Estimation in Threshold Autoregressive Models with Nonstationarity

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September 27, 2009
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1. Existing TAR Models

- Threshold auto–regressive (TAR) models have been quite popular in the literature, since its invention mainly by Howell Tong (1983).

- The TAR model has been generalized to the Smooth Transition Auto–regressive (STAR) model mainly by Clive Granger & Timo Teräsvirta (1993).
Consider a threshold auto-regressive (TAR) model of the form

\[ y_t = \alpha_1 y_{t-1} I[z_t \in C_\tau] + \alpha_2 y_{t-1} I[z_t \in C^c_\tau] + e_t, \quad (1.1) \]

where

- \( C_\tau \) is a subset of \( \mathbb{R}^1 = (-\infty, \infty) \) indexed by \( \tau \),
- \( \{z_t\} \) is a stationary threshold variable,
- \( -\infty < \alpha_1, \alpha_2 < \infty \) are unknown parameters,
- \( \{e_t\} \) is an error with \( E[e_1] = 0 \) and \( 0 < \sigma^2_e = E[e^2_1] < \infty \),
- \( \{e_t\} \) and \( \{y_s\} \) are mutually independent for all \( s < t \), and
- \( 1 \leq t \leq n \), \( n \) is the sample size of the time series.
1. Many of the threshold models used have been stationary models, i.e., models for which $|\alpha_1| < 1$ and $|\alpha_2| < 1$. 
2. Econometrics literature proposes testing whether

\[ H_0 : \alpha_1 = \alpha_2 = 1 \]

in a model of the form

\[
y_t - y_{t-1} = (\alpha_1 - 1) y_{t-1} I[z_t \in C_\tau] \\
+ (\alpha_2 - 1) y_{t-1} I[z_t \in C_\tau^c] + e_t,
\]

(1.2)

where

- the parameters \( \alpha_1 \) and \( \alpha_2 \) are then estimated under

\[ H_0 : y_t = y_{t-1} + e_t. \]
• A closely related paper is by Caner & Hansen (2001); and

• The authors also point out that there are several nonstationary alternatives when $H_0$ does not hold.
The main aim of this paper is then to estimate the unknown parameters involved in a nonstationary alternative.

This research aim falls into the recent research agenda of this research group on

*Estimating Unknown Parameters and Functions After a Hypothesis or Specification Testing*
2. New TAR Model

The rest of this paper is interested in a new TAR model of the form:

\[ y_t = \alpha_1 y_{t-1} I[y_{t-1} \in C_\tau] + \alpha_2 y_{t-1} I[y_{t-1} \in C^c_\tau] + e_t \]

\[ = \begin{cases} 
\alpha_1 y_{t-1} + e_t & \text{if } y_{t-1} \in C_\tau, \\
\alpha_2 y_{t-1} + e_t & \text{if } y_{t-1} \in C^c_\tau,
\end{cases} \quad (2.1) \]

where

- \( C_\tau \) is a subset of \( R^1 = (-\infty, \infty) \),
- \( |\alpha_1| < 1 \) or \( |\alpha_1| > 1 \), \( \alpha_2 \equiv 1 \), and
- \( \{e_t\} \) is an independent error with \( E[e_1] = 0 \), \( 0 < \sigma_e^2 = E[e_1^2] < \infty \) and \( E[e_1^4] < \infty \).
It is obvious that $\alpha_1$ and $\alpha_2$ can be estimated by the ordinary least squares estimators

\[
\hat{\alpha}_1 = \hat{\alpha}_1(\tau) = \frac{\sum_{t=1}^{n} y_t y_{t-1} I[y_{t-1} \in C_\tau]}{\sum_{t=1}^{n} y_{t-1}^2 I[y_{t-1} \in C_\tau]}
\]

(2.2)

\[
\hat{\alpha}_2 = \hat{\alpha}_2(\tau) = \frac{\sum_{t=1}^{n} y_t y_{t-1} I[y_{t-1} \in C_c]}{\sum_{t=1}^{n} y_{t-1}^2 I[y_{t-1} \in C_c]}
\]

(2.3)
Observe that model (2.1) can be written as

\[ y_t - y_{t-1} = (\alpha_1 - 1)y_{t-1}I[y_{t-1} \in C_\tau] + e_t \]
\[ \equiv u_t + e_t, \]

(2.4)

where \( u_t = (\alpha_1 - 1)y_{t-1}I[y_{t-1} \in C_\tau] \neq 0 \) unless \( \alpha_1 = 1 \).

