Exclusion in multidimensional screening models

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Motivation: private information and monetary equilibria

Most of market transaction in modern economies rely on a monetary exchange.

Recently, rapid progress has been made in understanding conditions that lead to emergence of monetary equilibria.

However, the current literature has a weak point: it rules out a possibility of credit by assumption.

Idea: a positive measure of agents will be unable to secure credit.

The key to our result is an observation made by Armstrong (1996) that in principal-agent models with hidden information a positive measure of agents will not be served at equilibrium, provided that private information has more than one dimension.
Motivation: private information and monetary equilibria

To capitalize on this result we have to:

1. develop a model of credit, where potential borrowers differ along two-dimensions: rate of time preference and default probability.

2. Generalize the Armstrong’s result to make it applicable in the monetary framework:
   a). Relax the joint convexity assumption
   b). Allow for oligopolistic market structure

An important step towards 2(b) is to allow for a type dependent outside option
Introduction

One of the most celebrated results in the theory of multi-dimensional screening, due to Armstrong (1996), is that the monopolist will find it **optimal not to serve some consumers at equilibrium**.

The intuition behind this result is rather simple:

*If the monopolist increases the tariff by \( \varepsilon > 0 \) she will earn extra profits of order \( O(\varepsilon) \) on the consumers who still buy the product,*

*but will lose only the consumers, whose surplus was below \( \varepsilon \). The measure of this set of the consumers is \( O(\varepsilon^m) \),*

*i.e. for \( m > 1 \) it is always profitable to increase the tariff.*
Introduction

The proof presented by Armstrong, however, depends on some strong technical assumptions.

In this paper we argue that this assumptions can be relaxed in the monopoly case.

Furthermore, the results can be generalized to the case of oligopoly and industry with free entry.

Therefore, the inability of some consumers to purchase the good at acceptable terms is solely driven by the multi-dimensional nature of private information rather than by market conditions or nature or distribution of the consumers’ tastes.
Basic set up

Assume a monopolist who faces a continuum of consumers produces $n$ goods, i.e. its production can be captured by a vector $x \in \mathbb{R}^n_+$. 

The cost of production is given by a strictly convex twice differentiable utility function $c(\cdot) : \mathbb{R}^n_+ \rightarrow \mathbb{R}$. 

A consumer’s utility is given by $v(\alpha, x) - t$, 

where $\alpha \in \Omega \subset \mathbb{R}^m$ is her unobservable type distributed according to a strictly positive, continuous density function $f(\cdot)$, 

$t$ is the amount of money transferred to the monopolist, 

$v : \Omega \times \mathbb{R}^n_+ \rightarrow \mathbb{R}$ is a continuous function, strictly increasing in both arguments. 

Consumers have an outside option of value $s_0(\alpha)$, which is assumed to be continuously differentiable and implementable
The monopolist’s problem

The monopolist selects a continuous $t : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ to solve

$$\max_{t(\cdot)} \int \left( t(x(\alpha)) - c(x(\alpha)) \right) f(\alpha) d\alpha,$$  \hspace{1cm} (1)

where $c(x)$ is the cost of producing a good with quality $x$ and $x(\alpha)$ satisfies

$$x(\alpha) \in \arg \max (v(\alpha, x) - t(x))$$
$$\max (v(\alpha, x) - t(x)) \geq s_0(\alpha)$$ \hspace{1cm} (2)
Armstrong’s result

One of the most celebrated results in the theory of multi-dimensional screening states that if:

1. \( \Omega \) is convex,

2. \( v(\cdot, x): \Omega \to \mathbb{R} \) is a convex, homogenous of degree one function for all \( x \),

3. \( s_0(\alpha) = 0 \) for all \( \alpha \).

Then the positive measures of consumers are excluded at the optimum

(1)+(2) have bite only jointly, (3) is a serious restriction

Assumptions are too strong to apply the result to monetary theory
Relaxing Armstrong’s Assumptions on the Geometry

We will relax these assumptions by assuming:

**Assumption 1** The boundary of the set \( \Omega \) is piecewise smooth, i.e. there exists \( k \in \mathbb{N} \) and \( k \) continuously differentiable functions \( g_i : \overline{\Omega} \to \mathbb{R} \) such that

\[
\partial \Omega = \bigcup_{i=1}^{k} \Sigma_i ,
\]

where

\[
\Sigma_i = \{ \alpha \in \Omega : g_i(\alpha, \beta) = 0 \},
\]

where \( \beta \in R^J \ (J \geq 1) \) are some parameters and for all \( x \in R^n_+ \) and all \( i = 1, k \) there exists \( \beta_0 \in R^J \) such that

\[
\text{rank}( \begin{vmatrix} \nabla_{\alpha}u(\alpha, x) \nabla_{\alpha}g_i(\alpha, \beta_0) \end{vmatrix} ) = 2.
\]
Relaxing Armstrong’s Assumptions on the Preferences

