Exclusion in multidimensional screening models

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Abstract.

One of the most celebrated results in the theory of multidimensional screening, due to Armstrong (1996) is that the monopolist will find it optimal not to serve some consumers at equilibrium. The proof presented by Armstrong, however, depends on some strong technical assumptions. Barelli (2006) had shown that this assumptions can be relaxed in the monopoly case. The cost of relaxing these assumptions is that the result now will hold generically, rather than for all fundamentals satisfying the conditions of the Theorem. Rochet and Stole (2003) have shown that it is possible to have an empty exclusion set. Their example, however, is not generic since infinitely small perturbation of the type set will lead to a positive measure of excluded customers. We generalize the result further by proving that the results can be
generalized to the case of oligopoly and industry with free entry. Therefore, the inability of some consumers to purchase the good at acceptable terms is solely driven by the multi-dimensional nature of private information rather than by market conditions or nature or distribution of the consumers’ tastes.

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1 Introduction

Assume a monopolist, who faces a continuum of consumers, produces several goods. Tastes of consumers are parametrized with a vector of characteristics. The problem of devising the optimal selling mechanism when this vector has more than one component (e.g. the consumers differ by their marginal rates of substitutions between multiple goods) is known as multi-dimensional screening.

One of the most celebrated results in the theory of multi-dimensional screening, due to Armstrong (1996) is that the monopolist will find it optimal not to serve some consumers at equilibrium. The intuition behind this result is rather simple: If the monopolist increases the tariff by $\varepsilon > 0$ she will earn extra profits of order $O(\varepsilon)$ on the consumers who still buy the product, but will loose only the consumers, whose surplus was below $\varepsilon$. The measure of this set of the consumers is $O(\varepsilon^m)$, i.e. for $m > 1$ it is always profitable to increase the tariff.

The proof presented by Armstrong, however, depends on some strong technical assumptions. In this paper we argue that this assumptions can be relaxed in the monopoly case. Furthermore, the results can be generalized to the case of oligopoly and industry with free entry. Therefore, the inability of
some consumers to purchase the good at acceptable terms is solely driven by the multi-dimensional nature of private information rather than by market conditions or nature or distribution of the consumers’ tastes.

The paper is organized in the following way. In Section 2 we discuss genericity of exclusion by a monopolist who faces consumers with a type dependent outside option. We also relax the joint convexity and homogeneity assumptions used by Armstrong (1996). In Section 3 we generalize the results for the case of oligopoly and the market with free entry. In Section 4 we discuss how the results will be generalized if one drops the quasilinearity assumption. Section 5 concludes.

2 Genericity of exclusion in a monopolistic screening model

Assume a monopolist who faces a continuum of consumers produces $n$ goods, i.e. its production can be captured by a vector $x \in \mathbb{R}^n_+$. The cost of production is given by a strictly convex twice differentiable utility function
\[ c(\cdot) : R^n_+ \to R. \] A consumer’s utility is given by:

\[ v(\alpha, x) - t, \] (1)

where \( v(\cdot, \cdot) \) is continuously differentiable in both arguments, \( \alpha \in \Omega \subset R^m \) is her unobservable type distributed on an open, bounded, set \( \Omega \subset R^m \) according to a strictly positive, continuous density function \( f(\cdot) \), and \( t \) is the amount of money transferred to the monopolist, and \( v : \Omega \times R^n_+ \) is a continuous function, strictly increasing in both arguments. Consumers have an outside option of value \( s_0(\alpha) \), which is assumed to be continuously differentiable and implementable.\(^1\)

The monopolist is looking for a selling mechanism that would maximize her profits. The Revelation Principle implies that one can without loss of generality assume that the monopolist simply announces a non-linear tariff \( t(\cdot) \). The amount \( t(x) \) determines how much a consumer has to pay for a good with quality characteristics \( x \).

The above consideration can be summarized by the following model. The

\(^1\)For conditions of implementability of a surplus function see Basov (2005).
monopolist selects a continuous $t : R^n_+ \rightarrow R$ to solve

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha, \quad (2)$$

where $c(x)$ is cost of producing a good with quality $x$ and $x(\alpha)$ satisfies

$$x(\alpha) \in \arg \max_{x} (v(\alpha, x) - t(x)), \quad (3)$$

$$\max(v(\alpha, x) - t(x)) \geq s_0(\alpha). \quad (4)$$

One of the most celebrated results in the theory of multi-dimensional screening states that if $\Omega$ is convex, $v(\cdot, x) : \Omega \rightarrow R$ is a convex, homogenous of degree one function for all $x$, and $s_0(\alpha) = 0$ for all $\alpha$. We will relax these assumptions by assuming:

**Assumption 1** The boundary of the set $\Omega$ is piecewise smooth, i.e. there exists $k \in N$ and $k$ continuously differentiable functions $g_i : \overline{\Omega} \rightarrow R$ such that

$$\partial \Omega = \bigcup_{i=1}^{k} \Sigma_i, \quad (5)$$

where

$$\Sigma_i = \{ \alpha \in \Omega : g_i(\alpha, \beta) = 0 \}, \quad (6)$$
where $\beta \in \mathbb{R}^J$ ($J \geq 1$) are some parameters and for all $x \in \mathbb{R}^n_+$ and all $i = 1, \ldots, k$ there exists $\beta_0 \in \mathbb{R}^J$ such that

$$
\text{rank} \left( \begin{array}{c}
\nabla_\alpha u(\alpha, x) \\
\nabla_\alpha g_i(\alpha, \beta_0)
\end{array} \right) = 2.
$$

(7)

**Assumption 2** For any $x \in \mathbb{R}^n_+$ function $u(\cdot, x)$ defined by

$$
u(\cdot, x) = v(\cdot, x) - s_0(\cdot)
$$

is increasing.

In the case when the value of the outside option is type independent Assumption 2 reduces to a standard assumption that the utility is increasing in type. Let us provide some other examples when Assumption 2 seems natural.

**Example 1** Assume that every consumer has an option to buy nothing and pay nothing, i.e.

$$s_0(\alpha) = v(\alpha, 0).
$$

(9)
In this case Assumption 2 reduces to a *weak single-crossing condition*

\[ v(\cdot, x) - v(\cdot, 0) \]  

increases in \( \alpha \) for every \( x \in R^n_+ \). Though usually it is assumed that \( v(\cdot, 0) = 0 \), this need not be so. For example, consider a consumer who has wealth \( w \) in her account. Her second period wealth can take to values \( w_H \) or \( w_L \). Probability that \( w = w_H \) is equal to \( p \) and discount factor is \( \delta \). Here the private information of the consumer is characterized by a two-dimensional vector \( \alpha = (p, \delta) \). The individual’s preferences are given by:

\[ U(c_1, c_2) = u(c_1) + \delta Eu(c_2), \]  

where \( c_1 \) and \( c_2 \) are the consumption levels in periods one and two respectively. We will assume that wealth in not storable between periods. An individual may approach a bank for a loan \( X \). If she does so, she will be asked to repay \( t(X) \) is the next period, provided her wealth is high and default if it is low. If the individual chooses not to take the credit her expected
utility will be:

\[ U_0(p, \delta) = u(w) + \delta(pu(w_H) + (1 - p)u(w_L)), \]

(12)

which is type dependent. By an appropriate change of variables one may ensure that utility in this example satisfies Assumption 2.

**Example 2** Suppose a monopolist produces cars of high quality, i.e. \( x \in R^n_+ \).

The utility of a consumer is

\[ u(\alpha, x) = A + \sum_{i=1}^{n} \alpha_i x_i, \]

(13)

where \( A > 0 \) can be interpreted as utility of driving a car, and the second term in (13) is a quality premium. Suppose a consumer has three choices: to buy a car from the monopolist, to by a car from a competitive fringe, and to buy no car at all. We will normalize the utility of buying no car at all to be zero. Assume the competitive fringe serves low quality cars of quality \(-x_0\), where \( x_0 \in R^r_+ \) at price \( p \), that is the consumers are experiencing disutility from the quality of the cars of the competitive fringe and higher so, higher
their type. The utility of the outside option in this case is given by:

$$s_0(\alpha) = \max(0, \ A - p - \sum_{i=1}^{n} \alpha_i x_{0i})$$  \hspace{1cm} (14)$$

and is decreasing in $\alpha$. Therefore, Assumption 2 holds.

Let us redefine a consumer’s utility function by (8). Assumption 2 guarantees that $u(\cdot, x)$ is increasing and we can reformulate the monopolist’s problem in the following way: the monopolist selects a continuous $t : \mathbb{R}_+^n \rightarrow \mathbb{R}$ to solve

$$\max_{t(\cdot)} \int_{\Omega} (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha,$$

where $c(x)$ is cost of producing a good with quality $x$ and $x(\alpha)$ satisfies

$$x(\alpha) \in \arg\max(u(\alpha, x) - t(x))$$  \hspace{1cm} (16)$$

$$\max(u(\alpha, x) - t(x)) \geq 0.$$  \hspace{1cm} (17)$$

Let $t(\cdot)$ be the optimal tariff\footnote{See Basov (2005) for the conditions that ensure the existence of the solution.} and define the consumer surplus as:

$$s(\alpha) = \max(u(\alpha, x) - t(x)).$$  \hspace{1cm} (18)$$
Let
\[ \Omega_0 = \{ \alpha \in \Omega : s(\alpha) = 0 \}, \quad (19) \]
be the set of consumers who get their outside option, which we will call the exclusion set, and for any measurable set \( K \subset R^m \) let \( \lambda(K) \) denote its Lebesgue measure.

