves over time according to
\[
k_{t+1} = \phi(k_t)
\]
\[
= A^a \tilde{B} \frac{(1 - \gamma)}{\gamma} \left\{ \begin{array}{ll}
  \left[\frac{(1-\alpha)}{(1-a\gamma)^{\alpha}(1-\tau)^{\alpha}}\right]^a k_t^{\alpha a} & \text{if } k_t \geq \bar{k} \\
  1+\theta \left\{ (1-\gamma)\left[\frac{-a\gamma}{(1-\gamma)(1-a\gamma^{\alpha})}\right] - 1 \right\} k_t^{1+\theta} \left\{ 1-\theta \left\{ (1-\gamma)\left[\frac{-a\gamma}{(1-\gamma)(1-a\gamma^{\alpha})}\right] - 1 \right\} k_t \right\} & \text{if } k_t < \bar{k}
\end{array} \right.
\] (30)

where \( \tilde{B} = \left[ \left( \frac{a}{1-a} \right)^{1-a} + \left( \frac{1-a}{a} \right)^a \right] (1 - \beta)^{1-a} \).

The following properties of the equation of motion (30) ensure existence of steady equilibrium (a)
\[
\phi(0) = (Ab)\tilde{B} \frac{(1 - \gamma)}{\gamma} > 0 \quad \text{and} \quad \lim_{k_t \to \infty} \phi'(k_t) = 0 \quad (31)
\]

Furthermore, the equation of motion has curvature properties which we summarise with the following

**Proposition 1** \( \phi(k_t) \) is increasing and convex in \( k_t \) over the interval \((0, \bar{k})\); increasing and concave over the interval \((\bar{k}, \infty)\).

**Proof.** Refering to the Appendix, the equation of motion has the following curvature properties
\[
\phi'(k_t) > 0 \quad \forall k_t < \infty
\]
\[
\phi''(k_t) < 0 \quad \text{if } k_t \geq \bar{k}
\]
\[
\phi''(k_t) > 0 \quad \text{if } k_t < \bar{k}
\]

Although we have obtained the reduced form equation of motion, \( k_{t+1} = \phi(k_t) \), viewing the equation of motion in terms of the structural equations in (19) and (26) is a useful way to intuit the curvature of the equation of motion. The curvature of the equation of motion follows from the dynamic behaviour of savings and fertility.
First, we consider Phase 1, where the economy comprises skilled and unskilled workers. If \( k_t < \tilde{k} \), the equilibrium skilled wage rate is constant, by (29). Intuitively, an agent chooses to spend a fraction of their time in higher education in order to become skilled. In an equilibrium where both skilled and unskilled workers exist, agents are indifferent between becoming skilled or remaining unskilled. Equating the indirect utility of workers yields a skilled wage premium that, referring to the appendix, is constant over time. Referring to (2), capital complements skilled labour, raising its marginal product. The supply of skilled workers increases, lowering its marginal product. The offsetting effect of a rising fraction of skilled workers maintains a constant skilled wage premium.

Thus, the dynamics of overall savings and fertility is solely determined by a rising fraction of skilled workers. By the simple linear relationship in (25), the higher \( k_t \), the larger the fraction of skilled workers. Since a skilled worker saves more and has less children than an unskilled worker, overall savings will rise and overall fertility will fall, as the number of skilled workers increases. From (19), increasing savings and decreasing fertility implies capital per household accumulates at an increasing rate. That is, \( \phi(k_t) \) is strictly convex.

Now, consider Phase 2, where all workers are skilled. If \( k_t \geq \tilde{k} \), the skilled wage rate is increasing in \( k_t \), by (29). Intuitively, capital complements skilled labour, raising its marginal product. Fertility and savings are, in turn, increasing in the skilled wage rate. Over time, both savings and fertility will rise. Proportionate to the increase in the efficient bought in child care input, fertility rises less than does savings, ensuring capital per household accumulates: \( \phi'(k_t) > 0 \). From (19), increasing savings and increasing fertility implies capital per household accumulates at an decreasing rate. That is, \( \phi(k_t) \) is strictly concave.

\( \phi(k_t) \) is increasing in \( A \), the initial state of technology. Figure 2 depicts the dynamic system in which there exists a unique steady state equilibrium, \( k^* \), comprising skilled workers and a stationary fertility rate, \( f^* \). Figure 3 describes a dynamic system characterised by multiple stable steady state equilibria.

**Proposition 2** If the initial state of technology is sufficiently low, the economy converges to a poverty trap, \( k^*_L \), or high education steady state, \( k^*_H \).