**Assumption 2** For any $x \in \mathbb{R}^n_+$ function $u(\cdot, x)$ defined by

$$u(\cdot, x) = v(\cdot, x) - s_0(\cdot)$$

is increasing.

In the case when the value of the outside option is type independent Assumption 2 reduces to a standard assumption that the utility is increasing in type.

If every consumer has an option to buy nothing and pay nothing, i.e. $s_0(\alpha) = v(\alpha, 0)$. Assumption 2 reduces to a weak single-crossing condition $v(\cdot, x) - v(\cdot, 0)$ increases in $\alpha$ for every $x \in \mathbb{R}^n_+$. 
Some Preliminary Results

Lemma 1 (Barelli, 2006) Assume \( \lambda(\Omega_0) = 0 \) and let Assumption 2 hold, then \( \Omega_0 \subset \partial \Omega \).

Lemma 1 states that if the exclusion set has Lebesgue measure zero it should be part of the boundary of the type set. Assumption 2 is crucial for this result. If it does not hold it is easy to come up with counter-examples even in the unidimensional case. To formulate our second lemma let us define a restriction \( \mu \) of \((m-1)\) dimensional Lebesgue measure to set \( \partial \Omega \) by:

\[
\mu(C') = \int_{\partial \Omega} \chi^C(\alpha) d\alpha, \tag{7}
\]

where \( C' \subset \partial \Omega \) is a measurable set, \( \chi^C(\cdot) \) is its characteristic function, and integral in equation (7) is a scalar surface integral.
Some Preliminary Results

**Lemma 2** Assume $\lambda(\Omega_0) = 0$ and Assumption 1 holds. Then $\mu(\Omega_0) = 0$ for almost all $\beta$.

**Proof.** Let $s(\cdot)$ be the surplus function generated by the optimal tariff via (77). By Lemma 1, $\Omega_0 \subset \partial \Omega$. By Assumption 1, it implies that

$$\Omega_0 = \bigcup_{i=1}^{k} \Omega_{0i}, \quad (8)$$

where $\Omega_{0i} = \{ \alpha \in \overline{\Omega} : g_i(\alpha, \beta) = 0, s(\alpha) = 0 \}$. Since union in (8) is finite it is sufficient to prove that $\mu(\Omega_{0i}) = 0$ for almost all $\beta$ for all sets in (8). Assumption 1 and the Transversality Theorem (see, Mas-Colell, Whinston, and Green, 1995, Chapter 17D) imply that

$$\text{rank} \left( \begin{array}{c} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta) \end{array} \right) = 2 \quad (9)$$

for almost all $\beta$. Therefore, by the Implicit Function Theorem, $\Omega_{0i}$ is a manifold of dimension of $(m - 2)$ for almost all $\beta$ and the conclusion follows.
Theorem 1 Consider problem (1)-(2) and assume that $u(\cdot, \cdot)$ is twice continuously differentiable and strictly increasing in both arguments, $c(\cdot)$ is strictly convex and twice continuously differentiable, $\Omega$ is an open bounded set with a piecewise smooth boundary satisfying Assumption 1, $f(\cdot)$ is strictly positive on $\Omega$, and the exists $K > 0$ such that

$$u(\alpha, x) \leq K(\alpha \cdot \nabla_\alpha u(\alpha, x))$$

(10)

for all $x \in \mathbb{R}^n_+$ and all $\alpha \in \Omega$. Then for almost all $\beta$ a set of positive measure of consumers is excluded at the equilibrium.

Comments:

1. Barelli relaxed Armstrong’s assumptions on the geometry and preferences

2. He showed that Armstrong’s result continues to hold but with a qualification almost always (see Rochet and Stole’ s counterexample below)
Rochet and Stole’ s counterexample

Let $u : \Omega \times R_+ \rightarrow R$ be

$$u(\alpha, x) = (\alpha_1 + \alpha_2)x$$

(11)

and $\Omega$ be a rectangle with sides parallel to the 45 degrees and $-45$ degrees lines.