**Lemma 1** (Barelli, 2006) Assume \( \lambda(\Omega_0) = 0 \) and let Assumption 2 hold, then \( \Omega_0 \subset \partial \Omega \).

Lemma 1 states that if the exclusion set has Lebesgue measure zero it should be part of the boundary of the type set. Assumption 2 is crucial for this result. If it does not hold it is easy to come up with counter-examples even in the unidimensional case. To formulate our second lemma let us define a restriction \( \mu \) of \((m - 1)\) dimensional Lebesgue measure to set \( \partial \Omega \) by:

\[
\mu(C) = \int_{\partial \Omega} \chi_C(\alpha) d\alpha, \quad (20)
\]

where \( C \subset \partial \Omega \) is a measurable set, \( \chi_C(\cdot) \) is its characteristic function, and integral in equation (20) is a scalar surface integral.

**Lemma 2** Assume \( \lambda(\Omega_0) = 0 \) and Assumption 1 holds. Then \( \mu(\Omega_0) = 0 \) for almost all \( \beta \).
Proof. Let $s(\cdot)$ be the surplus function generated by the optimal tariff via (18). By Lemma 1, $\Omega_0 \subset \partial \Omega$. By Assumption 1, it implies that

$$\Omega_0 = \bigcup_{i=1}^{k} \Omega_{0i},$$  \hspace{1cm} (21)

where

$$\Omega_{0i} = \{ \alpha \in \Omega : g_i(\alpha, \beta) = 0, s(\alpha) = 0 \}. \hspace{1cm} (22)$$

Since union in (21) is finite it is sufficient to prove that $\mu(\Omega_{0i}) = 0$ for almost all $\beta$ for all sets (22). Assumption 1 and the Transversality Theorem (see, Mas-Colell, Whinston, and Green, 1995, Chapter 17D) imply that

$$\text{rank} \left( \begin{array}{c} \nabla_{\alpha} u(\alpha, x) \\ \nabla_{\alpha} g_i(\alpha, \beta) \end{array} \right) = 2 \hspace{1cm} (23)$$

for almost all $\beta$. Therefore, by the Implicit Function Theorem, $\Omega_{0i}$ is a manifold of dimension of $(m - 2)$ for almost all $\beta$ and the conclusion follows.

Q.E.D.

For any $a, b \in \mathbb{R}^m$ let $(a \cdot b)$ denote the inner product of $a$ and $b$. The following theorem holds.

**Theorem 1** Consider problem (15)-(17) and assume that $u(\cdot, \cdot)$ is twice
continuously differentiable and strictly increasing in both arguments, \( c(\cdot) \) is strictly convex and twice continuously differentiable, \( \Omega \) is an open bounded set with a piecewise smooth boundary satisfying Assumption 1, \( f(\cdot) \) is strictly positive on \( \Omega \), and there exists \( K > 0 \) such that

\[
    u(\alpha, x) \leq K(\alpha \cdot \nabla_\alpha u(\alpha, x))
\]

for all \( x \in \mathbb{R}_+^n \) and all \( \alpha \in \Omega \). Then for almost all \( \beta \) a set of positive measure of consumers is excluded at the equilibrium.

**Proof.** See Barelli (2006).

Rochet and Stole provided an example where the exclusion set is empty. In their example \( u : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R} \) has a form

\[
    u(\alpha, x) = (\alpha_1 + \alpha_2)x
\]

and \( \Omega \) is a rectangle with sides parallel to the 45 degrees and \(-45\) degrees lines. They argued that one can shift the rectangle sufficiently far to the right to have an empty exclusion region. Their result is driven by the fact that they allow only very special collection of type sets, rectangles with parallel sides. Formally, Assumption 1 fails in this case, since \( g_1(\alpha, \beta) = \alpha_1 + \alpha_2 - \beta = 0 \).
\[ \nabla_\alpha u(\alpha, x) = x x \Rightarrow rank(\nabla_\alpha u(\alpha, x)) = 1. \] (26)

3 Genericity of exclusion in an oligopolistic screening model

Consider a framework similar to the one of the previous Section but assume that the market is served by \( K \) producers. The production cost is identical among the producers. A pure strategy of a producer \( k \) is a non-linear tariff, i.e. a measurable mapping \( t_k : R^n \to R \). Let us assume that there exists a symmetric equilibrium at which all producers charge the same tariff. We will argue that at such an equilibrium a positive measure of the consumers are not served.