---

\( \text{From (26), } k_{t+1} \text{ will follow the path of the per unit child rearing cost function, } p(w^*_t, \beta) \) which, referring to the appendix, is increasing and concave in \( k_t \).
depending on $k_0$.\footnote{There is a third possibility, where $A$ is extremely low such that $\phi(k_0)$ intersects the 45° line once at $k^* < k_L^*$. In this case, a poverty trap is the only possible equilibrium.}

**Corollary 1 (to Proposition 2)** The poverty trap is characterised by high fertility due a high fraction of unskilled workers.

With an increasing fraction of skilled workers, capital accumulates at an accelerating rate. Initial conditions determine whether an economy converges to a high education state or a poverty trap. If $k_0 > \bar{k}$ such that $\phi(k_0) > k_0$, capital accumulates at an accelerating rate with the rise in fraction of skilled workers up until $\tilde{t}$. Fertility undergoes the baby bust to bounce-back sequence described in the previous Corollary. However, if $k_0 < \bar{k}$ such that $\phi(k_0) < k_0$, the economy converges to a poverty trap, where the low $k_L^*$ determines a low $\varphi_L^*$. The high fraction of unskilled workers implies high overall fertility, since unskilled workers have more children.

Referring again to Figure 2, the high $k^*$ equilibrium describes a developed or high income economy. We explore the behaviour of fertility in convergence to this equilibrium.

**Proposition 3** If the initial state of technology is sufficiently high, the economy converges to a high education steady state, $k^*$.

**Corollary 2 (to Proposition 3)** In transition to this steady state, overall fertility decreases over the interval $(0, \bar{k})$ and increases over the interval $(\bar{k}, k^*)$. Skilled workers bring about the baby bounce-back.

Referring to Figure 4, up until $\tilde{t}$, the fraction of skilled workers rises with the accumulation of capital per household. Skilled workers have less children than unskilled workers. Up until $\tilde{t}$, increasing weight given to the relatively low fertility of skilled workers reduces the overall fertility rate.

At time $\tilde{t}$, a sufficiently high level of capital per household has been accumulated such that the workforce is skilled. Overall fertility is given by the fertility of skilled workers. From $\tilde{t}$, the skilled wage rate rises with the accumulation of capital per household. As skilled wages rises, the opportunity cost of raising children rises, but so to does income. Substitution of bought in child care for parental time mitigates some of the rising opportunity cost for skilled workers, so that the income effect dominates substitution effect and a skilled worker choose to have more children as the skilled wage rate rises. After $\tilde{t}$, the overall fertility rate until it converges to the stationary rate, $f^*$. 

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4 Implications

For contrast, Figure 5 depicts the limiting case, $a \rightarrow 1$, when parental time is the sole child rearing input. In this special case, fertility stays at a constant, low level once the workforce is skilled and the economy begins to converge to a high income steady state. By implication, the lack of an alternative to parental time in child rearing may explain why some high income economies, such as Japan\textsuperscript{16} and Italy, have experienced a more prolonged period of low fertility. This is consistent with evidence that child care is particularly intensive in maternal time in Japan and Italy (Feyrer et al, 2008).

In our model, substitution to child rearing goods and services depends on policy (the rate of subsidy, $\beta$)\textsuperscript{17} and cultural factors (captured by $a$, the share of parental time in child rearing "production").

Allowing for the substitution of child rearing goods and services for parental time has striking implications for the path of fertility. However, the accelerating growth in output per capita associated with a rise in the fraction of skilled workers is a feature common to the model with and without an alternative to parental time. This feature enables multiple equilibria. Thus, our model predicts a rich array of fertility differences within and across low and high income countries, which we now summarise:

1. Within an economy, skilled workers have less children than unskilled workers.

2. Depending on the availability of a child rearing alternative to parental time, fertility of skilled workers increases with skilled wages.

3. If the initial technology is sufficiently low, an economy may converge to a poverty trap, with a high fraction of unskilled workers and high fertility.

4. As an economy transits through developing to developed, the fraction of skilled workers increases and overall fertility follows a non-monotonic path. Fertility initially declines. A high income economy with a skilled workforce may experience a fertility upturn.


\textsuperscript{17}In terms of further research, the model could be extended to analyse a range of family policies, such as paid parental leave and child payments.
5 Conclusion

By incorporating child rearing choice, the model presented in this paper yields a rich dynamic interplay between fertility and income per capita, as an economy moves through two phases distinguished by the skill composition of the workforce. In the initial phase, the fraction of skilled workers rises with capital accumulation. Because a skilled worker has less children than an unskilled worker, as the fraction of skilled workers rises, the overall fertility rate declines. With rising savings and declining fertility, capital per household accelerates, enabling multiple equilibria.

In the final phase, the availability of child rearing goods and services engenders a dramatically different path in growth and fertility for a high income economy with a skilled workforce. As the skilled wage rises, the opportunity cost of using parental time to rear children rises but so does household income. The use of child rearing goods and services limits the opportunity cost of child rearing, so that the income effect dominates the substitution effect and the fertility of skilled workers begins to rise. With rising savings and rising fertility, capital per household accumulates at a decelerating rate and we obtain convergence to a steady state equilibrium, as in the unit elastic version of Diamond (1965).