One can shift the rectangle sufficiently far to the right to have an empty exclusion region.

The result is driven by the fact that they allow only very special collection of type sets, rectangles with parallel sides.

Formally, **Assumption 1 fails in this case**, since $g_1(\alpha, \beta) = \alpha_1 + \alpha_2 - \beta = 0$ and

$$\frac{\nabla_\alpha u(\alpha, x)}{\nabla_\alpha g_i(\alpha, \beta)} = \begin{pmatrix} x \\ 1 \end{pmatrix} \Rightarrow \text{rank}(\begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta) \end{pmatrix}) = 1.$$
Genericity of exclusion and market stricture: the basic set-up

Consider a framework similar the previous one, but assume that the market is served by $K$ producers.

The production cost is identical among the producers.

A pure strategy of a producer $k$ is a non-linear tariff, i.e. a measurable mapping $t_k : \mathbb{R}^n_+ \rightarrow \mathbb{R}$.

Let us assume that there exists a symmetric equilibrium at which all producers charge the same tariff.

We will argue that at such an equilibrium a positive measure of the consumers are not served.
Genericity of exclusion and market stricture: the model

Assume at equilibrium producer \( k \) charges tariff \( t_k(\cdot) : R^n_+ \rightarrow R \). Then \( t_\ell(\cdot) \) solves

\[
\max_{t_\ell(\cdot)} \int_{\Omega} (t_\ell(x_\ell(\alpha)) - c_\ell(x_\ell(\alpha))) f(\alpha) d\alpha, \quad (13)
\]

subject to:

\[
x(\alpha) \in \text{arg max}(\max_{x \in R^n_+} (v(\alpha, x) - t(x)))
\]

\[
\max_{x \in R^n_+} (v(\alpha, x) - t(x)) \geq s_0(\alpha) \quad (14)
\]

where

\[
t(x) = \min \sum_{k=1}^{K} t_k(x_k), \text{ s.t. } \sum_{k=1}^{K} x_k = x, x_k \geq 0 \quad (15)
\]

and \( t_{-\ell}(x) \) solves problem (15) subject to an additional constraint \( x_\ell = 0 \) and

\[
s_0(\alpha) = \max\{s^*_0(\alpha), \max_{x \in R^n_+, x_\ell=0} (v(\alpha, x) - t_{-\ell}(x))\}.
\]
Some definitions

Let us define

\[ u(\alpha, x_\ell) = v(\alpha, x_\ell + \sum_{k=1, k \neq \ell}^K x_k) - \sum_{k=1, k \neq \ell}^K t_k(x_k) - s_0 \]

Then the problem of producer \( \ell \) becomes:

\[
\max_{t_\ell(\cdot)} \int_{\Omega} (t_\ell(x_\ell(\alpha)) - c_\ell(x_\ell(\alpha))) f(\alpha) d\alpha,
\]

\[ s.t. \quad x_\ell(\alpha) \in \arg \max (u(\alpha, x_\ell) - t_\ell(x_\ell)) \]

\[ \max_{x_\ell \in \mathbb{R}_+^n} (u(\alpha, x_\ell) - t_\ell(x_\ell)) \geq 0. \]

Let us impose the following form of single-crossing property:

\[
\frac{\partial^2 v}{\partial \alpha_i \partial x_j} \geq 0 \text{ for all } i, j
\]  \hspace{1cm} (16)

Then \( u(\alpha, x_\ell) \) is increasing in \( \alpha \) for all \( x_\ell \in \mathbb{R}_+^n \).
The main result

**Theorem 2** Under Assumptions of Theorem 1 a positive measure of consumers will be excluded in any symmetric equilibrium of the oligopolistic market.

**Proof.** Consider oligopolist 1. Given behavior of her competitors, her problem is isomorphic to the problem of the monopolist, with appropriately redefined utility. Therefore, Theorem 1 implies that she will find it optimal to exclude a positive measure of consumer. By symmetry, so will the other oligopolists.
Conclusions

In this paper we extended Armstrong’s (1996) result to argue that if private information of the consumers is multidimensional a positive measure of the consumers will generically not be served at the equilibrium.

The result holds for general utility functions and type sets and can be extended to the oligopolistic setting.

Another extension, worth pursuing is to allow for arbitrary outside options, i.e. relaxing Assumption 2

Another interesting extension, obtained in the literature, are for the auction-theoretic setting (Monteiro, Svaiter, and Page, 2001).