Assume at equilibrium producer \( k \) charges tariff \( t_k(\cdot) : R^n_+ \to R \). Then \( t_k(\cdot) \) solves

\[ \max_{t_k(\cdot)} \int_{\Omega} (t_k(x_\ell(\alpha)) - c_\ell(x_\ell(\alpha))) f(\alpha) d\alpha, \] (27)
subject to:

\[ x(\alpha) \in \arg \max (v(\alpha, x) - t(x)) \]  \quad (28)

\[ \max_{x \in R^+_n} (v(\alpha, x) - t(x)) \geq s_0(\alpha), \]  \quad (29)

where

\[ t(x) = \min \sum_{k=1}^{K} t_k(x_k) \]
\[ s.t. \sum_{k=1}^{K} x_k = x, \quad x_k \geq 0 \]  \quad (30)

and

\[ s_0(\alpha) = \max \{ s_0^*(\alpha), \max_{x \in R^+_n, x_{\ell} = 0} (v(\alpha, x) - t_{-\ell}(x)) \} \]  \quad (31)

and \( t_{-\ell}(x) \) solves problem (30) subject to an additional constraint \( x_{\ell} = 0 \).

Equation (31) states that the outside option of a consumer seen from the point of view of producer \( \ell \) is determined by her best opportunity outside the market and the best bundle she may purchase from the competitors.

Let us define

\[ u(\alpha, x_\ell) = v(\alpha, x_\ell + \sum_{k=1, k \neq \ell}^{K} x_k(\alpha)) - \sum_{k=1, k \neq \ell}^{K} t_k(x_k(\alpha)) - s_0(\alpha), \]  \quad (32)

where \( x_k(\alpha) \) is the equilibrium quantity purchased by the consumer of type
α from the producer k. Then the problem of producer ℓ becomes:

$$\max_{t_\ell(\cdot)} \int_{\Omega} (t_\ell(x_\ell(\alpha)) - c_\ell(x_\ell(\alpha))) f(\alpha) d\alpha,$$  \( (33) \)

subject to:

$$x_\ell(\alpha) \in \arg \max (u(\alpha, x_\ell) - t_\ell(x_\ell)) \quad (34)$$

$$\max_{x_\ell \in R^n_+} (u(\alpha, x_\ell) - t_\ell(x_\ell)) \geq 0. \quad (35)$$

Let us impose the following form of single-crossing property:

$$\frac{\partial^2 v}{\partial \alpha_i \partial x_j} \geq 0 \quad (36)$$

for all i, j. Then \( u(\alpha, x_\ell) \) is increasing in \( \alpha \) for all \( x_\ell \in R^n_+ \).

**Theorem 2** Under Assumptions of Theorem 1 a positive measure of consumers will be excluded in any symmetric equilibrium of the oligopolistic market.

**Proof.** Consider oligopolist 1. Given behavior of her competitors, her problem is isomorphic to the problem of the monopolist, with appropriately redefined utility. Therefore, Theorem 1 implies that she will find it optimal
to exclude a positive measure of consumer. By symmetry, so will the other oligopolists.

Q.E.D.

Note that though we can argue that there is a positive measure of consumers is excluded in any oligopolistic equilibrium, existence of such an equilibrium is a non-trivial issue. The reason is that if the monopolistic solution included bunching the profit functions of oligopolists become discontinuous (Champsuar and Rochet, 1989). Since conditions that rule out bunching in the multidimensional case are unknown we are forced to leave the questions of existence open.³

Now let us assume that the number of producers is not fixed but there is a positive entry cost $F > 0$. It is easy to see that this problem can be reduced to the previous one, since equilibrium number of the producers is always finite. Indeed, with $K$ producers the profits of an oligopolist in a symmetric equilibrium are bounded by $\pi^m/K$, where $\pi^m$ are the profits of a monopolist. Therefore, at equilibrium $K \leq \pi^m/F$ and a positive measure of the consumers will be excluded from the market.

³Bunching may be, however, not as typical in the multidimensional case as suggested by Rochet and Chone (1998). See discussion in Basov (2005).
4 Conclusions

In this paper we extended Armstrong’s (1996) result to argue that if private information of the consumers is multidimensional a positive measure of the consumers will generically not be served at the equilibrium. The result holds for general utility functions and type sets and can be extended to the oligopolistic setting. Another extension, worth pursuing is to allow for arbitrary outside options. Another interesting extension, obtained in the literature, are for the auction-theoretic setting (Monteiro, Svaiter, and Page, 2001).

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