Our analysis identifies different issues for an economy at either endpoint of the reverse-J fertility curve. For a low income economy, the state of technology and initial capital stock will determine whether an economy is caught in a poverty trap with a predominantly unskilled workforce and high fertility. For a high income economy, with a skilled workforce, the possibility of a fertility upturn depends on the availability of a child rearing alternative to parental time, which in turn depends on policies, such as child care subsidies, and cultural factors.

This paper predicts a fertility upturn in developed economies where the workforce is skilled and alternatives to parental time in child rearing are readily available. This prediction challenges the conventional wisdom that better educated parents will continue to have less children and offers optimism for high income countries concerned with the problems presented by ageing populations.
6 Appendix

6.1 Fertility of a skilled worker is increasing in wages

Proof of (13b).

\[
\frac{\partial n_s^i}{\partial w_s^i} = \frac{\gamma (1 - \tau) [p(.) - w_s^i \partial p(.)/\partial w_s^i]}{[p(w_s^i, \beta)]^2}
\]

\[
\frac{\partial n_u^i}{\partial w_s^i} = \frac{\gamma (1 - \tau) [w_u^i \partial p(.)/\partial w_s^i + (1 - \beta) \partial p(.)/\partial (1 - \beta) - w_s^i \partial p(.)/\partial w_s^i]}{[p(w_s^i, \beta)]^2}
\]

\[
\frac{\partial n_s^i}{\partial w_u^i} = \frac{\gamma (1 - \tau) (1 - \beta) \hat{x}}{[p(w_s^i, \beta)]^2} > 0
\]

6.2 Lower Fertility of Skilled Worker implies equilibrium skilled wage premium

From (10) and (12),

\[n_s^i < n_u^i \iff \frac{p(w_s^i, \beta)/w_s^i}{p(w_u^i, \beta)/w_u^i} > (1 - \tau)\]

Noting that \(\tau \in (0, 1); \gamma \in (0, 1),\)

\[
\frac{p(w_s^i, \beta)/w_s^i}{p(w_u^i, \beta)/w_u^i} > (1 - \tau) > (1 - \tau)^{1/\gamma}
\]

where

\[
\frac{p(w_s^i, \beta)/w_s^i}{p(w_u^i, \beta)/w_u^i} > (1 - \tau)^{1/\gamma} \iff w_s^i > w_u^i
\]

6.3 Child Rearing of Cobb-Douglas form

Dropping the time subscript for convenience, input demands for time and goods are, respectively

\[\hat{z}^i = \left[\frac{(1 - \beta)a}{w^i(1 - a)}\right]^{1-a}\]

\[\hat{x}^i = \left[\frac{w^i(1 - a)}{(1 - \beta)a}\right]^a\]

The per unit cost function for child rearing is

\[p(w^i, \beta) = [w^i]^a (1 - \beta)^{1-a} B\]
where \( B = \left( \frac{a}{1-a} \right)^{1-a} + \left( \frac{1-a}{a} \right)^a \).

A skilled worker chooses fertility

\[
 n^s = \gamma (1 - \tau) \frac{(w^s)^{1-a}}{(1 - \beta)^{1-a}B}
\]

and total time spent rearing children

\[
 \hat{z}n^s = \gamma (1 - \tau) \frac{(a/(1-a))^{1-a}}{(a/(1-a))^{1-a} + ((1-a)/a)^a}
\]

\[
 \equiv \gamma (1 - \tau) \lambda
\]

where

\[
 \lambda = \left( \frac{a}{1-a} \right)^{1-a} \left[ \left( \frac{a}{1-a} \right)^{1-a} + \left( \frac{1-a}{a} \right)^a \right]
\]

\[
 = \left[ \left( \frac{a}{1-a} \right)^{1-a} + \left( \frac{1-a}{a} \right)^a \right]^{-1}
\]

\[
 = \left[ 1 + \left( \frac{1-a}{a} \right) \right]^{-1}
\]

Further substituting for \( p(w^i, \beta) = [w^i]^a (1 - \beta) - a B \) in (15), the skilled wage differential is given by

\[
 \frac{w^u}{w^s} = (1 - \tau)^{1/(1-a\gamma)}
\]

### 6.4 Equation of Motion (30)

Substituting the second line of (29) in (28) and noting that \( w^u = Ab \) yields

\[
k_{t+1} = \frac{1 - \gamma}{\gamma} A^a B (1-\beta)^{1-a} \left[ \frac{b + \left\{ \frac{\theta^1-a(1-\tau)^{1-a}(1-a)}{(1-a\gamma)^{1-a}} - b\theta \right\} k_t}{b^{1-a} + \left\{ \frac{\theta^1-a(1-\tau)^{1-a}(1-a)^{1-a}}{(1-a\gamma)^{a(1-a)}} - b^{1-a}\theta \right\} k_t} \right]
\]