- This shows that \( \{y_t\} \) does not follow a standard random walk model.
- It has been shown that \( \{y_t\} \) is a \( \beta \)-null recurrent Markov chain with \( \beta = \frac{1}{2} \).
Theorem 2.1 Assume that model (2.1) holds. Then as \( n \to \infty \)

\[
\sqrt{T(n)} (\hat{\alpha}_1 - \alpha_1) \to_D N \left( 0, \sigma_e^2 \sigma_1^{-2} \right), \quad (2.5)
\]

\[
n (\hat{\alpha}_2 - 1) \to_D \frac{(Q^2(1) - \sigma_e^2)}{2 \int_0^1 Q^2(r) dr}, \quad (2.6)
\]

where

1. \( T(n) \approx \sqrt{n} \) is the (random) number of visits of \( \{y_t\} \) to a particular set in the time period \([0, n]\),

2. \( \sigma_1^2 = \int_{-\infty}^{\infty} y^2 I[y \in C_\tau] \pi_s(dy) \) with \( \pi_s(\cdot) \) being an invariant measure, and

3. \( Q(r) = \sigma_e B(r) + m_u M_{\frac{1}{2}}(r) \), in which
   - \( m_u = (\alpha_1 - 1) \mu_1 \) with \( \mu_1 = \int_{-\infty}^{\infty} y I[y \in C_\tau] \pi_s(dy) \) and
   - \( M_{\frac{1}{2}}(r) \) is a known random process.
Remark 2.1. Theorem 2.1 shows that

1. the rate of convergence of $\hat{\alpha}_1$ to $\alpha_1$ is proportional to $\sqrt{\sqrt{n}} = n^{1/4}$, and
   - the reason is that one can only get $\lfloor \sqrt{n} \rfloor$ number of observations in the stationary regime.

2. the rate of convergence of $\hat{\alpha}_2$ to 1 is proportional to $n$
   (this is because one can have $n - \lfloor \sqrt{n} \rfloor$ number of observations in the nonstationary regime).
3. Estimation in The TAR Model

Let

\[ \hat{e}_t(\tau) = y_t - \hat{\alpha}_1 y_{t-1} I[y_{t-1} \in C_\tau] - \hat{\alpha}_2 y_{t-1} I[y_{t-1} \in C^c_\tau] \]

and then define the estimated variance by

\[ \hat{\sigma}^2(\tau) = \frac{1}{n} \sum_{t=1}^{n} \hat{e}_t^2(\tau). \] (3.1)

The \( \tau \) and \( \alpha_i \) can finally be estimated by

\[ \hat{\tau} = \operatorname{arg\,min}_{\tau} \hat{\sigma}^2(\tau), \] (3.2)

\[ \tilde{\alpha}_1 = \hat{\alpha}_1(\hat{\tau}) = \frac{\sum_{t=1}^{n} y_t y_{t-1} I[y_{t-1} \in C_\hat{\tau}]}{\sum_{t=1}^{n} y_{t-1}^2 I[y_{t-1} \in C_\hat{\tau}]}, \] (3.3)

\[ \tilde{\alpha}_2 = \hat{\alpha}_2(\hat{\tau}) = \frac{\sum_{t=1}^{n} y_t y_{t-1} I[y_{t-1} \in C^c_\hat{\tau}]}{\sum_{t=1}^{n} y_{t-1}^2 I[y_{t-1} \in C^c_\hat{\tau}]}. \] (3.4)
4. Examples of Implementation

1. Consider model (2.1) with $e_t \sim N(0, 1)$;

2. Consider the case of $n = 1000$, $2000$, $5000$ and $10000$;

3. Let $N = 1000$ be the number of replications; and

4. $\tilde{\alpha}_i(j)$ and $\hat{\tau}(j)$ be the respective value of $\tilde{\alpha}_i$ and $\hat{\tau}$ at the $j$–th replication throughout Examples 4.1 and 4.2 below.
• Calculate the standard deviations of the form

\[
\text{std}(\tilde{\alpha}_i) = \sqrt{\frac{1}{N-1} \sum_{j=1}^{N} \left( \tilde{\alpha}_i(j) - \overline{\tilde{\alpha}_i} \right)^2}
\]

\[
\text{std}(\hat{\tau}) = \sqrt{\frac{1}{N-1} \sum_{j=1}^{N} \left( \hat{\tau}(j) - \overline{\hat{\tau}} \right)^2}
\]

for \( i = 1, 2 \), where

\[
\overline{\tilde{\alpha}_i} = \frac{1}{N} \sum_{j=1}^{N} \tilde{\alpha}_i(j) \quad \text{and} \quad \overline{\hat{\tau}} = \frac{1}{N} \sum_{j=1}^{N} \hat{\tau}(j).
\]
Example 4.1  Consider an asymmetrical (bounded) form of $C_{\tau} = [\tau_1, \tau_2]$ with

- Case A: $\alpha_1 = \frac{1}{2}, \quad \alpha_2 = 1, \quad \tau_1 = -3$ and $\tau_2 = 2.5$;

- Case B: $\alpha_1 = \frac{3}{2}, \quad \alpha_2 = 1, \quad \tau_1 = -1.5$ and $\tau_2 = 1$.