Further substituting for \( \theta^{-a} \), using (24), and manipulating yields (30).
6.5 Proof of Proposition 1

6.5.1 Cobb-Douglas Child Rearing Production Function

Differentiating (30) and using (24) gives

\[ \phi'(k_t) = A^a B \left( \frac{1-\gamma}{\gamma} \right) \left\{ a \alpha \frac{1-a}{(1-a)\gamma} \right\} k_t^{a\alpha - 1} > 0 \quad \text{if } k_t \geq \bar{k} \]

\[ v^a \theta \left\{ \left( 1-\theta \right) \left( \frac{1-a}{\gamma} \right) k_t \right\} \left\{ \left( 1-\theta \right) \left( \frac{1-a}{\gamma} \right) \right\} > 0 \quad \text{if } k_t < \bar{k} \]

Further differentiating, we obtain

\[ \phi''(k_t) = A^a B \left( \frac{1-\gamma}{\gamma} \right) \left\{ -(1-aa\alpha) a \alpha \frac{1-a}{(1-a)\gamma} \right\} k_t^{a\alpha - 2} < 0 \quad \text{if } k_t \geq \bar{k} \]

\[ v^a 2$$a$$ \theta^2 \left\{ \left( 1-\theta \right) \left( \frac{1-a}{\gamma} \right) k_t \right\} \left\{ \left( 1-\theta \right) \left( \frac{1-a}{\gamma} \right) \right\} > 0 \quad \text{if } k_t < \bar{k} \]

Moreover, we show that convexity and concavity, throughout Phase 1 and Phase 2, respectively, is not specific to a Cobb-Douglas functional form.

6.5.2 General Form Child Rearing Production Function

For instance, consider \( \phi(k_t) \) if \( k_t \geq \bar{k} \). Referring to (26), \( k_{t+1} \) is proportional to child rearing costs per pair of children, \( p(w^a_s t) \). By Euler’s Theorem, \( \frac{\partial p(w^a_s t)}{\partial w^a_s t} = \hat{z} > 0 \). Concavity of the per unit child-rearing cost function follows from the fact that demand for parental time is downward sloping:

\( \frac{\partial \hat{z}}{\partial w^a_s t} = \frac{\partial p}{\partial w^a_s t^2} < 0 \).

6.5.3 Parental Time Only

The special case where a fraction of time, \( \hat{z} \), is the only child rearing input is the limit as \( a \to 1 \) of the Cobb-Douglas child rearing production function

\[ n_t = J(z_t)^{a} (x_t)^{1-a} \]

where \( J > 1 \) is the level of production technology, analogous to \( A \) in (1).
Letting $a = 1$ in (30) and $1/J = \hat{z}$, we obtain

$$
\phi(k_t) = A\hat{z}\frac{(1 - \gamma)}{\gamma} \begin{cases} 
\frac{(1 - \alpha)}{(1 - \gamma)^{\alpha}(1 - \tau)^{\beta}} k_t^\alpha & \text{if } k_t \geq \bar{k} \\
1 + \theta \left\{ (1 - \tau)^{1 - \gamma} - 1 \right\} k_t & \text{if } k_t < \bar{k} 
\end{cases}
$$

$$
\phi'(k_t) = A\hat{z}\frac{(1 - \gamma)}{\gamma} \begin{cases} 
\frac{\alpha(1 - \alpha)}{(1 - \gamma)^{\alpha}(1 - \tau)^{\beta}} k_t^{\alpha - 1} > 0 & \text{if } k_t \geq \bar{k} \\
\theta \left\{ (1 - \tau)^{1 - \gamma} - (1 - \tau) \right\} k_t & \text{if } k_t < \bar{k} 
\end{cases}
$$

$$
\phi''(k_t) = A\hat{z}\frac{(1 - \gamma)}{\gamma} \begin{cases} 
-\frac{(1 - \alpha)^2}{(1 - \gamma)^{\alpha}(1 - \tau)^{\beta}} k_t^{\alpha - 2} < 0 & \text{if } k_t \geq \bar{k} \\
2\theta^2 \frac{\tau (1 - \tau)^{1 - \gamma}}{(1 - \tau)(1 - \gamma) - (1 - \tau)} & \text{if } k_t < \bar{k} 
\end{cases}
$$
References


Figure 1: Total Fertility Rate (births per woman)

Source: World Bank (2009); National Sources
Figure 2: Evolution of capital per household - Phase 1 and 2 (high A)
Figure 3: Evolution of capital per household - Phase 1 and 2 (low A)
Figure 4: Fertility when Child Rearing uses Goods and Time
Figure 5: Fertility when Child Rearing uses only Time