Table 4.1 Simulation Results for Cases A and B

<table>
<thead>
<tr>
<th>Case</th>
<th>std($\tilde{\alpha}_1$)</th>
<th>std($\tilde{\alpha}_2$)</th>
<th>std($\hat{\tau}_1$)</th>
<th>std($\hat{\tau}_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n = 1000</td>
<td>0.0694</td>
<td>0.0208</td>
<td>0.2506</td>
<td>0.1396</td>
</tr>
<tr>
<td>n = 2000</td>
<td>0.0503</td>
<td>0.0074</td>
<td>0.2029</td>
<td>0.1186</td>
</tr>
<tr>
<td>n = 5000</td>
<td>0.0362</td>
<td>0.0024</td>
<td>0.1634</td>
<td>0.0754</td>
</tr>
<tr>
<td>n = 10000</td>
<td>0.0359</td>
<td>0.0008</td>
<td>0.1401</td>
<td>0.0659</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n = 1000</td>
<td>0.7606</td>
<td>0.0028</td>
<td>0.2825</td>
<td>0.3146</td>
</tr>
<tr>
<td>n = 2000</td>
<td>0.7438</td>
<td>0.0015</td>
<td>0.2501</td>
<td>0.2937</td>
</tr>
<tr>
<td>n = 5000</td>
<td>0.6596</td>
<td>0.0006</td>
<td>0.2155</td>
<td>0.2799</td>
</tr>
<tr>
<td>n = 10000</td>
<td>0.6168</td>
<td>0.0003</td>
<td>0.1938</td>
<td>0.2535</td>
</tr>
</tbody>
</table>
Example 4.2  Consider a TAR model of the form

\[ y_t = \alpha_1 y_{t-1} I[y_{t-1} \leq \tau] + \alpha_2 y_{t-1} I[y_{t-1} > \tau] + e_t, \]

(4.1)

where

- Case A: \( \alpha_1 = \frac{1}{2}, \ \alpha_2 = 1, \ \tau = 3; \)
- Case B: \( \alpha_1 = \frac{3}{2}, \ \alpha_2 = 1, \ \tau = 3. \)
Table 4.2 Simulation Results for Case A and Case B

<table>
<thead>
<tr>
<th>Case A</th>
<th>std($\tilde{\alpha}_1$)</th>
<th>std($\tilde{\alpha}_2$)</th>
<th>std($\hat{\tau}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1000$</td>
<td>0.0413</td>
<td>0.1177</td>
<td>0.1595</td>
</tr>
<tr>
<td>$n = 2000$</td>
<td>0.0373</td>
<td>0.0475</td>
<td>0.1133</td>
</tr>
<tr>
<td>$n = 5000$</td>
<td>0.0192</td>
<td>0.0155</td>
<td>0.0677</td>
</tr>
<tr>
<td>$n = 10000$</td>
<td>0.0169</td>
<td>0.0052</td>
<td>0.0556</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case B</th>
<th>std($\tilde{\alpha}_1$)</th>
<th>std($\tilde{\alpha}_2$)</th>
<th>std($\hat{\tau}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1000$</td>
<td>0.2798</td>
<td>0.0034</td>
<td>0.1830</td>
</tr>
<tr>
<td>$n = 2000$</td>
<td>0.1712</td>
<td>0.0012</td>
<td>0.1530</td>
</tr>
<tr>
<td>$n = 5000$</td>
<td>0.1551</td>
<td>0.0005</td>
<td>0.1453</td>
</tr>
<tr>
<td>$n = 10000$</td>
<td>0.1340</td>
<td>0.0002</td>
<td>0.1247</td>
</tr>
</tbody>
</table>
Tables 4.1 and 4.2 show that the proposed estimation method works well numerically with

- the rate of $\tilde{\alpha}_1$ is proportional to $n^{-\frac{1}{4}}$, and
- the rate of $\tilde{\alpha}_2$ is proportional to $n^{-1}$. 
Example 4.3  The series under study are 2–year \((x_{1t})\) and 30–year \((x_{2t})\) Australian government bonds, representing short–term and long–term series in the term structure of interest rates.

1. The analysis will be based on the transformed versions of \(y_{it} = \ln(x_{it})\) for \(i = 1, 2\).

2. The time frame of the study is January 1957 to March 2009, with 627 monthly collected observations for each of \(y_{it}\).
Figure 1: A: Plot of the logged series $y_{1t}$; B: Plot of the logged series $y_{2t}$
Our estimation method suggests using

\[ y_{1t} = 1.1173 \, y_{1,t-1} \, I(y_{1,t-1} \leq 1.5439) \]
\[ + 0.9995 \, y_{1,t-1} \, I(y_{1,t-1} > 1.5439) + e_t \]

(4.2)

for \{y_{1t}\}, where \( \hat{\tau} = 1.5439 \) and \( \hat{\sigma}^2 = 0.0023 \), and

\[ y_{2t} = 1.0004 \, y_{2,t-1} \, I(y_{2,t-1} \leq 1.6101) \]
\[ + 0.9955 \, y_{2,t-1} \, I(y_{2,t-1} > 1.6101) + e_t \]

(4.3)

for \{y_{2t}\}, where \( \hat{\tau} = 1.6101 \) and \( \hat{\sigma}^2 = 0.0016 \).
Model (4.2) implies

\[ y_{1t} - y_{1,t-1} = 0.1173 \ y_{1,t-1} \ I(y_{1,t-1} \leq 1.5439) - 0.0005 \ y_{1,t-1} \ I(y_{1,t-1} > 1.5439) + e_t, \]

(4.4)

and model (4.3) implies

\[ y_{2t} - y_{2,t-1} = 0.0004 \ y_{2,t-1} \ I(y_{2,t-1} \leq 1.6101) - 0.0045 \ y_{2,t-1} \ I(y_{2,t-1} > 1.6101) + e_t. \]

(4.5)
1. While model (4.5) indicates that \( \{y_{2t}\} \) may be modeled by a simple random walk model,

2. model (4.4) shows that \( \{y_{1t}\} \) is nonstationary but does not necessarily follow a random walk process, since the value of 0.1173 >> 0.

3. This provides support from an empirical application point of view that there is some need to study a nonstationary threshold model of the form (1.2).
5. Conclusions and Discussion

• Conclusions include

  – This paper has considered a class of threshold autoregressive models with possible non-stationarity.
  – The slope parameters have been consistently estimated.
  – Both simulated and real data examples are used to support the theory.
• Discussion

– 1st issue is how to establish an asymptotic theory for $\hat{\tau}$ in this kind of nonlinear and nonstationary situation.

– 2nd issue is possible extensions to higher-order models as well as vector threshold auto-regressive (VTAR) models.

– 3rd issue is possible extensions to threshold cointegration models with nonstationarity.
Estimation in an autoregressive model of the form

\[ y_t = g(y_{t-1}, \theta) + e_t, \quad (5.1) \]

where \( g(\cdot, \cdot) \) is a known function, and \( \{y_t\} \) is nonstationary but not just \( I(1) \).

Note that model (5.1) is an autoregressive counterpart of the nonlinear regression model

\[
\begin{align*}
y_t &= g(x_t, \theta) + e_t, \\
x_t &= x_{t-1} + \epsilon_t.
\end{align*}
\quad (5.2)
\]

Estimation results for (5.2) are available in the literature by Park & Phillips (2001).
6. Appendix

**Definition:** Let $T(n)$ denote the complete number of regenerations in the time interval, and its sample version is

$$T_C(n) = \sum_{k=0}^{n} I_C(y_k)$$

(6.1)

for some compact set $C$.

It has been shown that

$$\frac{T(n)}{\sqrt{n}} \rightarrow_D \xi$$

(6.2)

for some random variable $\xi > 0$. 
Proposition: Consider an autoregressive model of the form

\[ y_t = g(y_{t-1}) + e_t, \quad (6.3) \]

where

- \( \{e_t\} \) is an i.i.d. with density function \( f(\cdot) \);
- \( \inf_{u \in \mathcal{C}} f(u) > 0 \) for all compact subsets \( \mathcal{C} \) in \( \mathbb{R}^1 \);
- \( g(y) \) is bounded on all compact subsets in \( \mathbb{R}^1 \).

Then, \( \{y_t\} \) is a \( \beta \)-null recurrent Markov chain with \( \beta = \frac{1}{2} \).
Proposition: Consider model (6.3). The sequence \( \{y_t\} \) is \( \beta \)-null recurrent if and only if

\[
P(S_n > n) = \frac{1}{\Gamma(1 - \beta)n^\beta L_s(n)} (1 + o(1)), \quad (6.4)
\]

where \( S_n = \sum_{t=1}^{n} y_t \), and \( L_s(\cdot) \) is a slowly-varying function.
References


